Asymptotic distribution of the pseudo-orbits and the generalized Euler constant γ_Δ for a family of strongly chaotic systems

R. Aurich and F. Steiner
II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 2000 Hamburg 50, Federal Republic of Germany
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Dynamical zeta functions, defined as Euler products over classical periodic orbits, have recently received enhanced attention as an important tool for the quantization of chaos. Their representation as a Dirichlet series over pseudo-orbits has proven to be particularly useful, since these series seem to possess in the general case much better convergence properties than the original Euler product. The convergence of the Dirichlet series depends crucially on the asymptotic distribution of the pseudo-orbits and thus on the ergodicity of the underlying dynamical system. It is shown that the lengths l_n (or rather $\exp l_n$) of the classical periodic orbits play mathematically the role of generalized prime numbers. Based on the theory of Beurling's generalized prime numbers, we derive an exact law for the proliferation of psuedo-orbits for the Hadamard-Gutzwiller model, which is one of the main testing grounds of our ideas about quantum chaos. The strength of growth of the pseudo-orbits is determined by the ratio $Z(2)/Z'(1)$, where $Z(s)$ denotes the Selberg zeta function. Two explicit, complementary representations are given that allow the computation of this ratio solely from the length spectrum $\{l_n\}$ of the classical periodic orbits, or from the quantal energy spectrum $\{E_n\}$. One of these relations depends exponentially on the generalized Euler constant γ_Δ, which is therefore also studied. The formulas are applied to two strongly chaotic systems. It turns out that our asymptotic law describes the mean proliferation of pseudo-orbits very well not only in the asymptotic region, but also surprisingly well down to the shortest pseudo-orbit.

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I. INTRODUCTION

In the past decades great efforts have been undertaken to unravel the mystery of chaos in quantum mechanics. One main goal has been to find a semiclassical quantization rule for classically chaotic systems involving classical quantities only. Based on Selberg's trace formula [1] and, in the general case, on Gutzwiller's periodic-orbit theory [2,3], recently several quantization rules have been proposed that allow the determination of the quantal energies, at least in principle, from a knowledge of the lengths of the periodic orbits of the underlying classical dynamical system. One formulation starts with the dynamical zeta function $Z(s)$ defined by a Euler product over the classical periodic orbits [4–9] whose zeros are tightly connected with the quantal energies. This formulation meets, however, a serious obstruction, since the zeros occur in general in a region where the Euler product diverges. The divergence, which at first sight seems to be merely a mathematical subtlety, turns out to be deeply connected with the ergodicity of the classical dynamical system and is determined [10] by the famous topological (Kolomogorov-Sinai) entropy and the metric entropy. Recently, it has been noticed that the Euler product can be rewritten as a Dirichlet series and that the latter may possess a larger region of convergence than the original Euler product [5–7,11]. Thus the formulation in terms of a Dirichlet series seems to be able to quantize a much larger class of chaotic systems. Furthermore, these Dirichlet series are an important ingredient in the recently suggested so-called Riemann-Siegel lookalike formula [9] (see also [7]). The role played by the periodic orbits as the main input in the Euler product formulation is now played by the so-called pseudo-orbits. The lengths of the pseudo-orbits entering the Dirichlet series are given by a certain linear combination of the lengths of the periodic orbits. The connection between the physical periodic orbits and pseudo-orbits turns out to be analogous to the well-known mathematical relation between primes and integers in a sense to be explained below. For the application of the quantization rules, it is important to know the proliferation of the pseudo-orbits with increasing length, which is unknown *a priori* in contrast to the proliferation of the periodic orbits.

It is the purpose of this paper to derive the law describing the asymptotic proliferation of the pseudo-orbits for the Hadamard-Gutzwiller model, which is one of the main testing grounds of our ideas about quantum chaos. Since it seems unlikely that one will be able to derive the asymptotic distribution of the pseudo-orbits for a general chaotic system in the near future, it is important to study a dynamical system that allows a rigorous derivation, but yet is general enough that one can abstract from it the expected behavior in the general case. The Hadamard-Gutzwiller model consists of a particle sliding freely on a compact Riemann surface of genus $g \geq 2$ having constant negative Gaussian curvature. An introduction to this strongly chaotic system can be found in [3,12] and in our earlier papers [13,4,14]. In all applications studied so far [13,15,4,14,7,16], the simplest realization of such a compact Riemann surface having genus $g=2$ is considered, i.e., a surface with the topology of a sphere with two handles. According to Poincaré's model for non-Euclidean geometry, we can represent the compact Riemann surfaces as fundamental domains $\mathcal{F}$ in the Poincaré disk $\mathcal{D}$, which consists of the interior of the unit circle in the complex z plane ($z=x_1+ix_2$) endowed with the hyperbolic metric

$$g_{ij}=\frac{4}{(1-x_1^2-x_2^2)^2}\delta_{ij},\quad i,j=1,2 \tag{1}$$

corresponding to constant negative Gaussian curvature $K=-1$. (Here the length scale R is put equal to 1.) On the surfaces with genus $g\geq 2$, the free motion of a point particle with mass m is determined by the Hamiltonian

$$H=\frac{1}{2m}p_i g^{ij}p_j,\quad p_i=mg_{ij}\frac{dx^j}{dt}\ , \tag{2}$$

which defines a conservative Hamiltonian system with two degrees of freedom whose classical motion is strongly chaotic (K system). (The energy $E=H$ is the only constant of motion.) The motion takes place along geodesics on a fundamental domain $\mathcal{F}\subset\mathcal{D}$, where $\mathcal{F}$ is a hyperbolic polygon with $4g$ geodesic edges, where opposite sides must be identified ("glued together") in order to obtain a closed Riemannian manifold. The area A of $\mathcal{F}$ is given by $A(\mathcal{F})=4\pi(g-1)$ due to the Gauss-Bonnet theorem.

The corresponding quantum-mechanical system is governed by the Schrödinger equation

$$-\frac{\hbar^2}{2m}\Delta\Psi_n(z)=E_n\Psi_n(z),\quad z\in\mathcal{F}\ , \tag{3}$$

where Δ denotes the Laplace-Beltrami operator, which reads on the Poincaré disk

$$\Delta=\tfrac{1}{4}(1-x_1^2-x_2^2)^2\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right]\ . \tag{4}$$

The eigenfunctions $\Psi_n(z)$ have to be orthonormal with respect to the invariant measure determined by the metric (1)

$$\int\int_{\mathcal{F}} dx_1 dx_2\frac{4}{(1-x_1^2-x_2^2)^2}\Psi_m^*(z)\Psi_n(z)=\delta_{mn}\ . \tag{5}$$

The Schrödinger equation (3) has to be solved with periodic boundary conditions

$$\Psi_n(b(z))\equiv\Psi_n(z)\quad\text{for all } b\in\Gamma\ , \tag{6}$$

where Γ is a discrete subgroup of $G\equiv\mathrm{SU}(1,1)/\{\pm 1\}$. Here the action of a "boost" $b\in\Gamma$,

$$b=\begin{bmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{bmatrix},\quad |\alpha|^2-|\beta|^2=1\ , \tag{7}$$

is defined by the Möbius transformation

$$b(z)=\frac{\alpha z+\beta}{\beta^* z+\alpha^*}\ . \tag{8}$$

Under the periodic boundary conditions (6), the quantal energy spectrum $\{E_n\}$ of (3) is discrete with $0=E_0<E_1\leq E_2\leq\cdots$. Asymptotically, the number $\mathcal{N}(E)$ of quantal energies E_n less than or equal to E is determined by Weyl's famous law

$$\mathcal{N}(E)\sim\frac{A(\mathcal{F})}{4\pi}E,\quad E\to\infty\ . \tag{9}$$

(Units $\hbar=2m=R=1$ are used from now on.)

