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HIGGS- AND YUKAWA-THEORIES ON THE LATTICE

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Recent developments in pure scalar ϕ^4 and scalar-fermion models are reviewed.

1. INTRODUCTION

The quartic self-interaction of a complex doublet scalar field and its Yukawa-couplings to leptons and quarks are basic ingredients in the Higgs sector of the Standard Model. The nonperturbative investigations of pure scalar or scalar-fermion Yukawa-models are motivated by the possibility of strong quartic and/or Yukawa-couplings, corresponding to heavy Higgs-bosons and/or heavy fermions. At present only a lower limit of about 50 GeV is known for the mass of the Higgs-boson (see, for instance, ref. [1]). Although the unknown top quark mass is consistently predicted by 1-loop perturbation theory in the range of 150 GeV [2], which still corresponds to a perturbative Yukawa-coupling, the existence of heavy fermions with strong Yukawa-couplings is not excluded by present phenomenology. In fact, a few additional heavy fermion families with heavy neutrinos and small mass splittings within doublets are still consistent with overall fits of the data by 1-loop perturbation theory [3,4]. Of course, perturbation theory is not really applicable to strongly interacting heavy bosons and fermions, therefore the results of such fits have to be taken with some caution.

Another source of motivation for a nonperturbative lattice formulation of the Standard Model is to provide a consistent mathematical framework, which is valid beyond perturbation theory. In fact, since the Standard Model is based on a chiral gauge theory, its lattice formulation is a highly nontrivial problem, due to the Nielsen-Ninomiya theorem [5].

In this review a summary of recent results, obtained mainly during the last year, will be presented. In the next section pure scalar models will be considered, whereas section 3 is devoted to scalar-fermion models with Yukawa-couplings. For previous work on this field see also the recent reviews [6–8].

2. SCALAR HIGGS MODELS

Neglecting all gauge interactions and Yukawa-couplings in the Higgs sector of the Standard Model one is left with a four real component O(4)-symmetric ϕ^4 theory. ϕ^4 models on the lattice are simple, nice quantum field theories. In case of the simplest lattice actions their solutions are essentially known, in the sense that one can calculate in a good approximation all physical quantities for arbitrary values of the bare parameters.

An important feature of the solution is that the continuum limit is *trivial*: there is a lattice spacing dependent upper limit on the renormalized quartic coupling, which goes to zero in the continuum limit $a \to 0$. Moreover, for small enough lattice spacing, for instance, if the physical mass in lattice units satisfies $am_H \leq 0.5$, the renormalized coupling is always small enough for the applicability of perturbation theory. In this sense the ϕ^4 theory is always perturbative. The only nonperturbative feature is the mapping between bare and renormalized parameters, which can be called *renormalization mapping.*

Due to the equivalence of O(4) to $SU(2)_L \otimes SU(2)_R$, there are different ways to represent the Higgs scalar field. The four real components in the O(4) basis ϕ_{Rx}, $(R = 0,1,2,3)$, which are also often denoted by $\sigma_x \equiv \phi_{0x}$ and $\pi_{rx} \equiv \phi_{rx}$, $(r = 1,2,3)$, can be put in a $2 \otimes 2$ matrix φ_x with the help of the isospin Pauli-matrices τ_r:

$$\varphi_x \equiv \sigma_x + i\tau_r\pi_{rx}$$
$$= \begin{pmatrix} \sigma_x + i\pi_{3x} & i\pi_{1x} + \pi_{2x} \\ i\pi_{1x} - \pi_{2x} & \sigma_x - i\pi_{3x} \end{pmatrix} . \tag{1}$$

The realness of ϕ_{Rx} can be expressed in terms of the matrix field φ_x by the relation

$$\varphi_x^+ = \epsilon^{-1}\varphi_x^T\epsilon = \tau_2\varphi_x^T\tau_2 . \tag{2}$$

The relation of the matrix field to the complex doublet Φ_{Ax} $(A = 1,2)$ is

$$\Phi_{Ax} = \varphi_{A2,x} , \quad \tilde{\Phi}_{Ax} = \varphi_{A1,x} , \tag{3}$$

where the doublet field with hypercharge $Y = -1$ is

$$\tilde{\Phi}_{Ax} \equiv \epsilon_{AB}\Phi^*_{Bx} = i\tau_{2,AB}\Phi^*_{Bx} . \tag{4}$$

The global $SU(2)_L \otimes SU(2)_R$ transformation of the matrix field is

$$\varphi'_x = U_L^{-1}\varphi_x U_R , \quad U_{L,R} \in SU(2)_{L,R} . \tag{5}$$

Therefore, (3) implies that Φ_{Ax} is an $SU(2)_L$ doublet, as required.

The simplest lattice action in terms of the matrix field φ_x can be defined as

$$S[\varphi] = \sum_x \Big\{ \frac{\mu_\phi}{2}\mathrm{Tr}\,(\varphi_x^+\varphi_x) + \lambda \left[\tfrac{1}{2}\mathrm{Tr}\,(\varphi_x^+\varphi_x)\right]^2$$
$$-\kappa \sum_{\mu=1}^{4} \mathrm{Tr}\,(\varphi_{x+\hat\mu}^+\varphi_x) \Big\} . \tag{6}$$

Here the normalization of the scalar field is left general. In lattice perturbation theory it is convenient to fix the normalization freedom by $\kappa = \frac{1}{2}$, and then the bare mass squared in lattice units is $\mu_0^2 = 2\mu_\phi - 8$. In numerical simulations the best choice is

$$\mu_\phi = 1 - 2\lambda . \tag{7}$$

Namely, in this case the limit $\lambda \to \infty$ is smooth, only the length of the field is frozen to unity:

$$\tfrac{1}{2}\mathrm{Tr}\,(\varphi_x^+\varphi_x) = \Phi^*_{Ax}\Phi_{Ax} = \phi_{Rx}\phi_{Rx} = 1 . \tag{8}$$

2.1. *Analytical results*

A recent addition to the approximate analytical solution of lattice ϕ^4 models is the study of the O(4) symmetric model in a variational cumulant expansion [9]. The field expectation value, renormalized mass, its critical behaviour and the effective potential are calculated up to the third order. The results agree well with previous numerical simulation results (see, for instance, [10]), and with the Lüscher-Weisz solution [11], based on high order hopping parameter expansion ("high temperature expansion") in the symmetric phase and perturbative renormalization group equations.

Another contribution is in the 4 dimensional Ising model (one component ϕ^4 model at infinite bare quartic coupling λ) the application of high order "low temperature expansion" in the broken phase [12]. In a range of renormalized masses about $am_H \simeq 0.3 - 0.6$, the results of up to 34th orders give a good agreement both with high precision numerical data [13] and with the Lüscher-Weisz solution [11].

2.2. Improved upper bounds on the Higgs-mass

The renormalized quartic coupling in the broken phase of an O(4) symmetric ϕ^4 model can be defined by the ratio of the Higgs boson mass m_H ("σ-mass") to the renormalized vacuum expectation value v_R. Possible definitions, in different normalization conventions, are

$$\frac{m_H^2}{v_R^2} \equiv 8\lambda_R \equiv \frac{g_R}{3} \equiv 2g_{ren} . \tag{9}$$

In lowest order in the $\mathrm{SU(2)_L}$ gauge couling $g_{SU(2)}$ the W-boson mass is

$$m_W^2 = \frac{g_{SU(2)}^2 v_R^2}{4} + O(g_{SU(2)}^4) , \tag{10}$$

therefore the ratio of the Higgs-mass to W-mass R_{HW} is given by

$$R_{HW}^2 \equiv \frac{m_H^2}{m_W^2} = \frac{32\lambda_R}{g_{SU(2)}^2} . \tag{11}$$

This shows how the upper limit on the quartic coupling in pure ϕ^4 theory gives an upper limit on R_{HW}.

Previous numerical and analytical investigations gave for the lowest possible cut-off's upper limits of the order

$$\lambda_R \leq 0.9 , \quad \frac{m_H}{v_R} \leq 2.7 , \quad R_{HW} \leq 8 , \tag{12}$$

quite independently of the form of lattice action [6]. Compared to the tree unitarity limit

$$\lambda_R \leq \frac{2\pi}{5} \simeq 1.26 , \tag{13}$$

this is not a very strong coupling, indeed. In QCD with two light quarks, which is a reasonably good approximation both to the real world and to an $\mathrm{SU(2)_L} \otimes \mathrm{SU(2)_R}$ symmetric σ-model with higher dimensional couplings, the mass of the σ "particle" is in the range $m_\sigma \simeq (6-8)f_\pi$. Since in the present context $m_\sigma \equiv m_H$ and $f_\pi \equiv v_R$, this would correspond to a substantially higher value of the quartic coupling than the upper limit in (12).

The question is, can one choose a reasonably simple lattice action, which would mimic low energy QCD, and give a higher upper limit on the Higgs mass? In order to find the answer, Heller, Neuberger and Vranas [14] investigated the upper limit for λ_R in the fixed length ($\lambda = \infty$) limit of a lattice action, with at most four lattice derivatives. The results of the $N \to \infty$ calculation and preliminary numerical data suggest an upper limit which is by about 30% higher than (12), if the upper limits for different lattice actions are compared at roughly the same amount of cut-off effects in physical amplitudes. This is consistent with the general expectation that the absolute upper limit on the Higgs-mass should be higher, if a more general class of lattice actions is considered. In physical units the new absolute upper limit would roughly correspond to

$$m_H \leq 800\, GeV . \tag{14}$$

This is close to the tree unitarity limit (13).

