Nuclear Physics B 384 (1992) 352–380
North-Holland

NUCLEAR
PHYSICS B

Unconstrained SU(2) and SU(3) Yang–Mills classical mechanics

Bernd Dahmen [1] and Bernd Raabe

II. Institut für Theoretische Physik der Universität Hamburg, Luruper Chaussee 149, 2000 Hamburg 50, Germany

Received 27 January 1992
Accepted for publication 8 July 1992

A systematic study of constraints in SU(2) and SU(3) Yang–Mills classical mechanics is performed. Except for the SU(2) case with vanishing spatial angular momenta they turn out to be non-holonomic. Using Dirac's constraint formalism we achieve a complete elimination of the unphysical gauge and rotational degrees of freedom. This leads to an effective unconstrained formulation both for the full SU(2) Yang–Mills classical mechanics and for the SU(3) case in the subspace of vanishing spatial angular momenta. We believe that our results are well suited for further explicit dynamical investigations.

1. Introduction

During the last decade there has been considerable interest in classical pure SU(N) Yang–Mills theory with spatially constant fields $A_{a\mu} \equiv A_{a\mu}(t)$. In the temporal "gauge" ($A_{a0} \equiv 0$) field theory then reduces to a nonlinear mechanical system, sometimes called Yang–Mills classical mechanics (YMCM).

Its investigations were initiated by Matinyan et al. [1] who studied YMCM for the gauge group SU(2). Making a special ansatz for the fields they obtained a two-dimensional system

$$L = \tfrac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 - x_1^2 x_2^2\right) \tag{1.1}$$

for which they found stochastic oscillations thereby disappointing earlier hopes that the non-abelian gauge theories might be integrable as the non-linear σ-models. Motivated by this unexpected result the dynamics of (1.1) was analysed by many other authors [2] using different methods of modern non-linear mechanics. Apart from being a rather simplified version of QCD, the system (1.1) has also served as a toy model for attempts to understand the semiclassical limit of quantum mechanical systems whose classical counterpart is completely chaotic [3]. That in this model islands of stability exist in some

[1] Present address: Deutsches Elektronen-Synchrotron DESY, Notkestrasse 85, 2000 Hamburg 52, Germany. Supported by Studienstiftung des deutschen Volkes.

0550-3213/92/$05.00

tiny fraction of phase space was only recently demonstrated by Dahlqvist and Russberg [4].

Due to Gauss' law YMCM is a constrained dynamical system. For the case of the gauge group SU(2), Villarroel [5] addressed the question of the holonomy of this constraint and proved that it is not integrable. This means that the physically accessible configuration space is *not* reduced by the constraint, analogous to the well-known motion of a wheel without slipping (see e.g. ref. [6]). In apparent contradiction to this result Asatryan and Savvidy [7] explicitly carried out such a reduction by means of the customary singular value decomposition.

To clarify this paradoxical situation and remove the apparent contradiction is the first aim of our paper. This is achieved by extending Villarroel's point of view and regarding *all* constants of motion as constraints on the dynamics.

Starting our investigations with the study of the SU(2) case, we show that for vanishing spatial angular momenta the combined constraints arising from gauge and rotational symmetry are holonomic and lead to a reduction of the initially nine-dimensional configuration space to a three-dimensional submanifold. This allows an immediate identification of the physically relevant degrees of freedom and their dynamics which for two components is determined by the lagrangian (1.1). In the general case, however, where arbitrary non-zero spatial angular momenta are admitted, the constraints are non-holonomic and the configuration space stays nine-dimensional.

Thereafter we treat the case of SU(3) YMCM most relevant in the description of hadronic phenomena. We find that for this model even in the case of vanishing spatial angular momenta no reduction takes place. Consequently, there is also no distinguished unconstrained system contrary to the case of SU(2) YMCM.