For the Riemann surfaces under consideration the asymptotic behavior of the number $N(\ell)$ of primitive periodic orbits with length l_n less than or equal to ℓ was derived by Huber [17] (see also [18])

$$N(\ell)=\mathrm{li}(e^{\ell})+\sum_{n=1}^{M}\mathrm{li}(e^{s_n\ell})+O\left[\frac{e^{3\ell/4}}{\sqrt{\ell}}\right],\quad \ell\to\infty\ , \tag{10}$$

where the logarithmic integral is defined by the Cauchy principal value

$$\mathrm{li}(x)\equiv P\int_0^x\frac{dt}{\ln t},\quad x>0\ .$$

In Eq. (10) the sum is over the M so-called "small" eigenvalues $E_n\equiv s_n(1-s_n)$, defined as eigenvalues satisfying $0<E_n<\frac{1}{4}$, i.e., $\frac{1}{2}<s_n<1$. For a compact Riemann surface of genus $g>2$ one has $M\leq 4g-3$ [19,20], while for $g=2$ there is at most one small eigenvalue [21]. Randol has shown [22] that the remainder in Eq. (10) can be replaced by $O(e^{3\ell/4}/\ell)$.

From Eq. (10) we obtain the *exact* leading asymptotic behavior $N(\ell)=e^{\ell}/\ell+\cdots$, which has to be compared with the *expected* behavior $N(\ell)=e^{\tau\ell}/\tau\ell+\cdots$ for general chaotic systems, where $\tau>0$ denotes the topological entropy. While the latter behavior, and in particular the numerical value for τ, can in general not be derived analytically, it is an exceptionally nice feature of the Hadamard-Gutzwiller model that this behavior can be rigorously derived including the subdominant terms given in Eq. (10) and the exact value $\tau=1$ for the topological entropy. In the general case this behavior can only be tested numerically by computing a large number of periodic orbits, and in case the data turn out to be consistent with the expected behavior, the topological entropy can be obtained numerically from a fit. For most chaotic systems studied in the past, people have not succeeded in enumerating enough periodic orbits, which would allow such a consistency check and thus the determination of τ. A rare exception is the hyperbola billiard, for which a large number of periodic orbits could be determined systematically [23] (see also [5]).

It is instructive to express Eq. (10) in the variable $x\equiv e^{\ell}$ using the definition $\pi_{\mathcal{P}}(x)\equiv N(\ell)$,

$$\pi_{\mathcal{P}}(x)=\mathrm{li}(x)+\sum_{n=1}^{M}\mathrm{li}(x^{s_n})+O\left[\frac{x^{3/4}}{\ln x}\right],\quad x\to\infty\ . \tag{11}$$

Here Randol's improved rest term has been used. Equation (11) reveals a striking similarity with the Riemann–von Mangoldt formula for the prime numbers discovered by Riemann [24] and proved by von Mangoldt

[25]. In analogy with the famous prime number theorem, the first term in Eqs. (10) and (11) can be called a "prime geodesic theorem" [26] or a "prime orbit theorem" [27]. For an explicit expression of the remainder term in Eqs. (10) and (11), see [16].

As already mentioned, some of the new rules for quantizing chaos are based on the so-called dynamical zeta functions. These functions arise very naturally if one starts from the generalized version [10] of Gutzwiller's trace formula [2] and considers the regularized trace of the energy-dependent Green's function following exactly the derivation of the Selberg zeta function described in [28]. As an illustrative example, let us consider the hyperbola billiard [23] for which the dynamical zeta function reads [5]

$$Z(s)=\prod_{\gamma}\prod_{k=0}^{\infty}(1-\sigma_\gamma^k b_{\gamma,k}e^{-sl_\gamma-[k+(1/2)]u_\gamma-i(\pi/2)\nu_\gamma}) . \tag{12}$$

Here $s=-ip$, and p denotes the (complex) momentum $E=p^2=-s^2$. In Eq. (12) the "Euler product" over γ runs over all primitive periodic orbits of the hyperbola billiard. To each orbit γ there belongs a well-defined length l_γ, an instability exponent $u_\gamma>0$, and a (scaled) Lyapunov exponent $\lambda_\gamma\equiv u_\gamma/l_\gamma$. σ_γ is the sign of the monodromy matrix for the periodic orbit γ, and ν_γ is the maximal number of conjugate points along the periodic orbit plus twice the number of reflections on the boundary, where Dirichlet boundary conditions are demanded. For a generic orbit one has $b_{\gamma,k}=1$.

The infinite product (12) converges absolutely for $\mathrm{Re}s>\sigma_a\equiv\tau-(\bar{\lambda}/2)$ ("entropy barrier"), where the topological entropy $\tau>0$ measures the exponential proliferation of the periodic orbits as a function of their length l_γ according to the asymptotic behavior $N(\ell)=e^{\tau\ell}/\tau\ell+\cdots$, $\ell\to\infty$. The quantity $\bar{\lambda}$ denotes the average of the Lyapunov exponents λ_γ and is also called the metric entropy because it measures the exponential spreading of the trajectories, i.e., the rate at which phase space is distorted in the neighborhood of a periodic orbit [3].

The important properties of the dynamical zeta function (12) can be deduced from the following representation for the logarithmic derivative of $Z(s)$:

$$\frac{1}{2s}\frac{Z'(s)}{Z(s)}=B+\Phi(s)+\sum_{n=1}^{\infty}\left[\frac{1}{\tilde{E}_n+s^2}-\frac{1}{\tilde{E}_n}\right] , \tag{13}$$

which can be derived [5] from the generalized version [10] of Gutzwiller's trace formula. Here the function $\Phi(s)$ originates from the "zero-length" contribution to the trace formula and is directly related to the mean level density as determined by Weyl's law (including higher corrections). B is a constant that is the analog of the generalized Euler constant γ_Δ to be discussed below in the case of the Hadamard-Gutzwiller model. The energies $\tilde{E}_n$ denote the semiclassical approximations to the true quantal energies E_n of the hyperbola billiard. [Relation (13) is only valid semiclassically, since the Gutzwiller trace formula has been derived under the assumption $\hbar\to 0$.] From Eq. (13) one deduces that the dynamical zeta function $Z(s)$ has an analytic continuation to complex s values and, in particular, that it has zeros on the "critical line" $\mathrm{Re}s=0$ at points $s_n=\pm i\sqrt{\tilde{E}_n}$, $n=1,2,\ldots$. (For a discussion of the subtleties, see Ref. [5].) Thus the condition $Z(s)=0$ constitutes a semiclassical quantization rule. However, the zeros are located on the line $\mathrm{Re}s=0$, where the Euler product (12) does not converge absolutely, since the entropy barrier of the hyperbola billiard is at $\mathrm{Re}s=\sigma_a=0.2415$ [5]. In order to obtain an analytic continuation of $Z(s)$ beyond the entropy barrier, one transforms the Euler product (12) into a Dirichlet series with the help of Euler's identity [29]

$$\prod_{k=0}^{\infty}(1-yx^k)=1+\sum_{m=1}^{\infty}\frac{(-1)^m y^m x^{(1/2)m(m-1)}}{\prod_{r=1}^{m}(1-x^r)} ,$$
$$|x|<1,\quad y\in\mathbb{C} . \tag{14}$$