The question is, can one still substantially increase the upper limit by considering an even broader class of relatively simple lattice actions? Of course, it is clear that the complexity of lattice actions has to be somehow restricted, certainly from practical, but also from general theoretical point of view. The dependence of the upper limit at low cut-off's on the lattice action also underlines, that the truely interesting upper limits are those belonging to high cut-off's.

2.3. Goldstone bosons in finite volumes

An important and interesting aspect of the numerical simulations in the pure ϕ^4 scalar sector is the presence of massless Goldstone bosons (π_r, $r = 1, 2, 3$) in the physically relevant phase with broken global symmetry. In finite volumes the Goldstone bosons give rise to finite size effects, which do not become small even on very large lattices. These effects have to be corrected for, in order to extract the relevant infinite volume physical information. Fortunately, on an L^4 lattice, in the range of lattice volumes satisfying

$$L \gg m_\sigma^{-1} \,, \quad L \leq m_\pi^{-1} \tag{15}$$

one can use *chiral perturbation theory* to calculate the finite volume effects in terms of some parameters on the $L = \infty$ lattice [15] (see also the contribution of Peter Weisz to this proceedings [16]). Measuring these effects can, in fact, be used to determine important quantities like, for instance, the renormalized vacuum expectation value v_R.

Comparisons of chiral perturbation theory for finite size effects due to Goldstone bosons were performed in case of the *constraint effective potential* [17] in refs. [18,19]. Lowest order chiral perturbation theory turns out to describe the numerical simulation data in four dimensions well. The extracted values of v_R and of the wave function renormalization factor Z_π agree well with the known values obtained previously by other methods. The advantage of the method based on the constraint effective potential is, that the introduction of an external source term in the action is not necessary.

Another important quantity, which is also defined without external source fields, is the *constraint correlation function*, which is defined at a fixed value of the averaged field. Such correlation functions can be measured numerically by determining the direction of the averaged field ϕ in O(4) space, and then separating parallel ("longitudinal") and perpendicular ("transversal") components of the spins. Chiral perturbation theory gives information on the large volume behaviour of the constraint correlations, too [20], which can be exploited in numerical simulations of pure Higgs and Yukawa-models.

2.4. Multivariable β-functions

Pure ϕ^4 systems and Higgs systems with gauge fields, obtained by switching on some gauge interactions, are well known by now. Therefore, they are ideally suited as theoretical laboratories for testing and developping new analytical and numerical methods. One important information in lattice quantum field theories with several bare parameters is contained in the *lines of constant physics (LCP's)*. These are curves in bare parameter space, where dimensionless physical quantities are kept fixed, only the lattice spacing is changing. The LCP's and the corresponding *multivariable β-functions* are important, for instance, if one is trying to control scale breaking lattice artifacts, or to calculate thermodynamic quantities etc.

The numerical determination of LCP's by interpolating the results obtained in several different points, is usually rather cumbersome, especially if the number of bare parameters is large. In order to illustrate a direct method based on high statistics simulations in a few points, let us consider the fundamental ("standard") Higgs-model, which is obtained by gauging the $\mathrm{SU(2)_L}$ symmetry in the above $\mathrm{SU(2)_L \otimes SU(2)_R \equiv O(4)}$ symmetric ϕ^4 model action (6). The lattice action is

$$S[U,\phi] = S_g[U] + S_\phi[U,\varphi] \,, \tag{16}$$

where the pure gauge part S_g is a sum over plaquettes

$$S_g[U] = \beta \sum_{pl} \left(1 - \frac{1}{2} \mathrm{Tr}\, U_{pl} \right) \,, \tag{17}$$

with $\beta \equiv 4/g^2_{SU(2)}$. The gauged scalar part is, in the normalization corresponding to (7),

$$S_\phi[U,\varphi] = \sum_x \{ \tfrac{1}{2} \mathrm{Tr}\, (\varphi_x^+ \varphi_x)$$
$$+ \lambda \left[\tfrac{1}{2} \mathrm{Tr}\, (\varphi_x^+ \varphi_x) - 1 \right]^2$$

$$-\kappa \sum_{\mu=1}^{4} \mathrm{Tr}\left(\varphi^{+}_{x+\hat{\mu}} U_{x\mu} \varphi_{x}\right) \Big\} . \tag{18}$$

This model has 3 independent bare parameters (β, κ, λ). On anisotropic lattices occuring in thermodynamic applications the number is already increased to 5: $(\beta_\sigma, \beta_\tau, \kappa_\sigma, \kappa_\tau, \lambda)$.

Let us now first consider in general n relevant bare parameters $g_1, g_2, \ldots, g_n$ [21]. In order to specify the LCP's one has to keep $(n-1)$ independent dimensionless physical quantities $F_2, F_3, \ldots, F_n$ constant. The points of a specific LCP can be parametrized, for instance, by the first bare coupling g_1: $g_j = g_j(g_1; F_2, F_3, \ldots, F_n)$ $(j = 2, 3, \ldots, n)$. In this case we have

$$\frac{dg_j(g_1)}{dg_1} = \frac{\det^{[1,j]}_{n-1}(\partial F/\partial g)}{\det^{[1,1]}_{n-1}(\partial F/\partial g)} . \tag{19}$$

Here $\det^{[i,k]}_{n-1}(\partial F/\partial g)$ denotes the $(n-1)\otimes(n-1)$ subdeterminant of the $n \otimes n$ derivative matrix $\partial F/\partial g$, which belongs to the matrix element $(\partial F/\partial g)_{ik} = \partial F_i/\partial g_k$.

Another possibility is to parametrize the points of an LCP by the value of some reference physical quantity F_1. For instance, it is reasonable to take F_1 as some physical mass in lattice units. This can be defined, for instance, by an "effective mass" obtained from the ratio of some connected correlation function:

$$F_1 \equiv am \equiv am_{(t,t+1)}$$
$$\equiv \log \frac{\langle s_0 s_t \rangle - \langle s_0 \rangle^2}{\langle s_0 s_{t+1} \rangle - \langle s_0 \rangle^2} . \tag{20}$$

In this case the differential equations for $g_i(F_1)$ $(i = 1, 2, \ldots, n)$ are

$$\frac{dg_i(F_1)}{dF_1} = \frac{\det^{[1,i]}_{n-1}(\partial F/\partial g)}{\det_n(\partial F/\partial g)}$$
$$\equiv \beta_{g_i/F_1}(F_1, g_2, g_3, \ldots, g_n) . \tag{21}$$

On the right hand side β_{g_i/F_1} is a generalized Callan-Symanzik β-function, which is considered here as a function of the reference quantity F_1 and the bare parameters $g_2, g_3, \ldots, g_n$. The direction of the LCP flow is determined by (19), whereas the generalized β-functions give the rate of scale change along the LCP's.

As an example of the partial derivatives appearing in (19) and (21) one can consider

$$\frac{\partial F_1}{\partial g_k} = \left[\langle s_0 s_{t+1}\rangle - \langle s_0\rangle^2\right]^{-1} \Big\{ \langle s_0 s_{t+1} \frac{\partial S}{\partial g_k}\rangle$$
$$- \langle s_0 s_{t+1}\rangle \langle \frac{\partial S}{\partial g_k}\rangle - 2\langle s_0\rangle \langle s_0 \frac{\partial S}{\partial g_k}\rangle$$
$$+ 2\langle s_0\rangle^2 \langle \frac{\partial S}{\partial g_k}\rangle \Big\} - (\, s_{t+1} \longrightarrow s_t \,) . \tag{22}$$

That is, the partial derivatives are expressed by some connected 3-point correlations containing parts of the lattice action, which multiply the corresponding bare parameters.

The question is, how difficult is to measure numerically these correlations? Returning now to the fundamental Higgs-model, let us identify the reference quantity F_1 by the W-mass in lattice units:

$$F_1 = am_W \equiv am_{W(2,3)} . \tag{23}$$

The second form indicates that on a 12^4 lattice one can take, for instance, the effective mass obtained from distances 2 and 3. For the dimensionless quantities $F_{2,3}$ one can take the Higgs-W mass ratio

$$F_2 = R_{HW} \equiv \frac{am_H}{am_W} \equiv \frac{am_{H(2,3)}}{am_{W(2,3)}} , \tag{24}$$

respectively, the renormalized gauge coupling defined by the Wilson-loop expectation values $\langle W_{R,T}\rangle$ $(R = 2, T = 6)$ as

$$F_3 = \alpha_W \equiv \alpha_W^{(R,T)}$$
$$\equiv \frac{4R(R-1)e^{am_W R}}{3T(1 + Re^{am_W} - R)} \log \frac{\langle W_{R-1,T}\rangle}{\langle W_{R,T}\rangle} . \tag{25}$$

I performed a test run on a 12^4 lattice at the bare parameter values $g_1 \equiv \kappa = 0.307$,

$g_2 \equiv \lambda = 1.0$, $g_3 \equiv \beta = 2.3$ with 10^5 Metropolis sweeps. The required triple correlations could be determined with errors of about 10-20%. This means that a reasonably good determination of the LCP flows requires of the order of 10^6 sweeps at this point.