To nevertheless eliminate the non-dynamical gauge and rotational degrees of freedom we have to broaden the scope of our analysis and pass to a description in phase space. The constraint formalism for hamiltonian systems invented by Dirac [8] allows us to isolate canonical pairs of variables which are no longer subject to constraints therefore representing the truly physical coordinates and momenta of the theory. By constructing the reduced hamiltonian in these variables, we obtain effective formulations for the general SU(2) YMCM and for the SU(3) case in the subspace of vanishing spatial angular momenta that contain all dynamical properties of the corresponding system. Hence those are no longer obscured by the presence of constraints but can be studied directly.

Our paper is organized as follows: In sect. 2 we investigate the holonomy properties of the constraints for the SU(2) as well as the SU(3) model and discuss the consequences of our results. After a brief review of the most relevant ideas and concepts to treat hamiltonian systems with constraints in sect. 3, we construct the completely reduced hamiltonian of full SU(2) YMCM in sect. 4.

In the case of SU(3) YMCM a direct reduction unfortunately could not be carried out. Nevertheless, after a consistent gauge fixing of the constraints we obtain a parametrization of the reduced phase space in sect. 5. Though the resulting effective hamiltonian is rather complicated we are able to classify the low-dimensional invariant subsystems. The simplest case of genuine SU(3) dynamics is represented by a four-dimensional system which therefore should be the starting point for further explicit dynamical investigations of the Yang–Mills theory for this gauge group.

2. Constraint analysis in configuration space

The lagrangian for SU(N) YMCM reads

$$L = \tfrac{1}{2}\dot{A}_{ai}\dot{A}_{ai} - \tfrac{1}{4}g^2 f_{abe}f_{ecd}A_{ai}A_{bj}A_{ci}A_{dj}\,, \tag{2.1}$$

where $i, j = 1, 2, 3$ and $a, b, c, d, e = 1, 2, \dots, \dim(\mathrm{SU}(N)) = N^2 - 1$. Here the f_{abc} are the structure constants of the gauge group and g is the coupling constant, which will be set equal to 1 in the following. The vector potentials A_{ai} parametrize the configuration space manifold $\mathrm{M} = \mathbb{R}^{3(N^2-1)}$.

The invariance of (2.1) under both SU(N) gauge transformations and O(3) rotations in ordinary space leads to $(N^2 - 1)$ conserved colour angular momenta

$$n_a \equiv f_{abc}A_{bi}\dot{A}_{ci} \tag{2.2}$$

and three conserved spatial angular momenta

$$m_i \equiv \varepsilon_{ijk}A_{aj}\dot{A}_{ak}\,, \tag{2.3}$$

respectively. As only gauge invariant quantities are of physical interest the fields have to obey Gauss' law, i.e. the dynamics of the system is restricted to a subspace of the tangent bundle TM defined by the $(N^2 - 1)$ functions

$$n_a = 0\,. \tag{2.4}$$

Consequently, only those solutions of the equations of motion derived from the lagrangian (2.1) are of physical interest which also fulfill these constraints.

Now the question is whether they lead to a reduction of the configuration space M. To answer it, it proves advantageous to rephrase the Gauss' law constraints (2.4) in the language of differential forms which emphasizes their geometrical nature: As any system of first-order ordinary differential equations, $n_a = 0$ can be represented by a system of vanishing one-forms (a Pfaff system), namely

$$\omega_a \equiv f_{abc}A_{bi}\mathrm{d}A_{ci} = 0\,.$$

This formulation allows to use methods from differential geometry, in particular

Frobenius' theorem [6]: Let $\{\omega_a\}$ be a set of p independent one-forms on a manifold M. Then the system

$$\omega_a = 0 \qquad \text{for all } a \tag{2.5}$$

is completely integrable iff

$$\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_p \wedge \mathrm{d}\omega_a = 0 \qquad \text{for all } a\,. \tag{2.6}$$

Complete integrability of such a Pfaff system is equivalent to the existence of p independent functions F_a on M, which define an $(n-p)$-dimensional maximal integral manifold $\Sigma \subset \mathrm{M}$ of the system (2.5) through the conditions

$$F_a(x_1, \ldots, x_n) = \text{const.}, \qquad x \in \mathrm{M}\,.$$

In this case the constraints are called *holonomic* and Σ represents the physically accessible configuration space. (It goes without saying that the notion of "integrability of constraints" must be carefully distinguished from that of the "integrability of the equations of motion".)