One then obtains

$$Z(s)=1+\sum_{N=1}^{\infty}A_N e^{-sL_N} , \tag{15}$$

where the sum over N runs through all "pseudo-orbits" with "pseudolength"

$$L_N\equiv\sum_i m_i l_{\gamma i},\quad m_i\in\mathbb{N},\quad L_N\le L_{N+1} .$$

[For an explicit expression of the coefficients A_N, see Eq. (11) in Ref. [5].] Obviously, the convergence of the Dirichlet series (15) depends crucially on the proliferation of the pseudo-orbits with increasing length, and one is thus led to study the asymptotic behavior of the staircase function

$$N_{\mathcal{P}}(L)\equiv\mathcal{N}\{L_N|L_N\le L\} , \tag{16}$$

which counts the number of pseudo-orbits with pseudolength L_N smaller than or equal to L. Making the simplest assumptions, one expects the asymptotic behavior $N_{\mathcal{P}}(L)=Ae^{\tau L}+\cdots$, $L\to\infty$, to hold. It is a challenge to seek for a sound derivation of the asymptotic distribution of the pseudo-orbits and to find an explicit expression for the strength A of the growth of pseudo-orbits for general chaotic systems like the hyperbola billiard. It is the purpose of this paper to provide a rigorous derivation in the case of the Hadamard-Gutzwiller model.

For the Hadamard-Gutzwiller model the Selberg zeta function [1] plays the role of the dynamical zeta function. It is defined as an Euler product over the length spectrum $\{l_n\}$ of the primitive periodic orbits

$$Z(s)=\prod_{\{l_n\}}\prod_{k=0}^{\infty}(1-e^{-(s+k)l_n}),\quad \mathrm{Re}s>1 . \tag{17}$$

Here the variable s is defined as $s\equiv\frac{1}{2}-ip$, where the relation between the momentum p and the energy E is slightly different from the hyperbola billiard and is given by $E=p^2+\frac{1}{4}=s(1-s)$. A comparison between the two zeta functions (17) and (12) shows that the Selberg zeta function is especially simple. Indeed, for the Hadamard-

Gutzwiller model all Lyapunov exponents are equal to 1, and for all primitive periodic orbits $\sigma_\gamma = b_{\gamma,k} = 1$, $\nu_\gamma = 0$.

As a consequence of Huber's law (10), the product (17) converges absolutely only for $\mathrm{Re}s > 1$, corresponding to the fact that the topological entropy τ is equal to 1. Starting from the Selberg trace formula [1], one can derive [28] the following equation:

$$\frac{1}{2s-1}\frac{Z'(s)}{Z(s)} = \gamma_\Delta + \frac{1}{s(s-1)} + 2(g-1)\Psi(s) + \sum_{n=1}^{\infty}\left[\frac{1}{E_n + s(s-1)} - \frac{1}{E_n}\right], \quad (18)$$

which relates the logarithmic derivative of the Selberg zeta function to the regularized trace of the resolvent of the Schrödinger operator, i.e., to the quantal energy spectrum $\{E_n\}$. Here $\Psi(s) = \Gamma'(s)/\Gamma(s)$ is the digamma function, and γ_Δ denotes the "generalized Euler constant" of the Laplacian on the considered Riemann surface defined by [28]

$$\gamma_\Delta \equiv 2(g-1)\gamma - 1 + \frac{1}{2}\frac{Z''(1)}{Z'(1)} . \quad (19)$$

[$\gamma = -\Psi(1)$ denotes Euler's constant.] The *exact* relation (18) for the Hadamard-Gutzwiller model should be compared with the *semiclassical* relation (13) for the hyperbola billiard. The similarity between the two relations is striking. From Eq. (18) it is not difficult to derive that $Z(s)$ has an analytic continuation to all $s \in \mathbb{C}$ (in fact, the Selberg zeta function is an entire function of s), whose "trivial" zeros are exactly known: $s = 1$ is a simple zero, $s = 0$ is a zero of multiplicity $2g - 1$, and $s = -k$, $k \in \mathbb{N}$, are zeros with multiplicity $2(g-1)(2k+1)$. Furthermore, the small eigenvalues with $0 < E_n < \frac{1}{4}$ lead to zeros on the real line between 0 and 1, located at $s_n = \frac{1}{2} \pm \sqrt{1/4 - E_n}$, $n = 1, \ldots, M$. Most importantly, $Z(s)$ has an infinite number of "nontrivial" zeros located at $s_n = \frac{1}{2} \pm i\sqrt{E_n - 1/4}$ corresponding to the quantal energies $E_n \geq \frac{1}{4}$, i.e., the nontrivial zeros lie on the critical line $\mathrm{Re}s = \frac{1}{2}$, and thus the analog of the Riemann hypothesis holds for the Selberg zeta function if one discards a finite number of zeros on the real line with $0 \leq s_n \leq 1$.

The remarkable connection between the nontrivial zeros $\{s_n\}$ of $Z(s)$ and the quantal energies $\{E_n\}$ is the basis of the following *exact rule for quantizing chaos* [4,7] (see also [6]):

$$Z(s) = 0 . \quad (20)$$

Unfortunately, as in the case of the hyperbola billiard, the Euler product (17) cannot be directly used for a determination of the energy levels via (20), since it converges absolutely only to the right of the line $\mathrm{Re}s = 1$ (entropy barrier [7]). In order to make use of the quantization rule (20), we need an analytic continuation of $Z(s)$ that is valid on the critical line $\mathrm{Re}s = \frac{1}{2}$. To this end we again transform the Euler product (17) into a Dirichlet series with the help of Euler's identity (14). We then obtain

$$Z(s) = \prod_{\{l_n\}}\left[1 + \sum_{m=1}^{\infty} a_{mn} e^{-ml_n s}\right] \quad (21)$$

with

$$a_{mn} \equiv (-1)^m \frac{e^{-[(1/2)m(m-1)l_n]}}{\prod_{r=1}^{m}(1 - e^{-rl_n})} . \quad (22)$$

Expanding the product over the primitive periodic orbits in (21), we again arrive [7] at the generalized Dirichlet series (15)

$$Z(s) = 1 + \sum_{N=1}^{\infty} A_N e^{-sL_N} ,$$

where the sum over N runs through the pseudo-orbits with pseudolength

$$L_N \equiv \sum_m m_i l_{n_i}, \quad m_i \in \mathbb{N} , \quad (23)$$

$L_N \leq L_{N+1}$, and $A_N \equiv \prod_i a_{m_i n_i}$. In [7] we have shown that for the Hadamard-Gutzwiller model the abscissa σ_a of absolute convergence of the Dirichlet series (15) is given by $\sigma_a = 1$. In addition, we have numerically studied the abscissa σ_c of conditional convergence of two compact Riemann surfaces with genus $g = 2$. In the two cases studied, there is strong evidence that $\sigma_c < 1$ and thus that the Dirichlet series (15) provides an analytic continuation of $Z(s)$ into the critical strip. As in the case of the hyperbola billiard, it is clear from Eq. (15) that the convergence of the Dirichlet series depends crucially on the proliferation of the pseudo-orbits with increasing length, and thus we have to study the asymptotic behavior of the staircase function (16)

$$N_{\mathcal{P}}(L) \equiv \mathcal{N}\{L_N | L_N \leq L\} ,$$

which counts the number of pseudo-orbits with pseudolengths L_N smaller than or equal to L. The main result of our paper consists in a derivation of the leading asymptotic term of $N_{\mathcal{P}}(L)$ for the Hadamard-Gutzwiller model in the limit $L \to \infty$. To derive the asymptotic behavior of the pseudolength spectrum $\{L_N\}$, we shall make use of the fact that the length spectrum $\{l_n\}$ of primitive periodic orbits can be interpreted as generalized prime numbers in the framework of Beurling's theory [30], while the pseudolength spectrum $\{L_N\}$ can be identified with generalized integers.