3. SCALAR-FERMION YUKAWA-MODELS

The Yukawa interaction of pions and nucleons was originally introduced to explain the low energy nuclear force [22]. In the Standard Model the Yukawa-couplings of quarks and leptons to the Higgs scalar field provide the masses of the elementary fermions. Strong Yukawa-couplings correspond to heavy fermions on the scale of the electroweak symmetry breaking, therefore the primary concern of nonperturbative studies is to investigate the influence of heavy fermions on the Higgs sector of the Standard Model.

3.1. Lattice actions

There is a great deal of arbitrariness in putting the Yukawa interactions on the lattice. One can use Wilson- or staggered fermions, the Yukawa-couplings can be defined locally on a single lattice point, or on extended regions as hypercubes, the global symmetry can be, for instance, $SU(2) \otimes SU(2)$ or $U(1) \otimes U(1)$ etc. This implies that in recent studies many different lattice actions were considered. From the physical point of view the most interesting models are those with "chiral" global symmetries, which can be gauged in order to describe a chiral gauge theory similar to the electroweak sector of the Standard Model. Most of the recent studies of lattice Yukawa-models with explicit chiral symmetry were concentrated either on the formulation with a Wilson-Yukawa coupling [23], or with mirror pairs of fermion fields in the action [24]. These will be separately considered in subsequent subsections. (For recent reviews see also refs. [6–8].)

Due to the large variety of different possible lattice formulations it is not easy to find a representative prototype of the lattice Yukawa-model actions. For definiteness let us start here by defining a $U(1) \otimes U(1)$ symmetric model with two Wilson-fermions (ψ_{Ax} and ψ_{Bx}) in the limit of $\lambda = \infty$, when the complex scalar field ϕ_x has a fixed unit length: $\phi_x^* \phi_x = 1$. We shall see later, that this model is related in one way or another to many of the models studied up to now. Its action is:

$$S = \sum_x \Bigg\{ -\kappa \sum_{\mu=\pm 1}^{\pm 4} \phi^*_{x+\hat\mu} \phi_x + \sum_{F=A,B} (\overline{\psi}_{Fx} \psi_{Fx})$$
$$- \sum_{F=A,B} K_F \sum_{\mu=\pm 1}^{\pm 4} (\overline{\psi}_{Fx+\hat\mu} [1 + \gamma_\mu] \psi_{Fx})$$
$$+G \left[\phi_x^* (\overline{\psi}_{Bx} \psi_{Ax}) + \phi_x (\overline{\psi}_{Ax} \psi_{Bx}) \right] \Bigg\} \ . \qquad (26)$$

$K_{A,B}$ denote the two hopping parameters, and G is the bare Yukawa-coupling. If K_A and K_B are equal, then S is a sum of two adjoint pieces, therefore the popular Hybrid Monte Carlo algorithm is applicable. The $U(1)_A \otimes U(1)_B$ global symmetry expresses the conservation of the two types of fermion numbers. The symmetry transformations of the fields are ($F = A, B$):

$$\psi'_{Fx} = e^{-i\alpha_F} \psi_{Fx} \ , \quad \overline{\psi}'_{Fx} = \overline{\psi}_{Fx} e^{i\alpha_F} \ ,$$
$$\phi'_x = e^{-i(\alpha_A - \alpha_B)} \phi_x \ . \qquad (27)$$

Since ϕ_x transforms as a $(\bar{B}A)$ bound state, it has a nonzero "A-B" fermion number, but a zero total "A+B" fermion number.

A similar $SU(2)_A \otimes SU(2)_B \otimes U(1)_F$ symmetric model can be constructed with the $2 \otimes 2$ matrix scalar field φ_x satisfying the reality condition (2), if the fermions ψ_{Ax} and ψ_{Bx} have an extra doublet index. The Yukawa-coupling is then

$$G\left[(\overline{\psi}_{Bx}\varphi_x^+\psi_{Ax}) + (\overline{\psi}_{Ax}\varphi_x\psi_{Bx})\right] \ . \qquad (28)$$

Due to its reality, the scalar field φ_x does not have fermion number here. There is only a total fermion number conservation expressed by $U(1)_F$.

The above form of the lattice action (26) is well suited for the description of the light states in the model for weak and medium strong Yukawa-couplings. For very strong couplings $|G| \gg 1$ another "neutral fermion formulation" is better [25]. In the present case one can define either "B-neutral" or "A-neutral" fermion fields. For definiteness, let us take the "B-neutral" (and "A-charged") ones:

$$\psi_{Ax}^{(n)} \equiv \sqrt{G}\psi_{Ax}, \quad \psi_{Bx}^{(n)} \equiv \sqrt{G}\phi_x\psi_{Bx} \ . \qquad (29)$$

These transform according to

$$\psi_{Fx}^{(n)\prime} = e^{-i\alpha_A}\psi_{Fx}^{(n)}, \quad \overline{\psi}_{Fx}^{(n)\prime} = \overline{\psi}_{Fx}^{(n)}e^{i\alpha_A} \ , \qquad (30)$$

therefore the symmetry can be considered as $U(1)_{F\equiv A} \otimes U(1)_{\phi\equiv(A-B)}$.

In terms of these fields the fermionic part of the action (26) is

$$S_f = \sum_x \Bigg\{ G^{-1} \sum_{F=A,B} (\overline{\psi}_{Fx}^{(n)}\psi_{Fx}^{(n)}) + (\overline{\psi}_{Bx}^{(n)}\psi_{Ax}^{(n)}) + (\overline{\psi}_{Ax}^{(n)}\psi_{Bx}^{(n)}) - \sum_{\mu=\pm1}^{\pm4} \Bigg[\frac{K_A}{G}(\overline{\psi}_{Ax+\hat{\mu}}^{(n)}[1+\gamma_\mu]\psi_{Ax}^{(n)}) + \frac{K_B}{G}[\phi_{x+\hat{\mu}}^*\phi_x]^{-1}(\overline{\psi}_{Bx+\hat{\mu}}^{(n)}[1+\gamma_\mu]\psi_{Bx}^{(n)}) \Bigg] \Bigg\} \ . \qquad (31)$$

The peculiarity of this form is that the previous mass terms and Yukawa-coupling terms now all look like mass terms. The scalar field appears only in the hopping term of $\psi_B^{(n)}$, in a form reminescent of a gauge interaction on links. The determinant of the mass term is $G^{-2} - 1$, therefore one fermion combination becomes massless at $G^2 = 1$.

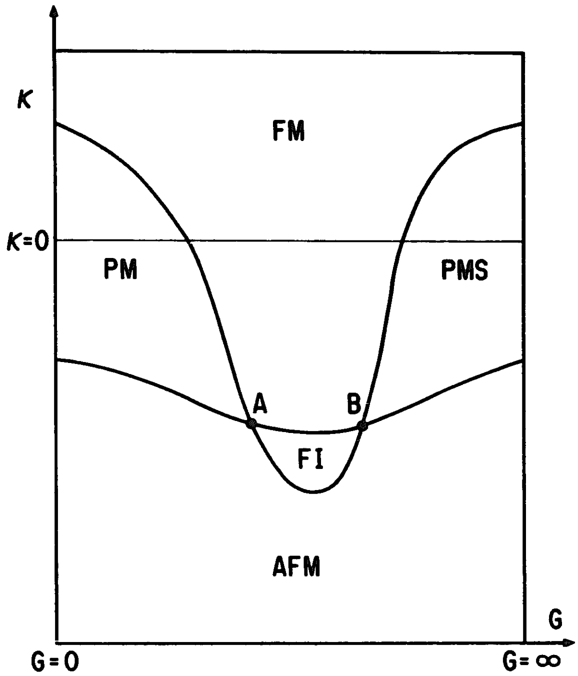

Fig. 1. The generic phase structure of Yukawa-models on the lattice: PM = paramagnetic (or symmetric), FM = ferromagnetic, AFM = antiferromagnetic, FI = ferrimagnetic and PMS = strong paramagnetic phase.

This suggests that at strong bare Yukawa-couplings G^2 has to be tuned in the continuum limit as a fermion mass term. For tuning the scalar mass to zero one can use the scalar hopping parameter κ.

3.2. Phase structure

Already the two forms (26) and (31) of the lattice action make plausible, that the lattice Yukawa-models have a rich phase structure with a certain kind of replication of the weak coupling behaviour at very strong bare Yukawa-couplings. The general picture can be represented in the plane of bare Yukawa-coupling (G) and scalar hopping parameter (κ) by fig. 1, and was already discussed at Lattice '90 [6,7].

Besides the expectation value of the scalar field ϕ_x, another important order parameter is the expectation value of the *staggered scalar field* defined by

$$\hat{\phi}_x \equiv e^{i\pi(x_1+x_2+x_3+x_4)}\phi_x \,. \tag{32}$$

In the PM and PMS phases we have $\langle\phi_x\rangle = 0$, $\langle\hat{\phi}_x\rangle = 0$, in the FM phase $\langle\phi_x\rangle \neq 0$, $\langle\hat{\phi}_x\rangle = 0$, in the AFM phase $\langle\phi_x\rangle = 0$, $\langle\hat{\phi}_x\rangle \neq 0$, and in the FI phase $\langle\phi_x\rangle \neq 0$, $\langle\hat{\phi}_x\rangle \neq 0$. Particularly interesting points of the phase space are points "A" and "B", where four phases can meet. The right hand side of the diagram in fig. 1, beyond the FI phase, can be understood in terms of the "neutral" fermion fields (for a recent contribution concerning this see ref. [26]). This part is actually absent in some staggered fermion models with overlapping hypercubic Yukawa-couplings [27].