In this section we will use Frobenius' theorem to analyse the holonomy properties of the constraints in SU(2) and SU(3) YMCM.

Villarroel [5] studied Gauss' law in isolation for the case of SU(2) YMCM. He found that these constraints are completely non-holonomic such that the physically accessible configuration space keeps on being nine-dimensional. However, though the possible values of the spatial angular momenta (2.3) are not fixed, they are nevertheless conserved,

$$m_i = k_i = \text{const.} \qquad \text{with } \boldsymbol{k} \in \mathbb{R}^3\,. \tag{2.7}$$

Hence, in addition to Gauss' law (2.4) there are in fact three more constraints on the dynamics and so it seems natural to treat gauge and rotation symmetries on equal footing.

In a first step we therefore investigate the question of holonomy for the particularly interesting case of vanishing spatial angular momenta $k_i = 0$. Thus we consider the dynamical system (2.1) for the gauge group SU(2) ($f_{abc} = \varepsilon_{abc}$) subject to the six constraints

$$n_a = \varepsilon_{abc} A_{bi} \dot{A}_{ci} = 0\,, \tag{2.8}$$

$$m_i = \varepsilon_{ijk} A_{aj} \dot{A}_{ak} = 0\,, \tag{2.9}$$

which correspond to the Pfaff system

$$\omega_a = \varepsilon_{abc} A_{bi} \mathrm{d}A_{ci} = 0\,, \tag{2.10}$$

$$\eta_i = \varepsilon_{ijk} A_{aj} \mathrm{d}A_{ak} = 0\,. \tag{2.11}$$

To check whether they fulfill Frobenius' condition (2.6) is facilitated by expressing them in coordinates $(\varphi_n, x_n, \theta_n)$ resulting from a *singular-value decomposition* for the real matrices $A \equiv (A_{ai})$ (cf. e.g. Atkinson [9], §7.9),

$$A = O\,\hat{A}\,R^{\mathrm{T}}\,, \tag{2.12}$$

where*

$$\hat{A} \equiv (\delta_{an} x_{[n]}) = \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}$$

and $O(\varphi_1, \varphi_2, \varphi_3), R(\theta_1, \theta_2, \theta_3) \in \mathrm{SO}(3)$. The matrix $O \in \mathrm{SO}(3)$ can be viewed as the adjoint representation of a gauge transformation $U \in \mathrm{SU}(2)$ leading to the interpretation of the Euler angles φ_n as gauge degrees of freedom analogous to the interpretation of the θ_n as ordinary rotational degrees of freedom. Consequently, eq. (2.12) effects a decomposition of the configuration manifold of SU(2) YMCM reflecting the symmetries of the system.

We expand the constraint one-forms (2.10) in the new basis $\{\mathrm{d}\varphi_n, \mathrm{d}x_n, \mathrm{d}\theta_n\}$:

$$\omega_a = \varepsilon_{abc} A_{bi} \left(\frac{\partial A_{ci}}{\partial \varphi_n} \mathrm{d}\varphi_n + \frac{\partial A_{ci}}{\partial x_n} \mathrm{d}x_n + \frac{\partial A_{ci}}{\partial \theta_n} \mathrm{d}\theta_n \right).$$

Making use of the orthogonality of the matrices O and the complete antisymmetry of the ε-tensor one shows that the coefficient in front of $\mathrm{d}x_n$ vanishes,

$$\begin{aligned} \varepsilon_{abc} A_{bi} \frac{\partial A_{ci}}{\partial x_n} &= \varepsilon_{abc} R_{b\mu} \delta_{\mu j} x_{[j]} O_{ij} R_{ck} \delta_{[k]n} O_{ik} \\ &= \varepsilon_{abc} R_{b\mu} x_{[j]} \delta_{\mu j} \delta_{jk} \delta_{[k]n} R_{ck} \\ &= \varepsilon_{abc} R_{bk} x_{[k]} \delta_{[k]n} R_{ck} \\ &= 0 . \end{aligned}$$