Our paper is organized as follows. In Sec. II we shall summarize the necessary definitions and theorems of the theory of Beurling's generalized prime numbers. These theorems will be applied in Sec. III to the length spectra $\{l_n\}$ and $\{L_N\}$. As a result, we shall obtain an exact expression for the asymptotic behavior of the pseudolength spectrum. The leading asymptotic term is completely determined for any compact Riemann surface by the associated Selberg zeta function $Z(s)$ evaluated as $s = 1$ and $s = 2$, respectively. Explicitly, the relevant parameter is given by the ratio $Z(2)/Z'(1)$. To apply our formula to a given Riemann surface, we therefore require a formula for this ratio. In Sec. IV we shall derive two explicit complementary representations for $Z(2)/Z'(1)$ involving the length spectrum $\{l_n\}$ and the quantal energy spectrum $\{E_n\}$, respectively. The second relation involves the generalized Euler constant γ_Δ defined in Eq. (19). We therefore devote Sec. V to a study of this extremely interesting number. Again we shall present two complementary formulas for γ_Δ, the first involving the energy spectrum $\{E_n\}$ only, the second the length spectrum $\{l_n\}$ only. In Sec. VI we shall apply our results to two compact Riemann surfaces of genus $g=2$. We shall show that our asymptotic law describes the proliferation of the pseudolengths very well not only in the asymptotic regime, but also in the mean surprisingly well down to the first pseudolength. Finally, our results are summarized in Sec. VII.

II. THEORY OF BEURLING'S GENERALIZED PRIME NUMBERS

The concept of "generalized primes" was introduced already by Landau [31], but a systematic study was carried out only later by Beurling [30]. Let us start with a definition of generalized primes and generalized integers.

Definition 1. A sequence $\mathcal{P}$ of positive real numbers $(p_1,p_2,p_3,\ldots)$ with $1<p_1\le p_2\le\cdots$, $p_j\to\infty$, is called a generalized prime number system or, briefly, a system of g primes.

Definition 2. Set $n_0=1$, $n_1=p_1$, and let $(n_0,n_1,n_2,\ldots)$ be the nondecreasing sequence of real numbers formed by the values $\prod_j p_j^{\nu_j}$, where each ν_j is allowed to range over all non-negative integers. The sequence $(n_0,n_1,n_2,\ldots)$ is called the generalized integers of the system $\mathcal{P}$ or, briefly, the g integers associated with $\mathcal{P}$. For our later application, it is important to notice that the real numbers p_j and n_j need not be all distinct, because degeneracies occur among the lengths of the periodic orbits as well as among the pseudo-orbits.

Definition 3. The abscissa of convergence $\tau>0$ of the system $\mathcal{P}$ is defined as the infimum of those real values σ for which $\sum_{j=1}^{\infty}p_j^{-\sigma}$ converges. It is easy to see that one can always redefine the system $\mathcal{P}$ in such a way that $\tau=1$. (One introduces $p_j'\equiv p_j^{\tau}$.) In the following we shall therefore assume that $\tau=1$.

Definition 4. The ζ function $\zeta_{\mathcal{P}}(s)$ associated with the system $\mathcal{P}$ is defined by the Euler product

$$\zeta_{\mathcal{P}}(s)\equiv\prod_{j=1}^{\infty}(1-p_j^{-s})^{-1},\quad \mathrm{Re}s>1\ . \tag{24}$$

Definition 5. The counting functions of the system $\mathcal{P}$ are defined by

$$\pi_{\mathcal{P}}(x)\equiv\mathcal{N}\{p_j\le x\}\ , \tag{25}$$

$$I_{\mathcal{P}}(x)\equiv\mathcal{N}\{n_j\le x\}\ . \tag{26}$$

They count the number of g primes and g integers, respectively, which are smaller than or equal to x. With these definitions at hand, we can now state the two theorems that we shall need for the derivation of the asymptotic behavior of $N_{\mathcal{P}}(L)$.

Theorem A [32]. If

$$\pi_{\mathcal{P}}(x)=\mathrm{li}(x)+O(xe^{-c_2\ln^{\beta}x}),\quad x\to\infty, \tag{27}$$

holds for $c_2>0$, $0<\beta<1$, then

$$I_{\mathcal{P}}(x)=Ax+O(xe^{-c_1\ln^{\alpha}x}),\quad x\to\infty \tag{28}$$

for some $A>0$, $c_1>0$ and $\alpha=\beta/(2+\beta)$.

Theorem B [33]. If

$$I_{\mathcal{P}}(x)=Ax+O(x\ln^{-\delta}x),\quad A>0,\ \delta>1,\ x\to\infty\ , \tag{29}$$

then

$$\zeta_{\mathcal{P}}(s)=\frac{A}{s-1}+\gamma_{\mathcal{P}}+O(s-1)\quad\text{for } \mathrm{Re}s>1\ , \tag{30}$$

with

$$\gamma_{\mathcal{P}}=A+\int_1^{\infty}dx\frac{I_{\mathcal{P}}(x)-Ax}{x^2}\ .$$

III. PROLIFERATION OF THE PSEUDO-ORBITS

In this section we shall apply Beurling's theory of the generalized prime number to the length spectrum of the primitive periodic orbits of the Hadamard-Gutzwiller model. It is worthwhile to notice that our interpretation of the lengths of periodic orbits as generalized primes is not restricted to the Hadamard-Gutzwiller model, but is always possible for general dynamical systems, if one considers the periods of the periodic orbits instead of their lengths, as long as all orbits are isolated and unstable and the system has a nonvanishing topological entropy $\tau>0$. (τ is then identical to the abscissa of convergence referred to in Definition 3 of Sec. II.) However, in order to apply Theorems A and B of Sec. II, the order of the remainder terms to the counting functions has to be known, see Eq. (27) and (29), which at present is completely unknown for chaotic systems like the hyperbola billiard. This shows again how important it is to study the Hadamard-Gutzwiller model, since much more information is available for this dynamical system and thus the proliferation of the pseudo-orbits can be derived for this model as will be shown in this section.