In most numerical studies up to now the fermion bare mass was put equal to zero (either for simplicity or as a consequence of the chiral symmetry). In a recent study in the mirror fermion model [28] also the dependence on the fermion mass was investigated, actually as a function of the fermion mirror fermion mixing mass $\mu_{\psi\chi}$, or the corresponding hopping parameter K. In the (K,κ)-plane, for fixed quartic and Yukawa-couplings, the phase structure can be schematically represented by fig. 2. In the subspace with zero fermion mirror fermion mixing mass a picture similar to fig. 1 emerges.

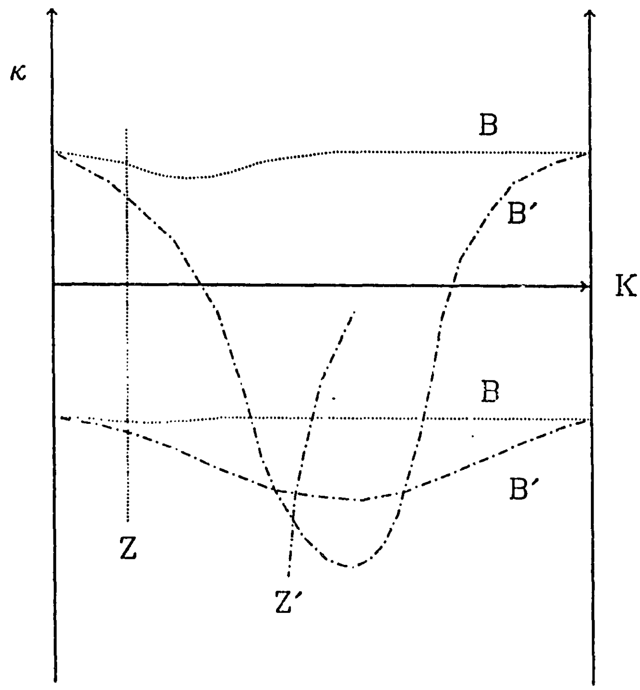

Fig. 2. The schematic phase structure of the $U(1)_L \otimes U(1)_R$ symmetric Yukawa-model with mirror pairs of fermion fields in the plane of the two hopping parameters. The phase boundaries are shown at small (dotted lines with label B) and large (dashed lines with label B') bare Yukawa-couplings. The lines Z, Z', which are not phase boundaries, represent the curves of zero fermion mixing mass for small and large Yukawa couplings, respectively.

Since the mass of the scalar field in lattice units becomes zero on the boundary separating the FM phase from PM or PMS, and similarly the mass belonging to the staggered scalar field (32) is zero on the boundary between AMF and PM or PMS, in the points "A" and "B" in fig. 1 there are two light scalar states. This means that, remarkably enough, *the Higgs scalar field is doubled* in continuum limits taken near these points! Although reflection positivity (and hence unitarity) usually cannot be proven near these points, because G is large and κ is negative [29], such continuum limits may have a relevance for a description of models with two Higgs doublets.

The doubling of the scalar field can be nicely seen in the momentum dependence of the scalar propagator. The Jülich group [30] investigated this with naive fermions. Near the point "B" in fig. 1, where the fermions become massive due to the strong Yukawa-coupling y, a typical result is shown in fig. 3. The light scalar states in the scalar inverse propagator can be seen at the two opposite corners of the Brillouin zone. In the vicinity of point "A" the momentum space prop-

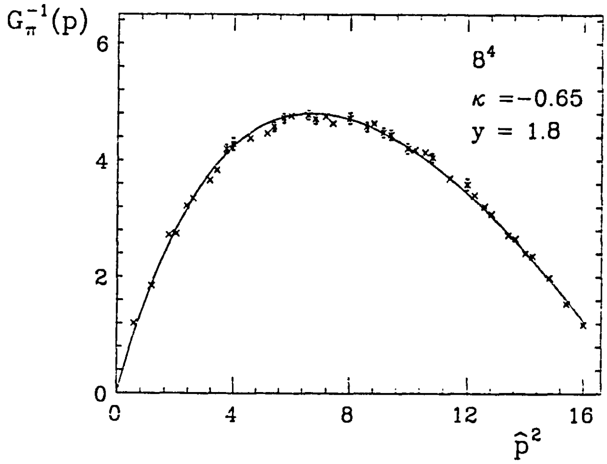

Fig. 3. The behaviour of the inverse scalar propagator near point "B" in the phase diagram, which shows two light states at the opposite corners of the Brillouin zone, where the inverse propagator is small.

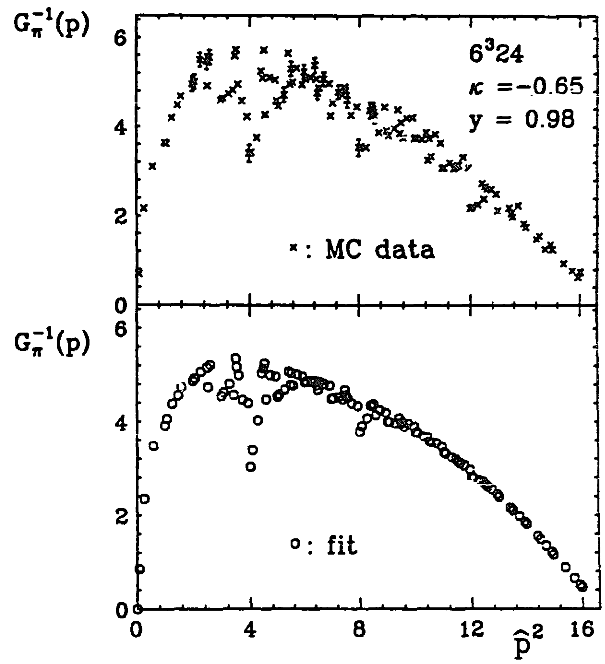

Fig. 4. The same as figure 3, near point "A". The upper part shows the Monte Carlo data, the lower one a fit by the contribution of fermion loop insertions.

agator also shows two light states but, in addition, there is a complicated structure at intermediate momenta (see fig. 4). This can be explained as due to the effect of light fermion doubler states in the fermionic self energy insertions.

The dependence of the phase structure on the bare quartic coupling λ was up to now not investigated in detail. Fig. 1 is based mainly on simulations and analytic approximations at $\lambda = \infty$. In a recent nonperturbative study, based on large N and numerical methods [31,32], at small λ an alternative phase structure emerged. As shown by fig. 5, point "A" is replaced by a first order phase transition line, before the FM and AFM phases come close to each other. The FI phase is not observed, at least not as a stable phase (metastability could be due to the fermion simulation algorithm). It is not excluded that the two pictures fig. 1 and fig. 5 will be unified as a function of λ, suggesting a possible nontrivial tricritical point at intermediate λ.

Unfortunately, many of the phase structure studies, in particular also fig. 5, was obtained by naive fermions, which offer some peculiar mechanisms for producing phase transitions. An example of such mechanisms, specific for naive fermions, is due to simple identities like

$$\phi_x \overline{\psi}_x \psi_x = \hat{\phi}_x \overline{\hat{\psi}}_x \hat{\psi}_x \; ;$$

$$\hat{\psi}_x \equiv e^{i\pi(x_1+x_2+x_3+x_4)} \psi_x \, . \tag{33}$$

This means that ordinary Yukawa-couplings also produce interactions among the staggered scalar field and fermions at the opposite corners of the Brillouin zone. For instance, if $\langle \hat{\phi}_x \rangle \neq 0$ then the second form in (33) produces an off-diagonal fermion mirror fermion mixing mass term (remember that doublers at the opposite corners of the Brillouin zone have opposite chiralities). If the doublers are heavy, such mixing mass terms are not important. In case of naive fermions,

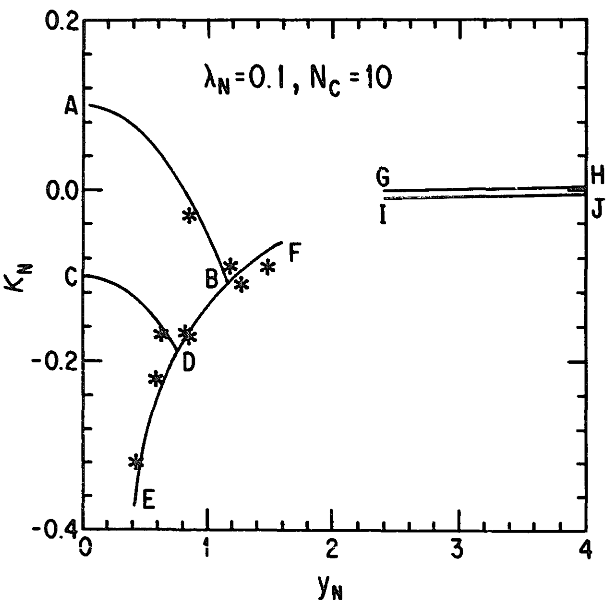

Fig. 5. The alternative phase structure, instead of fig. 1, at small quartic scalar coupling. The dashed line shows the position of a first order phase transition. The details in the middle of the diagram are not known at present.

however, when the doublers are light, they can qualitatively change the light fermion content. The statement that the naive fermion action describes 16 identical fermion flavours is in general not true in an interacting Yukawa-theory. In case of staggered fermion actions, depending e.g. on the lattice form of the Yukawa-coupling, similar mechanisms as the one discussed here may also be present.