Hence

$$\omega_a = \varepsilon_{abc} A_{bi} \left(\frac{\partial A_{ci}}{\partial \varphi_n} \mathrm{d}\varphi_n + \frac{\partial A_{ci}}{\partial \theta_n} \mathrm{d}\theta_n \right), \tag{2.13}$$

and similarly for the one-forms (2.11)

$$\eta_i = \varepsilon_{ijk} A_{aj} \left(\frac{\partial A_{ak}}{\partial \varphi_n} \mathrm{d}\varphi_n + \frac{\partial A_{ak}}{\partial \theta_n} \mathrm{d}\theta_n \right). \tag{2.14}$$

Thus, neither the ω_a nor the η_i have components proportional to $\mathrm{d}x_n$ and consequently the wedge product of all the constraints $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \eta_1 \wedge \eta_2 \wedge \eta_3$ is proportional to the six-form built from the basis forms $\mathrm{d}\varphi_n$ and $\mathrm{d}\theta_n$ only, viz. $\mathrm{d}\varphi_1 \wedge \mathrm{d}\varphi_2 \wedge \mathrm{d}\varphi_3 \wedge \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \mathrm{d}\theta_3$, which is unique up to ordering. Calculating the differentials we find furthermore that $\mathrm{d}\omega_a$ as well as $\mathrm{d}\eta_i$ are proportional to $\mathrm{d}y_m \wedge \mathrm{d}y_n$ where $\mathrm{d}y_m$ is arbitrary but $\mathrm{d}y_n \in \{\mathrm{d}\varphi_n, \mathrm{d}\theta_n\}$. As a consequence, one basis form out of $\{\mathrm{d}\varphi_n, \mathrm{d}\theta_n\}$ shows up twice in each summand of the eight-forms $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \mathrm{d}\omega_a$ and $\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \mathrm{d}\eta_i$, respectively. Hence Frobenius' conditions (2.6) are fulfilled and the constraints (2.8), (2.9) are *holonomic*. As a result we find that the non-holonomic Gauss'

* We do not sum over indices in square brackets.

law constraints $n_a = 0$ are rendered holonomic by imposing the additional constraints $m_i = 0^\star$.

Furthermore, the only non-trivial solution of the transformed constraint equations are given by constant angles φ_n and θ_n. Thus, in the special case of vanishing colour *and* spatial angular momenta the gauge and rotational degrees of freedom φ_n and θ_n freeze out. The coordinates $\{x_1, x_2, x_3\}$ parametrize the three-dimensional maximal integral manifold Σ of the constraints (2.8) and (2.9) and we get as the *reduced* lagrangian

$$L_R = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \tfrac{1}{2}(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2), \tag{2.15}$$

which completely determines the dynamics on Σ. Note that for $x_3 = \dot{x}_3 = 0$ the model (1.1) of Matinyan et al. [1] is reproduced.

In the next step we investigate the general case of arbitrary spatial angular momenta $m_i = k_i \in \mathbb{R}\backslash\{0\}$ leading to the slightly modified constraints

$$n_a = 0, \tag{2.16}$$

$$\tilde{m}_i \equiv m_i - k_i = 0. \tag{2.17}$$

Hence the three one-forms corresponding to the constraints (2.17) acquire additional terms proportional to $k_i\mathrm{d}t$

$$\tilde{\eta}_i \equiv \eta_i - k_i \mathrm{d}t = 0.$$

Unfortunately, a direct check of Frobenius' condition (2.6) using the calculus of forms is impractical. Instead, we evaluate the resulting eight-forms (2.6) on all possible distinct ordered sets of eight vectors $\{e_j\}$ taken from the canonical basis $\{\partial/\partial A_{ai}, \partial/\partial t\}$ dual to $\{\mathrm{d}A_{ai}, \mathrm{d}t\}$. Expanding the differentials of the constraints in terms of the basic one-forms $\xi_j \in \{\mathrm{d}A_{ai}, \mathrm{d}t\}$,

$$\mathrm{d}\omega_a = \Omega_{aij}\mathrm{d}\xi_i \wedge \mathrm{d}\xi_j \qquad \text{with } i < j$$