We define the g prime system $\mathcal{P}$ associated with the length spectrum $\{l_n\}$ of the primitive periodic orbits of the Hadamard-Gutzwiller model by

$$p_n\equiv e^{l_n}\ , \tag{31}$$

and the g integers associated with $\mathcal{P}$ by $(n_0\equiv1)$

$$n_N\equiv\prod_j p_j^{m_j}=e^{L_N}\ , \tag{32}$$

where $m_j\in\mathbb{N}_0$ and L_N denote the pseudolength defined in Eq. (23). With these identifications, it is obvious that the counting function (25) is identical to the staircase function $N(\ell)$ of the primitive periodic orbits, i.e., $N(\ell)=\pi_{\mathcal{P}}(e^{\ell})$, whose asymptotic behavior was already given in Eq. (11)

$$\pi_{\mathcal{P}}(x)=\mathrm{li}(x)+\sum_{n=1}^{M}\mathrm{li}(x^{s_n})+\left\{\frac{x^{3/4}}{\ln x}\right\}\ . \tag{33}$$

Because of $x \to \infty$,

$$\frac{x^{3/4}}{\ln x} < x^{3/4} = xe^{-[(1/4)\ln x]} < xe^{-c_2 \ln^{\beta} x} ,$$

with $0<\beta<1$, $c_2=\frac{1}{4}$, and

$$\mathrm{li}(x^{s_n}) = O\left[\frac{x^{s_n}}{\ln(x^{s_n})}\right], \quad xe^{-(1-s_n)\ln x} < xe^{-c_2' \ln^{\beta} x} ,$$

with $c_2' = 1-s_n > 0$ and $0<\beta<1$, all assumptions of Theorem A of Sec. II are satisfied, and we obtain [see Eq. (28)]

$$I_{\mathcal{P}}(x) = Ax + O(xe^{-c_1 \ln^{\alpha} x}), \quad x \to \infty, \tag{34}$$

with $A>0$, $c_1>0$, and $0<\alpha<\frac{1}{3}$. Clearly, the function $I_{\mathcal{P}}(x)$ can be identified via $N_{\mathcal{P}}(L) \equiv I_{\mathcal{P}}(e^L)$ with the staircase function counting the number of pseudo-orbits, see Eq. (16), and thus we obtain from Eq. (34) the asymptotic behavior

$$N_{\mathcal{P}}(L) = Ae^{L}[1+O(e^{-c_1 L^{\alpha}})], \quad L \to \infty . \tag{35}$$

There remains the calculation of the coefficient A, i.e., of the strength of growth of the pseudo-orbits. To this end we shall use Theorem B of Sec. II, which can be applied because of the following inequality ($c_1>0$, $0<\alpha<\frac{1}{3}$, $\delta>1$, and $x\to\infty$):

$$xe^{-c_1 \ln^{\alpha} x} < xe^{-\delta \ln \ln x} = x \ln^{-\delta} x .$$

Then theorem B, Eq. (30), tells us that the coefficient A is identical to the residue of the pole of the ζ function $\zeta_{\mathcal{P}}(s)$ at $s=1$. The ζ function $\zeta_{\mathcal{P}}(s)$ of $\mathcal{P}$ associated with the length spectrum $\{l_n\}$, see Eq. (24), is given by (Re$s>1$)

$$\zeta_{\mathcal{P}}(s) = \frac{1}{R(s)} \tag{36}$$

with

$$R(s) \equiv \prod_{n=1}^{\infty} (1-p_n^{-s}) = \prod_{\{l_n\}} (1-e^{-sl_n}) , \tag{37}$$

which is related to Selberg's zeta function $Z(s)$, see Eq. (12), by

$$Z(s) = \prod_{k=0}^{\infty} R(s+k) = \prod_{k=0}^{\infty} \zeta_{\mathcal{P}}(s+k)^{-1} , \tag{38}$$

and thus

$$R(s) = \frac{Z(s)}{Z(s+1)} , \qquad R'(s) = \frac{Z'(s)}{Z(s+1)} - R(s)\frac{Z'(s+1)}{Z(s+1)} . \tag{39}$$

As a consequence of the zero mode $E_0=0$, Selberg's zeta function $Z(s)$ has a simple zero at $s=1$ with $Z'(1)>0$. Furthermore, it follows from the Euler product (17) that $Z(s)>0$ for $s>1$, which leads with (39) to $R(1)=0$, $R'(1)=Z'(1)/Z(2)>0$, and thus to the expansion

$$R(s) = \frac{Z'(1)}{Z(2)}(s-1) + O((s-1)^2) . \tag{40}$$

Combining (40) with (36), we see that $\zeta_{\mathcal{P}}(s)$ has a simple pole at $s=1$, in agreement with Theorem B, whose residue is given by $A = Z(2)/Z'(1)$. We thus obtain with Eq. (35) our main result

$$N_{\mathcal{P}}(L) = \frac{Z(2)}{Z'(1)} e^{L}[1+O(e^{-c_1 L^{\alpha}})], \quad L \to \infty , \tag{41}$$

with $c_1>0$, $0<\alpha<\frac{1}{3}$.

We would like to emphasize that the leading asymptotic behavior of the length spectrum of the primitive periodic orbits, described by the first term in Huber's law (10), is the same for every Riemann surface. However, this does not imply, as Eq. (41) shows, that the pseudo-length spectra also possess identical asymptotic behavior. If one imagines a given length spectrum $\{l_n\}$ that obeys (10), then one can alter the shortest length l_1 without changing (10) being an asymptotical law. On the other hand, the number of pseudo-orbits $N_{\mathcal{P}}(L)$ up to a given L is the larger the shorter the first periodic orbit. In the limit $l_1 \to 0$, the number $N_{\mathcal{P}}(L)$ would diverge, see Eq. (63) below.

As already mentioned, at present we are unable to analytically derive the analog of the asymptotic behavior (41) for general chaotic systems, i.e., to show that $N_{\mathcal{P}}(L) = Ae^{\tau L} + \cdots$ holds in the general case. Numerical calculations show, however, that such a behavior seems to hold [11] for the hyperbola billiard, and thus one might speculate that this is the correct law for the proliferation of pseudo-orbits for generic chaotic systems.

IV. EXPLICIT REPRESENTATIONS FOR $Z(2)/Z'(1)$

In this section we shall derive two explicit representations that allow us to compute the crucial parameter $Z(2)/Z'(1)$ either from the classical length spectrum $\{l_n\}$ or from the quantal energy spectrum $\{E_n\}$.

We start with McKean's integral representation [34]

$$\frac{Z'(s)}{Z(s)} = (2s-1)\int_0^{\infty} dt\, e^{-s(s-1)t}\Theta^{(2)}(t) , \qquad \mathrm{Re}[s(s-1)]>0 , \tag{42}$$

where $\Theta^{(2)}(t)$ denotes the periodic-orbit contribution to the trace of the heat kernel $\Theta(t) \equiv \mathrm{Tr}e^{\Delta t}$, $t>0$ (see, e.g., [34,28])

$$\Theta(t) = \sum_{n=0}^{\infty} e^{-E_n t} = \Theta^{(1)}(t) + \Theta^{(2)}(t)$$
$$= \frac{A(\mathcal{F})e^{-t/4}}{(4\pi t)^{3/2}} \int_0^{\infty} db \frac{be^{-(b^2/4t)}}{\sinh(b/2)} + \frac{e^{-(t/4)}}{4\sqrt{\pi t}} \sum_{\{l_n\}} \sum_{k=1}^{\infty} \frac{l_n e^{-[(kl_n)^2/4t]}}{\sinh(kl_n/2)} . \tag{43}$$

For an approximate numerical computation of $\Theta^{(2)}(t)$ from a length spectrum that is known only up to a given cutoff length L, one should add to the truncated sum over the periodic orbits the following remainder term (see [7]):

$$R_1(L,t)=\tfrac{1}{2}\operatorname{erfc}\left[\frac{\sqrt{t}}{2}\left[\frac{L}{t}-1\right]\right] . \quad (44)$$

Due to the nondegenerate ground-state energy $E_0=0$, $\lim_{t\to\infty}\Theta(t)=1$ holds. Furthermore, one derives from Eq. (43) $\Theta^{(1)}(t)=O(e^{-t/4}/t^{3/2})$ for $t\to\infty$, and thus $\lim_{t\to\infty}\Theta^{(2)}(t)=1$.