3.3. Smit-Swift model

A possibility to reconcile the Wilson fermion formulation with chiral symmetry is to intruduce scalar fields at appropriate places in the Wilson-term. The obtained *Wilson-Yukawa coupling* has the rôle of removing the unwanted opposite chirality fermion doublers from the physical spectrum. This leads to the Smit-Swift approach for chiral lattice gauge theories [23], which has been the object of intensive scrutiny in recent years (see also the contributions to this Proceedings [33–36]).

The fermion part of the $SU(2)_L \otimes SU(2)_R$ symmetric Smit-Swift action is

$$S_f = \sum_x \Big\{ -K \sum_{\mu=\pm1}^{\pm4} (\overline{\psi}_{x+\hat{\mu}}\gamma_\mu\psi_x) + y\left[(\overline{\psi}_{Rx}\varphi_x^+\psi_{Lx}) + (\overline{\psi}_{Lx}\varphi_x\psi_{Rx})\right] + w\sum_{\mu=\pm1}^{\pm4}\left[(\overline{\psi}_{Rx}\varphi_x^+\psi_{Lx}) - (\overline{\psi}_{Rx+\hat{\mu}}\varphi_x^+\psi_{Lx}) + (\overline{\psi}_{Lx}\varphi_x\psi_{Rx}) - (\overline{\psi}_{Lx+\hat{\mu}}\varphi_{x+\hat{\mu}}\psi_{Rx})\right]\Big\} \ . \tag{34}$$

The fermion hopping parameter K can be chosen by the fermion field normalization, for instance, to $K=\frac{1}{2}$. y is the bare Yukawa-coupling and w is the Wilson-Yukawa coupling parameter.

In the vicinity of the *Gaussian fixed point* at $y=w=0$, up to the point "A" in the phase diagram (fig. 1), it is not expected that the fermion doublers are removed from the spectrum. In fact, as the detailed investigations of the Jülich group showed [36], the ratio of the doubler masses to the renormalized vacuum expectation value remains always well below 1 (see fig. 6). Therefore in the weak w region the continuum limit is not appropriate for describing the electroweak Standard Model.

At larger w, of the order of $w=0.5-1.0$, previous numerical and analytical results strongly suggest that the doubler masses can be kept at the cut-off scale, therefore they are removed from the physical spectrum in the continuum limit (for a recent addition in chiral U(1) and $SU(2)_L \otimes U(1)_Y$ symmetric models see [37]). In this region the *neutral fermion field*

$$\psi_x^{(n)} \equiv \varphi_x^+\psi_{Lx} + \psi_{Rx} \equiv \chi_{Lx} + \psi_{Rx} \ , \tag{35}$$

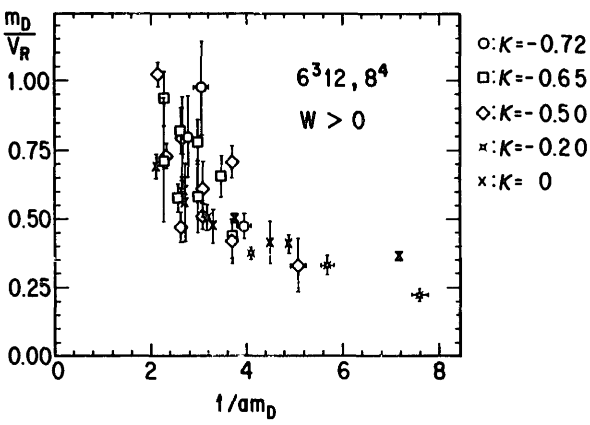

Fig. 6. The ratio of the doubler masses m_D to the renormalized vacuum expectation value v_R, as a function of $(am_D)^{-1}$ in the Smit-Swift model at small Wilson-Yukawa coupling w.

is the relevant fermionic degree of freedom [25]. $\psi^{(n)}$ is the combination of the right-handed component of the original fermion field ψ_R plus the left-handed component of a composite mirror fermion field χ_L. Both of them are singlets under $\mathrm{SU(2)_L}$, and doublets under $\mathrm{SU(2)_R}$.

In a recent paper [38] different analytical arguments based on the Golterman-Petcher shift symmetry [39], Schwinger-Dyson equations and $1/d$-expansion were collected, in order to construct a coherent picture of the continuum limit at $w = O(1)$. This continuum limit has to be taken at the intersection of the $y = 0$ line with the FM-PMS phase transition line. As already argued in section 3.1, two relevant mass parameters have to be tuned there. This is different from the situation in the Standard Model, where the scales for all masses are set by the vacuum expectation value of the scalar field. In addition, the renormalized 3-point Yukawa-coupling of the light neutral fermion is argued to vanish in the continuum limit as $y_{ren} = O(a^2)$ (not as $y_{ren} = O(\log^{-1}(am))$ expected in a theory with trivial continuum limit).

In the symmetric (PMS) phase one can also construct a *charged fermion field*

$$\psi_x^{(c)} \equiv \psi_{Lx} + \varphi_x \psi_{Rx} \equiv \psi_{Lx} + \chi_{Rx} , \tag{36}$$

which is a doublet under $\mathrm{SU(2)_L}$ and singlet under $\mathrm{SU(2)_R}$. Besides the left-handed component of the original field ψ_L, this contains the right-handed component of a composite mirror fermion field χ_{Rx}. A light bound state in the $\psi^{(c)}$ channel would be interesting from the point of view of constructing a continuum limit for a chiral $\mathrm{SU(2)_L}$ gauge theory. Nevertheless numerical simulations show [40,33] that the rest energy in the $\psi^{(c)}$ channel is, to a good approximation, equal to the sum of rest energies in the $\psi^{(n)}$ and scalar φ channels. This suggests that no $\psi^{(c)}$ bound state occurs.

In addition to the vanishing of the renormalized 3-point Yukawa-coupling in the continuum limit, Golterman, Petcher and Smit [38] argue, that in the $\mathrm{SU(2)_L}$-gauged model also the renormalized gauge coupling of the light $\psi^{(n)}$ fermion state vanishes. (In the hopping parameter expansion Aoki has a contradicting result [41], but the hopping parameter expansion is not convergent in the FM broken phase, therefore the relevance of his argument can be questionned.)

The final conclusion of these detailed investigations is, that in the large w continuum limit of the Smit-Swift model the renormalized 3-point Yukawa-coupling and the gauge coupling of the fermion most probably vanish, therefore the Smit-Swift model is not suitable for a lattice formulation of the Standard Model.

3.4. Models with mirror pairs of fermion fields

Another approach to chiral symmetric lattice gauge theories is based on the introduction of mirror pairs of fermion fields in the action [24]. Mirror fermion states not only appear in the fermion propagator as a consequence of the Nielsen-Ninomiya theorem [5], but also emerge in the strong bare Yukawa-coupling limit of lattice scalar-fermion models due to *dynamical fermion doubling* [42]. For instance, as discussed in section 3.3, in the PMS phase of the Smit-Swift model the light fermion dynamics is best described by the "neutral" and "charged" fields, which contain chiral components of a composite mirror fermion.

If mirror fermions with opposite chiral transformation properties are there anyway, it is better to introduce them explicitly as "elementary" fields in the action. Each fermion mirror fermion pair ψ_x - χ_x describes 32 fermions on the lattice. But it is possible to write down a generalized Wilson term $r\bar{\psi}_{R,x+\hat{\mu}}\chi_{Lx} + \ldots$, which mixes ψ and χ and is chirally invariant. This term removes 30 doublers by giving them masses of the order of the cut-off, and we are left with a single mirror pair.

The mirror fermion method has the following advantages: i) the remaining mirror pair constitutes the minimal possible doubling, ii) the mirror field χ is easier to control than the usual doublers since it is explicitly contained in the action and has its own couplings, iii) we have perturbation theory at our disposal to study the vicinity of the Gaussian fixed point, iv) reflection positivity can be proven in large parts of the bare parameter space [29].

The fermion part of a generic lattice action with a fermion mirror fermion pair is, in case of chiral $SU(2)_L \otimes SU(2)_R$ symmetry,

$$
\begin{aligned}
S_f = \sum_x \{ & \mu_{\psi\chi} \left[(\bar{\chi}_x \psi_x) + (\bar{\psi}_x \chi_x) \right] \\
& - \sum_{\mu=\pm 1}^{\pm 4} \left[K_\psi (\bar{\psi}_{x+\hat{\mu}} \gamma_\mu \psi_x) + K_\chi (\bar{\chi}_{x+\hat{\mu}} \gamma_\mu \chi_x) \right. \\
& \quad - K_r \left((\bar{\chi}_x \psi_x) - (\bar{\chi}_{x+\hat{\mu}} \psi_x) \right. \\
& \quad \left. \left. + (\bar{\psi}_x \chi_x) - (\bar{\psi}_{x+\hat{\mu}} \chi_x) \right) \right] \\
& + G_\psi \left[(\bar{\psi}_{Rx} \varphi_x^+ \psi_{Lx}) + (\bar{\psi}_{Lx} \varphi_x \psi_{Rx}) \right] \\
& + G_\chi \left[(\bar{\chi}_{Rx} \varphi_x \chi_{Lx}) + (\bar{\chi}_{Lx} \varphi_x^+ \chi_{Rx}) \right] \} \; . \qquad (37)
\end{aligned}
$$

A suitable fermion field normalization for numerical simulations can be defined by

$$
K_\psi = K_\chi \equiv K ; \quad K_r \equiv rK ;
$$

$$
\bar{\mu} \equiv \mu_{\psi\chi} + 8rK = 1 \; . \qquad (38)
$$

K is the common hopping parameter of fermion and mirror fermion, $\mu_{\psi\chi}$ is the corresponding fermion mirror fermion mixing mass. r is a Wilson-fermion parameter, which is usually chosen to be 1. The Yukawa-coupling of the fermion, respectively, mirror fermion is G_ψ and G_χ.