(which is unique once an ordering of basis elements has been chosen) and similarly for $\mathrm{d}\tilde{\eta}_i$, these eight-forms are given as sums of wedge products of one-forms. Because for arbitrary one-forms $\{\xi_j\}$ and vectors $\{e_j\}$ (not necessarily basis elements) one has the formula (cf. ref. [11], §7.E)

$$(\xi_1 \wedge \ldots \wedge \xi_n)(e_1, \ldots, e_n) = \det \begin{pmatrix} \xi_1(e_1) & \xi_2(e_1) & \cdots & \xi_n(e_1) \\ \xi_1(e_2) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \xi_1(e_n) & \cdots & \cdots & \xi_n(e_n) \end{pmatrix},$$

$\star$ See ref. [10] as quoted in ref. [6], pp.624–627, for another example, in which for the motion of a car subjected to the constraints of "steer" and "drive" the configuration space is not reduced, but the constraints "steer", "drive" and "wriggle" are completely integrable.

the calculation is thus reduced* to the computation of sums of determinants which we found to be non-zero in most cases. This implies the *non*-holonomy of the constraints (2.16) and (2.17).

One might hope that they at least *partially* reduce the configuration space, but we could show that this is also ruled out. As explained by Villarroel [5] one can use Frobenius' theorem in dual language to find the dimensionality of the physically accessible space: It is given by the number of dimensions r of the algebra generated by the vector fields which are annihilated by all the constraints (with the commutator as the algebra product). The integral manifold tangent to the vector fields in this algebra is the reduced configuration space.

In our case, the initial configuration space is $\{A_{ai}, t\} = \mathbb{R}^{10}$, so we first have to find $10 - 6 = 4$ independent vector fields which are annihilated by the 6 constraints (2.16) and (2.17). Expressed in the coordinates of the singular value decomposition, three such vector fields are the $\partial/\partial x_i$ dual to the one-forms $\mathrm{d}x_i$. Introducing the basis $\{e_{\omega_a}, e_{\eta_i}\}$ in the space of vector fields orthogonal to $\partial/\partial x_i$ as well as $\partial/\partial t$ and dual to the constraint one-forms $\{\omega_a, \eta_i\}$, a fourth vector field is given by

$$u \equiv k_i e_{\eta_i} + \partial/\partial t .$$

The coefficients of this last vector field with respect to the basis dual to the one-forms $\{\mathrm{d}\varphi_i, \mathrm{d}x_i, \mathrm{d}\theta_i, \mathrm{d}t\}$ are rather complicated and could only be evaluated with the help of the computer. Due to the simplicity of the first three vector fields it was nevertheless possible to calculate the twelve commutators

$$v_i \equiv [\partial/\partial x_i, u], \qquad w_{3(i-1)+j} \equiv [\partial/\partial x_i, v_j] .$$

We determined the dimensionality of the subspace of vector fields orthogonal to $\partial/\partial x_i$ spanned by u and the commutators v_i and $w_{3(i-1)+j}$ by a numerical singular value decomposition** and invariably found the rank to be maximal $(10 - 3 = 7)$ thereby proving that the constraints (2.16) and (2.17) are *completely* non-holonomic***.

In summary, we reached the following conclusions for the gauge group SU(2): In the general case, where the system is subjected to the Gauss' law constraints $n_a = 0$ and any values of the spatial angular momenta $m_i = k_i \neq 0$ are admitted, the configuration space keeps on being nine-dimensional. Nevertheless, in the especially interesting case of vanishing spatial angular momenta $k_i = 0$, the constraints are holonomic. The singular value decomposition is the appropriate transformation for the fields $A_{ai}(t)$ leading to constant Euler

* Our analysis was performed with the help of the algebra program REDUCE [12].
** We used the F02WEF-routine out of the NAG Fortran Library [13].
*** This especially implies the complete non-holonomy of the Gauss' law constraints in isolation and therefore corroborates Villarroel's result.

angles φ_n and θ_n. Consequently, the dynamics is completely determined by the effective lagrangian (2.15) which has been studied rather extensively in the literature [1–3].