Integration of Eq. (42) yields with $\lim_{s\to\infty}Z(s)=1$ the explicit representation of the Selberg zeta function ($\operatorname{Re}[s(s-1)]>0$)

$$Z(s)=\exp\left[-\int_0^\infty \frac{dt}{t}e^{-s(s-1)t}\Theta^{(2)}(t)\right] . \quad (45)$$

In order to derive a representation that is valid also in a neighborhood of $s=1$, we regularize the integral in (45) as follows:

$$\int_0^\infty \frac{dt}{t}e^{-s(s-1)t}\Theta^{(2)}(t)=\int_0^1 \frac{dt}{t}e^{-s(s-1)t}\Theta^{(2)}(t)+\int_1^\infty \frac{dt}{t}e^{-s(s-1)t}[\Theta^{(2)}(t)-1]+\int_1^\infty \frac{dt}{t}e^{-s(s-1)t} .$$

With

$$\int_1^\infty \frac{dt}{t}e^{-s(s-1)t}=E_1[s(s-1)]=-\gamma-\ln[s(s-1)]-\sum_{n=1}^{\infty}\frac{(-1)^n[s(s-1)]^n}{nn!} ,$$

where γ is Euler's constant, we finally obtain the following representation of the Selberg zeta function, which now holds also near $s=1$:

$$Z(s)=s(s-1)\exp\left[\gamma+\sum_{n=1}^{\infty}\frac{(-1)^n[s(s-1)]^n}{nn!}-\int_0^1\frac{dt}{t}e^{-s(s-1)t}\Theta^{(2)}(t)-\int_1^\infty\frac{dt}{t}e^{-s(s-1)t}[\Theta^{(2)}(t)-1]\right] . \quad (46)$$

After differentiation at $s=1$, we obtain

$$Z'(1)=\exp\left[\gamma-\int_0^1\frac{dt}{t}\Theta^{(2)}(t)-\int_1^\infty\frac{dt}{t}[\Theta^{(2)}(t)-1]\right] . \quad (47)$$

Combining this with the expression obtained from (46) at $s=2$, we end up with the explicit representation

$$\frac{Z(2)}{Z'(1)}=2\exp\left[\int_0^\infty\frac{dt}{t}(1-e^{-2t})[\Theta^{(2)}(t)-1]\right] , \quad (48)$$

which allows the computation of $Z(2)/Z'(1)$ solely from the classical length spectrum $\{l_n\}$.

An alternative representation of $Z(2)/Z'(1)$ can be derived from the following representation for $Z(s)$ [28]:

$$Z(s)=Z'(1)s(s-1)e^{\gamma_\Delta s(s-1)}\times[(2\pi)^{1-s}e^{s(s-1)}G(s)G(s+1)]^{2(g-1)}\times\prod_{n=1}^{\infty}\left[\left[1+\frac{s(s-1)}{E_n}\right]e^{-s(s-1)/E_n}\right] , \quad (49)$$

where $G(s)$ is the Barnes G function [35] and γ_Δ denotes the generalized Euler constant defined in Eq. (19). In the next section it is shown that γ_Δ can be expressed either in terms of the length spectrum $\{l_n\}$ or in terms of the quantal energy spectrum $\{E_n\}$. With $G(2)=G(3)=1$ we arrive at our second explicit representation

$$\frac{Z(2)}{Z'(1)}=\frac{2e^{2\gamma_\Delta+4(g-1)}}{(2\pi)^{2(g-1)}}\prod_{n=1}^{\infty}\left[\left[1+\frac{2}{E_n}\right]e^{-2/E_n}\right] , \quad (50)$$

which allows the computation of $Z(2)/Z'(1)$ solely from the quantal energy spectrum $\{E_n\}$. Equation (50) shows in combination with Eq. (41) that the strength of growth of the pseudo-orbits depends exponentially on the generalized Euler constant γ_Δ.

In summary, we have shown that the important parameter $Z(2)/Z'(1)$ determining the proliferation of the pseudo-orbits is determined completely either by the classical length spectrum via Eq. (48) or by the quantal energy spectrum via Eq. (50).

V. GENERALIZED EULER CONSTANT γ_Δ

In this section we shall derive two representations of the generalized Euler constant γ_Δ in terms of the length spectrum $\{l_n\}$ and in terms of the quantal energy spectrum $\{E_n\}$, respectively. Furthermore, a lower bound for γ_Δ is given and the behavior of γ_Δ in the limit $l_1\to 0$ is discussed, where l_1 is the length of the shortest periodic orbit. Our starting point is Eq. (18), which can be written as

$$\gamma_\Delta=\sum_{n=1}^{\infty}\left[\frac{1}{E_n}-\frac{1}{E_n+s(s-1)}\right]-2(g-1)\Psi(s)+\frac{1}{2s-1}\frac{Z'(s)}{Z(s)}-\frac{1}{s(s-1)} . \quad (51)$$

Since γ_Δ is a constant by definition, the right-hand side of Eq. (51) must be independent of s and thus can be evaluated at any s value that is convenient. The limit $s\to\infty$ reveals the dependence of γ_Δ on $\{E_n\}$, whereas the limit $s\to 1$ shows the dependence on $\{l_n\}$.

At first let us consider the limit $s\to\infty$. From the representation (15) of $Z(s)$ as a Dirichlet series, one derives the asymptotic behavior

$$Z(s)\sim 1-\frac{g_1e^{-l_1s}}{1-e^{-l_1}},\quad s\to\infty ,$$

and hence

$$\frac{1}{2s-1}\frac{Z'(s)}{Z(s)}=O\left[\frac{e^{-l_1 s}}{s}\right],\quad s\to\infty\ , \tag{52}$$

where g_1 denotes the number of periodic orbits with length l_1. With (52) one arrives at

$$\gamma_\Delta=\lim_{s\to\infty}\left[\sum_{n=1}^{\infty}\left[\frac{1}{E_n}-\frac{1}{E_n+s(s-1)}\right]-2(g-1)\Psi(s)-\frac{1}{s(s-1)}\right]. \tag{53}$$

To get a numerically useful formula, one must get rid of the infinite summation over the energy spectrum $\{E_n\}$, which is achieved in analogy with the derivation of the formula for Euler's original constant γ. Splitting the series at $N=s(s-1)$, Eq. (53) is equivalent to

$$\gamma_\Delta=\lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{E_n}-\sum_{n=1}^{N}\frac{1}{E_n+N}+\sum_{n=N+1}^{\infty}\left[\frac{1}{E_n}-\frac{1}{E_n+N}\right]-2(g-1)\Psi(\tfrac{1}{2}+\sqrt{N+1/4})-\frac{1}{N}\right]. \tag{54}$$

Now one observes that one can replace the quantal energies E_n by their asymptotic mean $\bar{E}_n$ in the second and third sum without altering the limit. The asymptotic mean is determined by Weyl's law (9)

$$\bar{E}_n=\frac{n}{g-1},\quad n\to\infty\ . \tag{55}$$

Inserting this into Eq. (54), the second and third summation can be carried out, leading to

$$\gamma_\Delta=\lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{E_n}+(g-1)[\Psi((g-1)N+1)-\Psi(N+1)]-2(g-1)\Psi(\tfrac{1}{2}+\sqrt{N+1/4})-\frac{1}{N}\right]. \tag{56}$$