First numerical studies were performed with the mirror fermion action in the $U(1)_L \otimes U(1)_R$ model [43,29], and recently also in the $SU(2)_L \otimes SU(2)_R$ model [44]. (For a review see also ref. [45].) It turned out that the doublers can be made heavy, indeed. In the symmetric phase the renormalized Yukawa-couplings could be driven to surprisingly large values, well beyond the tree unitarity bound (although, due to limitations in computer time, up to now only on small lattices). In the broken phase the mirror fermion masses were pushed up to (3-4)-times the renormalized vacuum expectation value, that is, in physical units into the TeV mass range. Mirror fermions with such high masses and small enough mixings to the three light fermion families are not excluded by present phenomenology [46,47]. Therefore the Standard Model can be formulated on the lattice to a sufficiently good approximation by the action with mirror pairs of fermion fields. Of course, the question remains, whether the mirror fermions have to be physical, or whether perhaps it is possible to decouple them completely in the continuum limit, leaving us with a purely chiral minimal Standard Model on the lattice.

Before discussing this question in section 3.6, let us remark that some simple limits of the electroweak Standard Model can be numerically simulated with present technique, using the mirror fermion action, without solving the difficult problem of exact chirality. The point is that the Yukawa-coupling of a degenerate fermion doublet is equivalent to the Yukawa-couling of a mirror doublet. This can be seen by applying charge conjugation to the mirror fermion field:

$$\chi_{cL(R)} = C\overline{\chi}^T_{R(L)} \,, \quad \overline{\chi}_{cL(R)} = \chi^T_{R(L)}C \,. \tag{39}$$

Therefore we have from (2)

$$\begin{aligned} &(\overline{\chi}_{Rx}\varphi_x\chi_{Lx}) + (\overline{\chi}_{Lx}\varphi_x^+\chi_{Rx}) \\ &= (\overline{\chi}_{cRx}\epsilon^{-1}\varphi_x^+\epsilon\chi_{cLx}) + (\overline{\chi}_{cLx}\epsilon^{-1}\varphi_x\epsilon\chi_{cRx}) \,. \end{aligned} \tag{40}$$

If the new fields

$$\chi'_x \equiv \epsilon\chi_{cx} \,, \quad \overline{\chi}'_x \equiv \overline{\chi}_{cx}\epsilon^{-1} \,, \tag{41}$$

are introduced, one obtains the Yukawa-coupling of a fermion doublet.

The equivalence of the Yukawa-coupling of a degenerate mirror doublet to the Yukawa-coupling of a doublet is very important from the point of view of the Hybrid Monte Carlo algorithm, where a flavour doubling has to be introduced. If the fermion matrix for the first flavour is Q, then the second flavour has Q^+, in order to have a positive matrix Q^+Q in the Gaussian action of pseudofermions. In the adjoint Q^+ the rôles of the fermion and mirror fermion are interchanged: if for the first flavour ψ_1 is the fermion and χ_1 the mirror fermion, then for the second flavour ψ_2 is the mirror fermion and χ_2 the fermion. Introducing the new basis ψ'_2, χ'_2 for the second flavour according to (41), the fermion fields will be ψ_1, ψ'_2 and the mirror fermion fields χ_1, χ'_2. In this form one has, obviously, a global SU(2) flavour symmetry interchanging the two pairs: ψ_1 with ψ'_2 and χ_1 with χ'_2.

Using the four fields $\psi_1, \chi'_1, \psi'_2, \chi_2$ one can describe four degenerate fermion doublets, which is equal to the fermion content of a degenerate standard fermion family. However, due to the ψ-χ mixing in the action with mirror pairs of fermion fields, there is no SU(4) Pati-Salam symmetry [48] acting on the four doublets. Even if the renormalized ψ-χ mixing angle, usually denoted by α_R [29], is tuned to zero, the $\mathrm{SU(4)_{Pati-Salam}}$ symmetry is only approximately valid at small momenta.

In order to exactly describe a degenerate fermion family, without gauge couplings, one has to introduce a further doubling, and consider the fermion matrix

$$\mathcal{Q} = \begin{pmatrix} Q & 0 & 0 & 0 \\ 0 & Q^+ & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & Q^+ \end{pmatrix} \,. \tag{42}$$

This can be simulated by the Hybrid Monte Carlo algorithm, and one has an exact SU(4) flavour symmetry acting simultaneously on the basis $(\psi_1, \psi'_2, \psi_3, \psi'_4)$ and $(\chi_1, \chi'_2, \chi_3, \chi'_4)$. Moreover at the special point

$$G_\chi = \mu_{\psi\chi} = 0 \tag{43}$$

there is an exact Golterman-Petcher fermion shift symmetry [39] in terms of the χ-fields [7,44]. This implies that in the continuum limit the χ-fields are completely decoupled and have no interactions at all. As a consequence, there is an $\mathrm{SU(4)_{Pati-Salam}}$ symmetry acting on the ψ-fields alone. This is, therefore, an exact description of a degenerate fermion family by the action with mirror pairs of fermion fields.

This way of decoupling the χ-fields was proposed by the Italian group [49]. Unfortunately, in case of nonzero gauge couplings it does not work, because the gauge couplings cannot vanish. The equivalence of the mirror doublet to an ordinary doublet does not work either, except for the SU(2) gauge coupling. The physical fermions are in complex representations of the $SU(3)_{colour} \otimes U(1)_Y$ symmetry group, which are not equivalent to their charge conjugates.

Neglecting the gauge couplings and the mass splittings within a fermion family can still be a reasonable approximation for some questions of nonperturbative nature. Such a question is the determination of the cut-off dependent *allowed region* in the space of renormalized quartic and Yukawa-couplings. If the continuum limit of Yukawa-models is trivial, then there are cut-off dependent upper bounds on both the renormalized quartic and Yukawa-couplings, which tend to zero in the continuum limit. In pure ϕ^4 models the upper bound is qualitatively well described by the 1-loop perturbative β-function, if the Landau-pole in the renormalization group equations is assumed to occur at the scale of the cut-off. The same might be true for scalar-fermion models with Yukawa-couplings. For instance, in the model with $SU(2)_L \otimes SU(2)_R$ symmetry and N_f degenerate fermion doublets the β-functions for the quartic (g_R) and Yukawa-($G_{R\psi}$) couplings are:

$$\beta_{g_R} = \frac{1}{16\pi^2}\left(4g_R^2 + 16N_f g_R G_{R\psi}^2 - 96N_f G_{R\psi}^4\right) ,$$

$$\beta_{G_{R\psi}} = \frac{1}{16\pi^2} \cdot 4N_f G_{R\psi}^3 . \tag{44}$$

Since in the region where $G_{R\psi}^2 \gg g_R$ the 1-loop β-function of the quartic coupling β_{g_R} is negative, besides the upper bounds there is also a lower bound on g_R for fixed $G_{R\psi}$, which is called

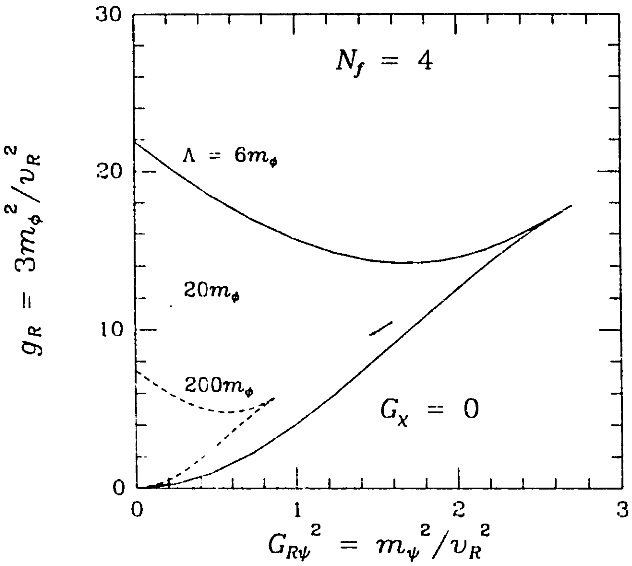

Fig. 7. The cut-off dependent allowed region in the $(G_{R\psi}^2, g_R)$ plane for cut-off values equal to some multiples of the Higgs-boson mass m_ϕ. In the Yukawa-model describing a degenerate fermion family without gauge couplings the perturbative 1-loop β-functions are assumed.

in the literature *vacuum stability bound* [50]. On the lattice, if one assumes the qualitative behaviour of the 1-loop β-function to be valid also nonperturbatively, the vacuum stability lower bound occurs at zero bare quartic coupling $\lambda = 0$, whereas the upper bound occurs at $\lambda = \infty$ [29,51,52]. Negative λ-values are excluded, because there the path integral is divergent. For the 1-loop β-functions in (44) the bounds in the plane of $(G_{R\psi}^2, g_R)$ are shown in fig. 7.

It has to be emphasized that the existence and shape of the allowed region for renormalized quartic and Yukawa-couplings in Yukawa-models is merely based on the assumption that the 1-loop β-function is correct also in the region where the higher loop contributions are large. At present there is no real support for this assumption. For instance, it is not at all true, that taking the 2-loop β-functions instead of the 1-loop ones, the allowed region is corrected only a little bit. In fact, according to the 2-loop β-functions there exists a nontrivial ultraviolet stable fixed point, which means that the continuum limit is nontrivial and every point in the $(g_R, G_{R\psi})$ plane is allowed [29]. Moreover, in the symmetric phase of Yukawa-models with mirror pairs of fermion fields there is some evidence that, unlike the renormalized quartic coupling in pure ϕ^4 models, the renormalized Yukawa-coupling may reach large values [43,44]. It is clear that under these circumstances numerical checks of the cut-off dependent allowed region, like in fig. 7, would be highly desirable.

The model with four degenerate fermion doublets still contains a relatively large number of fermions. This may have an important qualitative influence from the quantum field theory point of view. For instance, the tree unitarity upper bound on the renormalized Yukawa-couplings with N_f mirror pairs of fermions is, both in the $SU(2)_L \otimes SU(2)_R$ and $U(1)_L \otimes U(1)_R$ case,

$$G^2_{R\psi},\ G^2_{R\chi},\ G_{R\psi}G_{R\chi} \le \frac{4\pi}{N_f}\ . \tag{45}$$

This means that "strong" coupling occurs at a value proportional to $1/\sqrt{N_f}$. (The same conclusion can also be drawn from the 1-loop β-functions, as in (44).) Numerical simulations by the Hybrid Monte Carlo algorithm can be performed for $N_f = 2$, but in general not for $N_f = 1$. The case $G_\psi = G_\chi$ is an exception, because then the action with mirror pairs of fermion fields is equivalent to some simple action similar to the one discussed in section 3.1 (see section 3.6).

A possibility to simulate the $N_f = 1$ case in general has been recently pointed out by Münster and Plagge [53,45]. They numerically determined the phase ϕ_Q of the fermion determinant $\det Q$ in the $U(1)_L \otimes U(1)_R$ symmetric model, on a number of configurations generated by the Hybrid Monte Carlo algorithm with Q^+Q, in many different points of the bare parameter space. It turned out that on the $4^3 \cdot 8$ lattice the phase was always very small: $|\phi_Q| \le 10^{-3}$. This means that one can use the hybrid Langevin classical dynamics algorithm to generate configurations with the weight $|\det Q|$, and then either neglect the phase factor $\exp(i\phi_Q)$ completely, or take it into account as a small correction in the measurable quantities.

3.5. Staggered mirror fermions

There are several different possibilities to formulate Yukawa-models with staggered lattice fermions. (A quite recent one by Smit [54], which uses the staggered flavours to represent the chiral $SU(2)_L \otimes U(1)_Y$ symmetry, will not be discussed here.) In most cases the internal symmetry, say, $SU(2)_L \otimes SU(2)_R$ is introduced by extra indices at a lattice site. Such actions were studied recently in refs. [27,55].

In the case of a $U(1)$ chiral symmetry [27] the action is a staggered fermion transcription of the continuum action, which corresponds to the action considered in section 3.1. An important ingredient is the choice of the Yukawa-coupling on overlapping 2^4 hypercubes, which is motivated by previous studies of staggered Yukawa-models with different coupling definitions (for a review see [6]). In this case the fermionic part of the lattice action, with two species of staggered fermions and a complex scalar field $\phi_x \equiv \sigma_x + i\pi_x$, is:

$$S_f = \sum_x \left\{ \frac{1}{2} \sum_{f=1}^{2} \sum_{\mu=1}^{4} \alpha_{x\mu} \overline{\chi}^{(f)}_x (\chi^{(f)}_{x+\hat\mu} - \chi^{(f)}_{x-\hat\mu}) \right.$$

$$+ \frac{y}{16} \sum_{l=1}^{16} \left[\sigma_x \left((\overline{\chi}^{(1)}\chi^{(2)})_{x+\xi_l} + (\overline{\chi}^{(2)}\chi^{(1)})_{x+\xi_l} \right) \right.$$

$$+i\pi_x \left((\overline{\chi}^{(1)}\chi^{(2)})_{x+\xi_l} - (\overline{\chi}^{(2)}\chi^{(1)})_{x+\xi_l} \right) \Big] \Big\} \; . \quad (46)$$

$\alpha_{x\mu}$ is the usual staggered fermion sign factor, and y the bare Yukawa-coupling. The summation $\sum_l$ goes over the points of the 2^4 hypercubes, which are labelled by the shift vectors ξ_l.

Decomposing the action into a sum of two adjoint pieces one can see that this model can be simulated by Hybrid Monte Carlo, and that the exact symmetry on the lattice is $U(1)^{\otimes 3}$. In the continuum limit, due to the fact that the staggered fermion fields represent four fermion species, the expected global symmetry is $U(1)_{chiral} \otimes U(4)_{fermion} \otimes U(4)_{mirror}$, where the last two factors are vectorlike symmetries acting on the four fermions, respectively, four mirror fermions. The fermions and mirror fermions are defined on the usual "flavour basis" of staggered fermions, where the Yukawa-coupling has the form

$$\sigma_x \sum_{q=1}^{8} (\overline{\psi}_{qx}\psi_{qx}) + i\pi_x \sum_{q=1}^{8} (\overline{\psi}_{qx}\gamma_5 T_{qq}\psi_{qx}) \; , \quad (47)$$

with the diagonal matrix $T_{qq} = \{1, 1, -1, -1, -1, -1, 1, 1\}$. The fermions are the components belonging to the eigenvalues +1, the mirror fermions to the eigenvalues -1.

Introducing an extra SU(2) index on the fermion field, and using the equivalence of degenerate fermion doublets with mirror fermion doublets (see section 3.4), one can describe two degenerate fermion families by a staggered fermion action analogous to (46). The Pati-Salam symmetry interchanging different doublets [48] is not exact on the lattice. Its restoration in the continuum limit is a nontrivial question which can, however, be investigated by numerical methods.

The model with U(1) chiral symmetry and lattice action (46) was investigated by numerical simulation in ref. [27]. In particular, the important question was asked, what happens with the Goldstone boson (π) in the finite volume simulations. (For the same question in pure ϕ^4 models see section 2.3.) Two methods were compared: either a symmetry breaking mass term was introduced, in order to make the Goldstone boson massive, or the "constraint correlations" were considered without symmetry breaking term at a given value of the average scalar field. The results obtained by the two methods are reasonably compatible. The exact Ward-Takahashi identities following from the global symmetry were numerically checked, in order to judge the quality of the numerical simulations in the chosen points of the bare parameter space. As a result of calculating the π-boson correlation functions by fermion-antifermion sources, it turned out that the annihilation diagrams dominate. These could be effectively calculated by using "noisy estimators" in terms of the pseudofermion field.

3.6. The question of chirality

As discussed in section 3.3, the Smit-Swift model probably does not have an appropriate continuum limit, which would correspond to an acceptable regularization of the chiral gauge theory in the electroweak Standard Model. The action of the model with mirror pairs of fermion fields, considered in section 3.4, does not look "chiral" in the sense that the representation of the electroweak symmetry group on the fields present in the action is real ("vectorlike"). Indeed, for instance, if one goes to the symmetric phase of the $SU(2)_L \otimes SU(2)_R$ symmetric model and introduces, instead of the fermion doublet (ψ_x) and mirror fermion doublet (χ_x) fields, the new combinations

$$\psi_{Ax} \equiv \psi_{Lx} + \chi_{Rx} \; , \quad \psi_{Bx} \equiv \chi_{Lx} + \psi_{Rx} \; , \quad (48)$$

then the fermion part of the action in (37) becomes

$$S_f = \sum_x \{ (\overline{\psi}_{Ax}\psi_{Ax}) + (\overline{\psi}_{Bx}\psi_{Bx})$$

$$-K \sum_{\mu=\pm1}^{\pm4} [(\overline{\psi}_{A,x+\hat{\mu}}[r+\gamma_\mu]\psi_{Ax}) + (\overline{\psi}_{B,x+\hat{\mu}}[r+\gamma_\mu]\psi_{Bx})] + (\overline{\psi}_{Ax}[G_\alpha - G_\beta\gamma_5]\varphi_x\psi_{Bx}) + (\overline{\psi}_{Bx}[G_\alpha + G_\beta\gamma_5]\varphi_x^+\psi_{Ax})\} \ . \quad (49)$$

The new Yukawa-couplings are related to the old ones by

$$G_\alpha \equiv \tfrac{1}{2}(G_\psi + G_\chi) \ , \quad G_\beta \equiv \tfrac{1}{2}(G_\psi - G_\chi) \ . \quad (50)$$

In case of $G_\beta = 0$, $G_\alpha = G$, $r = 1$ and $K \equiv K_A = K_B$ this is the $SU(2)_A \otimes SU(2)_B \equiv SU(2)_L \otimes SU(2)_R$ symmetric variant of the action (26) considered in section 3.1. Gauging the global symmetry in the symmetric phase gives a vectorlike gauge theory with a pair of $SU(2)_A$- and $SU(2)_B$-doublet fermions, whith the same mass.