Using the asymptotic expansion of $\Psi(x)$, the last formula takes a form that is very similar to the formula for Euler's constant γ

$$\gamma_\Delta=\lim_{N\to\infty}\left[\sum_{n=1}^{N}\frac{1}{E_n}-(g-1)\ln N\right]+(g-1)\ln(g-1)\ . \tag{57}$$

Indeed, for genus $g=2$ it is exactly Euler's formula, if one replaces $\{E_n\}$ by $\{n\}$. From Eq. (57) one expects that γ_Δ increases in the mean with increasing genus g, because of the term $(g-1)\ln(g-1)$ and because of the fact that the quantal energies behave in the mean as $\bar{E}_n=n/(g-1)$, which implies that the sum in Eq. (57) is proportional to $g-1$ and thus that the limit of the expression in parentheses is also proportional to $g-1$ if one assumes for all g the same statistical properties of $\{E_n\}$.

Now let us study the other limit, $s\to1$, revealing the dependence of γ_Δ solely on $\{l_n\}$, because the series over $\{E_n\}$ in (51) vanishes then. To apply Eq. (42) in the limit $s\to1$, we first have to regularize the integral because of $\lim_{t\to\infty}\Theta^{(2)}(t)=1$ as follows:

$$\frac{1}{2s-1}\frac{Z'(s)}{Z(s)}=\int_0^\infty dt\, e^{-s(s-1)t}\Theta^{(2)}(t)=\int_0^\infty dt\, e^{-s(s-1)t}[\Theta^{(2)}(t)-1]+\int_0^\infty dt\, e^{-s(s-1)t}$$
$$=\int_0^\infty dt\, e^{-s(s-1)t}[\Theta^{(2)}(t)-1]+\frac{1}{s(s-1)}\ .$$

Inserting this in Eq. (51), we arrive at our second formula for the generalized Euler constant

$$\gamma_\Delta=2(g-1)\gamma+\int_0^\infty dt[\Theta^{(2)}(t)-1]\ . \tag{58}$$

To obtain a lower bound on γ_Δ, we rewrite the last integral with the help of Eq. (43)

$$\gamma_\Delta=2(g-1)\gamma+\int_0^T dt[\Theta^{(2)}(t)-1]+\int_T^\infty dt[\Theta(t)-1]-\int_T^\infty dt\,\Theta^{(1)}(t)\ . \tag{59}$$

Since $\Theta^{(2)}(t)>0$ and $\Theta(t)\ge1$, one obtains the inequality

$$\gamma_\Delta>2(g-1)\gamma+\sup_{T\in]0,\infty[}\left[-T-\int_T^\infty dt\,\Theta^{(1)}(t)\right], \tag{60}$$

which is sharpest for T_0 such that $\Theta^{(1)}(T_0)=1$. With the definition $\vartheta(t)\equiv\Theta^{(1)}(t)/(g-1)$ we get a function that is independent of the genus g and of the fundamental domain $\mathcal{F}$. In Table I we list the strongest lower bound for γ_Δ in dependence of the genus g determined by $\vartheta(T_0)=1/(g-1)$ and (60). One observes the remarkable fact that the generalized Euler constant is strictly positive for genus $g\ge4$.

Now we would like to discuss the behavior of γ_Δ and $Z(2)/Z'(1)$ in the limit $l_1\to0$. Consider the contribution B_1 of the shortest length l_1 to Eq. (58) using (43)

TABLE I. The lower bound for the generalized Euler constant γ_Δ is presented together with the optimal parameter T_0.

Genus g	T_0	$\gamma_\Delta>$
2	0.776 365 54	−0.692 721 33
3	1.311 041 24	−0.389 994 23
4	1.727 511 17	0.164 313 50
5	2.071 818 26	0.848 428 81
6	2.366 851 98	1.613 493 24

$$B_1 \equiv \sum_{k=1}^{\infty} \frac{g_1 l_1}{4\sqrt{\pi}\sinh(k l_1/2)} \times \int_0^\infty dt \frac{e^{-[(t/4)+(k^2 l_1^2/4t)]}}{\sqrt{t}} = g_1 l_1 \sum_{k=1}^{\infty} \frac{e^{-kl_1}}{1-e^{-kl_1}} . \quad (61)$$

With the expansion

$$\frac{1}{1-e^{-x}} = \frac{1}{x} + \frac{1}{2} + \frac{x}{12} + O(x^3)$$

one arrives, after performing the summation over k, at $B_1 = -g_1 \ln l_1 + O(1)$ and hence at

$$\gamma_\Delta = -g_1 \ln l_1 + O(1) \quad \text{for } l_1 \to 0 . \quad (62)$$

Similarly, one derives from Eq. (48)

$$\frac{Z(2)}{Z'(1)} = O\left[\frac{1}{l_1^{g_1}}\right] \quad \text{for } l_1 \to 0 . \quad (63)$$

VI. APPLICATION TO TWO DIFFERENT HADAMARD-GUTZWILLER MODELS

For the application of our formulas, we choose two different members of the family of Hadamard-Gutzwiller models. As explained in Sec. I, these models provide simple examples of dynamical systems with two degrees of freedom that are strongly chaotic. In these models one considers a particle sliding freely on a compact Riemann surface of genus $g \geq 2$. Choosing two different compact Riemann surfaces of genus $g=2$, we obtain two different members out of the infinite family of Hadamard-Gutzwiller models. (Different Riemann surfaces have completely different length spectra $\{l_n\}$ and energy spectra $\{E_n\}$, respectively.) The first surface to be considered is defined by the most symmetrical fundamental domain $\mathcal{F}$, called regular octagon, which is invariant under operations of the Dieder D_8 group. The regular octagon possesses a highly degenerate length spectrum $\{l_n\}$ with exponentially increasing multiplicities [13]. For this fundamental domain the length spectrum is completely known up to $L_{\max} = 18.092$ [36] and provides, therefore, a unique test system. The second Riemann surface is defined by an asymmetric octagon having a length spectrum with at most fourfold degeneracies. It obeys only time-reversal and parity symmetry, which is the least possible symmetry for a fundamental domain of genus $g=2$. Therefore, these two fundamental domains lie on opposite ends with respect to symmetry properties. The length spectrum of the asymmetric octagon has been computed up to $L_{\max} = 15$, but it is not complete since only periodic orbits having a representation of at most 12 generators have been taken into account (see [13]). For both fundamental domains the first 200 quantal energies have been computed by the method of finite elements.

To compute the generalized Euler constant γ_Δ from the first 200 quantal energies, it is more advantageous to use Eq. (56) than its asymptotic form (57). For the regular octagon we obtain $\gamma_\Delta = -0.600\,38$ and for the asymmetric one $\gamma_\Delta = -0.512\,07$. These values are in good agreement with the values computed from the length spectrum using (58), $\gamma_\Delta = -0.598\,65$ and $\gamma_\Delta = -0.522\,05$, for the regular and the asymmetric octagon, respectively. The agreement between the values for γ_Δ obtained from the two methods is better in the regular case, which is due to the fact that in this case the length spectrum is completely known up to $L_{\max} = 18.092$.