In the broken phase with nonzero expectation value of the scalar field φ_x the fermion spectrum is better described in the original form of the action (37). The fermion and mirror fermion are mixed with each other and have different masses. Tuning to small mixing ($|\alpha_R| \ll 1$) is possible by the hopping parameter K, and if $|G_\chi| \gg |G_\psi|$ is chosen, then a large mass splitting ($m_\chi \gg m_\psi$) can be achieved. First numerical simulations showed [29], that the mass of the mirror fermions can also be substantially larger than the renormalized vacuum expectation value: $m_\chi \simeq (3-4)v_R$. This means that the mirror fermions can be sufficiently decoupled from the light fermions. Mirror fermions in the few hundred GeV range with mixings satisfying $|\alpha_R| < 0.1$ are allowed by present experimental data [46,47]. Nevertheless, the mirror fermions are still there in the continuum limit as physical particles. Since for small mixing their masses are essentially given by spontaneous symmetry breaking, if the continuum limit is trivial, then the upper bounds on the renormalized Yukawa-couplings imply that there is a cut-off dependent upper bound on the mirror fermion masses, too. The only possibility to give the mirror femions a mass much higher than the electroweak scale would be if, perhaps, there was a nontrivial fixed point in the Yukawa-coupling, allowing for a nontrivial continuum limit. Once the mirror fermions are there, it seems very difficult to get rid of them!

To avoid mirror fermions in a lattice formulation with explicit gauge invariance is not possible if certain conditions, like perturbative renormalizability and absence of negative metric states, are fulfilled [56]. The Nielsen-Ninomiya theorem [5], implying the presence of mirror fermions in the lattice fermion propagator, can also be extended to nonhermitian actions, if chiral invariance is assumed [57].

A potential problem with lattice formulations of the electroweak Standard Model is the absence of baryon number violation by the $SU(2)_L$ anomaly [58]. A solution to this problem in the Smit-Swift model has been proposed in ref. [59] by defining an appropriate baryon number current (see also ref. [41]). However, since the Smit-Swift model does not have the assumed continuum limit, this solution is only formal.

In a model with mirror pairs of fermions the baryon number is obviously conserved: the violation in the baryon current of fermions is exactly compensated by an opposite violation in the baryon current of mirror fermions. It is, however, intuitively plausible that at temperatures much below the mass of the mirror fermions the asymmetry in the sphaleron dominated baryon number violating processes [60] has to be effectively reproduced. The lattice formulation of the Standard Model by mirror pairs of fermion fields is "vectorlike" only at high energies. In the "low" energy region near the electroweak symmetry breaking scale the "chirality" of the minimal Standard Model is reproduced.

The only lattice formulation of chiral gauge theories without mirror fermions is based on gauge fixing [61]. In a recent paper [62] it has been shown, that in lattice formulations of chiral gauge theories, in which the non-invariant terms in the action are made gauge invariant by the coupling to a scalar field, like in the Smit-Swift model, the scalar field can be decoupled only if the parameters are tuned so as to satisfy a set of BRST identities. This leads to a necessary gauge fixing and, therefore, to the approach of ref. [61]. This conclusion does not immediately apply to the formulation with mirror pairs of fermion fields, because if the mirror fermions are integrated out, the action becomes nonlocal as a consequence of the presence of the mirror fermions in the physical spectrum.

Unfortunately, the consistency of the gauge fixing approach of Borrelli et al. [61] is up to now only shown in 1-loop perturbation theory. This may not reveal all problems because, for instance, there are also power divergences in some mass counterterms. It is questionable whether in the space of these bare parameters the appropriate (Gaussian) fixed point can be found in the desired phase, which allows for the definition of the continuum limit of the target chiral gauge theory. The nonperturbative study of this question is difficult, due to the large number of bare parameters, and possibly also because of the problem of Gribov copies at nonperturbative gauge fixing.

In conclusion, the nonperturbative formulation of the Standard Model would be much simpler if in nature there were mirror pairs of fermions. The only way to give the mirror fermions a mass well above the TeV range is (perhaps) the existence of a nontrivial fixed point in the Yukawa-coupling. The obvious question is, where are the mirror fermions experimentally? It is rather strange that the minimal Standard Model, wich is so succesful in reproducing all known experimental data, up to now does not have a consistent and simple lattice formulation.

3.7. Composite models

A possible way out of the difficulties to formulate the electroweak Standard Model by elementary scalar and fermion fields corresponding to the observed light particles is, that there is a further layer of compositness. A popular set of models are based on *preons* bound together in the "conventional" way by QCD-like asymptotically free nonabelian gauge forces. For a recent work along these lines see ref. [63].

There is also another possibility to produce bound states, which goes back to the classical paper of Nambu and Jona-Lasinio [64], and was revived recently in the context of a heavy top quark (see ref. [65] and references therein). The strong force producing the bound states is in these models a short range four-fermion interaction. In fact, from the point of view of quantum field theory this way of strong binding is quite different from the one acting in QCD, because it is based not on asymptotically free, but on infrared free field theories. The couplings become stronger and stronger at short distances, and according to the wisdom of triviality, the scale of the strong interaction is roughly equal to the cutoff scale, which can never be infinite.

Examples of such behaviour producing bound states are well known in scalar-fermion models with very strong bare Yukawa-couplings. For instance, in the σ-model with Wilson lattice fermions at strong bare Yukawa-coupling composite mirror fermions occur [42]. In a staggered fermion model with U(1) global symmetry bosonic bound states with fermion number 0 and ± 2 are present in the strong coupling PMS phase [66]. The important feature of these systems is that the one-to-one correspondence between light states and fields in the action is lost. Since the continuum limit can be taken only at strong bare couplings, bare perturbation theory as a tool is lost. As a consequence of all these, renormalization group scaling in the continuum limit generally requires the extension of bare parameter space, similarly to strong coupling lattice QED [67], which is another case of an infrared free quantum field theory.

This new kind of bound state problem can be called *short range compositness*, which is based on *asymptotic slavery*, in contrast to asymptotic freedom in QCD-like theories.

The best studied examples of "short range compositness" are Nambu-Jona-Lasinio-type four-fermion models. As nonperturbative technical tool, $N \to \infty$ methods are usually applied. The four-fermion interaction is equivalent to taking $\kappa = \lambda = 0$ in scalar-fermion models, when the scalar field has no bare kinetic term and can, therefore, be trivially integrated out. Of course, the kinetic term of the "auxiliary" scalar field is generated by the strong Yukawa interaction. Due to the phase structure of lattice Yukawa-models there are, in principle, two distinct possibilities. Since in fig. 1 the critical line at the boundary of the FM phase has two intersections with $\kappa = 0$, one can take the continuum limit either near point "A" or near point "B". The corresponding, physically different, theories can be called "A-type" and, respectively, "B-type". The two types of continuum limits can be observed, for instance, in the hopping parameter expansion at $\kappa = 0$ [68]. Note that at small bare quartic coupling λ and large N there seems to be only a single intersection of the critical line with $\kappa = 0$ (see fig. 5), therefore there is only an A-type short range compositness.

Since the $\kappa = 0$ subspace in scalar-fermion models has one parameter less, there is a relation between the renormalized parameters, for instance, between the fermion mass and boson mass [65]. If, however, higher dimensional four-fermion couplings are allowed, this constraint is released and the number of independent renormalized parameters is restored [31,69].

An exciting possibility would be to use short range compositness to formulate chiral gauge theories on the lattice ("chiral short range compositness"). In fact, one of the earliest proposals for a chiral invariant version of Wilson lattice fermions by Eichten and Preskill [70] is based on four-fermion interactions, and hence on "short range compositness". However, as shown recently by Golterman [34], if one reintroduces the scalar field as an auxiliary field, the Eichten-Preskill model becomes very similar to a particular Smit-Swift-type model. Therefore it has a similar phase structure, which contradicts the phase structure anticipated by Eichten and Preskill. In other words, the Eichten-Preskill model presumably does not have an appropriate continuum limit, similarly to the Smit-Swift model.

4. SUMMARY AND CONCLUSIONS

Since the Tallahassee Lattice '90 Conference our understanding of nonperturbative Yukawa-models progressed substantially, even if there are still very important and tantalizing open questions. In a short summary:

– Pure ϕ^4 Higgs models are well understood. They are serving nowadays mainly as simple theoretical laboratories for trying out new analytical and numerical methods.

– In spite of the progress, important features of the scalar-fermion models are still to be discovered.

– An important prerequisite for future numerical studies is that there are simple reasonable limits of the electroweak Standard Model (with degenerate fermion families and without gauge couplings), which can be numerically studied by present fermion techniques, without solving the difficult chirality problem. An obvious question for numerical simulations is to investigate the influence of heavy fermions, for instance of a fourth heavy family, on the bounds of the Higgs boson mass.

– The list of open theoretical questions is long. Just to mention a few most important ones: Are there nontrivial fixed points in the Yukawa-coupling? How heavy can the mirror fermions be made? Is there a problem with baryon number nonconservation? Does gauge fixing work for chiral gauge theories on the lattice?

– In the composite model approach, is "short range compositness" a viable alternative dynamical scheme for the Higgs-Yukawa sector of the Standard Model?

Since the Higgs-Yukawa sector is a central and the least known piece of the Standard Model, one hopes that many of these questions will be understood in the future with the help of lattice quantum field theory techniques.

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