For the parameter $Z(2)/Z'(1)$ we obtain from Eq. (48) the values 0.3930 and 0.4274 for the regular and the asymmetric octagon, respectively. The first 200 quantal energies are not sufficient to obtain $Z(2)/Z'(1)$ directly from Eq. (50), because the limit is not yet reached at $N=200$. However, if a fit is made to the function

$$f(N) \equiv \frac{e^{2\gamma_\Delta^{(N)}+4}}{2\pi^2} \prod_{n=1}^{N} \left[\left[1+\frac{2}{E_n}\right] e^{-2/E_n} \right] \quad (64)$$

with

$$\gamma_\Delta^{(N)} \equiv \sum_{n=1}^{N} \frac{1}{E_n} - 2\Psi(\tfrac{1}{2}+\sqrt{N+1/4}) - \frac{1}{N}$$

$[\lim_{N\to\infty} f(N) = Z(2)/Z'(1)]$ using the parametrization $F(N) \equiv a + (b/N^c)$, the parameter a yields a good approximation to $Z(2)/Z'(1)$. We get $a=0.3922$ and $a=0.4261$ for the regular and the asymmetric case, respectively, which is in accordance with the values computed from $\{l_n\}$ using (48). Figure 1 shows $f(N)$ in comparison with the fit $F(N)$ for both octagons. This demonstrates that $Z(2)/Z'(1)$ can be well determined from a knowledge of the first 200 quantal energies only.

Using the values computed for $Z(2)/Z'(1)$ from Eq. (64), we present in Figs. 2 and 3 a comparison between the staircase $N_{\mathcal{P}}(L)$ and the theoretical prediction (41). Recall that the proliferation of the periodic orbits described by Huber's law (10) is determined by the ground-state energy $E_0=0$ and by the $M \leq 4g-3$ small eigenvalues, if they occur at all. (The two surfaces considered here possess no small eigenvalues.) This is in contrast to the proliferation of the pseudo-orbits, whose strength of

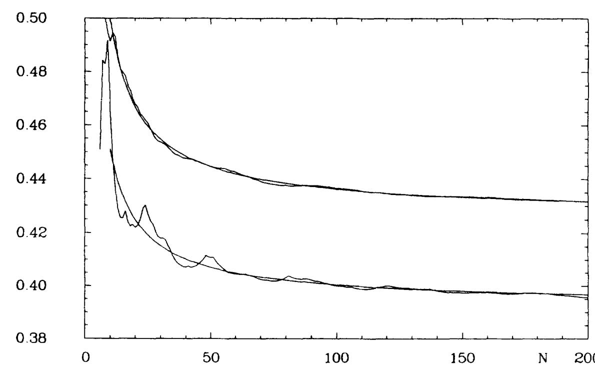

FIG. 1. Function $f(N)$, Eq. (64), is shown in comparison with the fit $F(N)$ described in the text for the regular octagon (lower curves) and for the asymmetric one (upper curves).

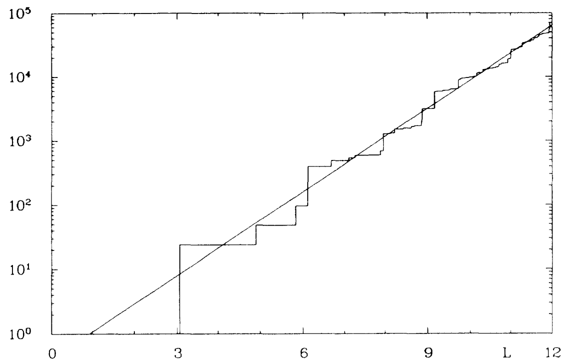

FIG. 2. Staircase $N_{\mathcal{P}}(L)$ is shown in comparison with the asymptotic behavior (41) for the regular octagon.

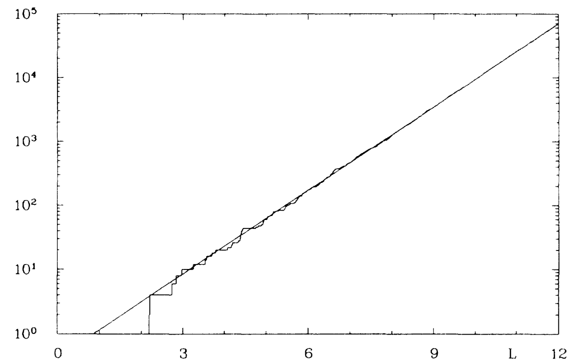

FIG. 3. Staircase $N_{\mathcal{P}}(L)$ is shown in comparison with the asymptotic behavior (41) for the asymmetric octagon.

growth is determined by Eq. (50) and thus requires infinitely many quantal energies. The very good agreement seen in Figs. 2 and 3 shows, however, that the proliferation of the pseudo-orbits is already well determined by the first 200 quantal energies. As can be seen by comparing Fig. 2 with Fig. 3, the staircase $N_{\mathcal{P}}(L)$ shows much larger steps in the regular case (Fig. 2), which are due to the exponentially increasing multiplicities of the length spectrum $\{l_n\}$ leading to exponentially growing multiplicities in the pseudolength spectrum $\{L_n\}$. Surprisingly, in both cases the asymptotic law (41) describes well the proliferation in the mean down to the shortest length. This is similar to Huber's law (10), which also yields a surprisingly good description of the staircase $N(\ell)$ of primitive periodic orbits down to the shortest length (see Fig. 2 in [36]).

VII. SUMMARY

The main purpose of this paper has been to derive the asymptotic behavior of the pseudolength spectrum of a strongly chaotic system, which is the necessary input in the recently proposed quantization rules based on the Dirichlet series representation of the relevant dynamical zeta functions. Since the Dirichlet series converges in a much larger region than the Euler product defining the dynamical zeta function, the Dirichlet series is expected to be applicable to a much wider class of chaotic systems.

The derivation of the asymptotic behavior of $N_{\mathcal{P}}(L)$ has been based on Beurling's theory of generalized primes. Equation (41) shows that $N_{\mathcal{P}}(L)$ increases universally in proportion to e^L, independent of the genus g and of the chosen fundamental domain $\mathcal{F}$. Only the proportionality factor, i.e., the strength of growth, is system dependent and has been shown to be given by $Z(2)/Z'(1)$, where $Z(s)$ is the dynamical zeta function, which in the case of the Hadamard-Gutzwiller model is identical to the Selberg zeta function. This factor can be computed from the length spectrum $\{l_n\}$ of the primitive periodic orbits [Eq. (48)] as well as from the quantal energy spectrum $\{E_n\}$ [Eq. (50)]. In the latter case, the so-called generalized Euler constant γ_Δ arises, which can also be computed from $\{l_n\}$ [Eq. (58)] or from $\{E_n\}$ [Eq. (57)]. A lower bound for γ_Δ has been given in (60), and it has been shown that γ_Δ diverges as $-g_1 \ln l_1$ in the limit $l_1 \to 0$. Finally, we have applied our formulas to two different members of the family of Hadamard-Gutzwiller models, for which the length spectra and the quantal energy spectra are partly known. As can be seen from Figs. 2 and 3, the asymptotic law (41) gives an excellent description of the average behavior of the two pseudolength spectra.

Since at present there seems to be no hope of giving an analogous derivation for the asymptotic behavior of the pseudo-orbits in the case of general systems, the Hadamard-Gutzwiller model plays again the role of a prototype example, as it has already played in the past, where the Selberg zeta function has been the first example of a dynamical zeta function.

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