We will now extend our investigations to the gauge group SU(3) relevant for the description of hadronic physics. The configuration space has 24 dimensions, Gauss' law leads to eight constraint equations (2.4) and in addition there are again three conserved spatial angular momenta (2.3). The singular value decomposition (2.12) of the 8×3 matrix $A = (A_{ai})$ consists of two orthogonal matrices $O \in \mathrm{SO}(8)$ and $R \in \mathrm{SO}(3)$ as well as an 8×3 matrix $\hat{A} = (\delta_{an} x_{[n]})$. However, because in general it is not possible to interpret an arbitrary element $O \in \mathrm{SO}(8)$ as the representation of a gauge transformation $U \in \mathrm{SU}(3)$ (cf. ref. [14]) this decomposition does not allow a direct identification of gauge and rotational degrees of freedom as in the SU(2) case. Obviously, a straightforward generalization of the previous results to SU(3) YMCM is not possible.

To analyse the holonomy properties of the constraints we checked Frobenius' condition (2.6) for the Gauss' law $n_a = 0$ in isolation as well as in combination with the special angular momentum constraints $m_i = 0$ by evaluating the corresponding 10- and 13-forms, respectively. Employing the computer as described above we found non-vanishing determinants in both cases thereby proving that analogous to the case of SU(2) YMCM Gauss' law represents *non*-holonomic constraints. But in contrast to the SU(2) case even in the subspace of vanishing spatial angular momenta no degrees of freedom freeze out because the constraints *stay* non-holonomic.

As a direct consequence of this last result, the configuration space of SU(3) YMCM cannot be reduced to a 13-dimensional submanifold and a distinguished unconstrained subsystem similar to (2.15) does not exist. Furthermore, for arbitrary gauge groups there is no decomposition of the configuration manifold of YMCM as claimed by Asatryan and Savvidy [7]. On the contrary, the obvious lack of such a decomposition gives rise to a much more complicated dynamical structure of SU(3) YMCM as compared to the SU(2) case, where an appropriate coordinate transformation is available via the singular value decomposition.

Hence, we found that for SU(2) YMCM with non-vanishing spatial angular momenta $m_i = k_i \neq 0$ as well as for SU(3) YMCM the constraints are non-holonomic. Even though the motion is restricted to some subspace of the tangent bundle TM a reduction of the configuration space M does not take place.

To nevertheless eliminate the unphysical degrees of freedom and obtain a description without constraints for these cases, we have to widen the scope of our analysis. Passing to a description in phase space Γ, which is the cotangent bundle T^*M of the configuration space M, the formalism for constrained hamiltonian systems invented by Dirac [8] (and elaborated by many others; see the monographs [15–17] and references cited therein) allows to isolate canonical pairs of variables which are no longer subject to constraints. They span the *reduced* phase space Γ_{R} and represent the truly dynamical degrees of freedom. Before carrying out such a reduction we will briefly review Dirac's constraint formalism in sect. 3.

3. Constraint formalism in phase space

In this section we briefly collect the necessary prerequisites for the derivation of an unconstrained description of SU(2) and SU(3) YMCM. In subsection 3.1 we review the vocabulary of primary and secondary, first and second-class constraints introduced by Dirac [8]. In subsection 3.2 we state and explain a couple of theorems taken from the book of Gitman and Tyutin [17], Ch. 1 and 2, which we will apply in the later sections.

The reader familiar with the subject may wish to skip this section and use it only for reference in the later parts of this work.

3.1. HAMILTONIAN SYSTEMS WITH CONSTRAINTS

We consider classical dynamical systems whose configurations are characterized by a set of generalized coordinates $\{q_a, a = 1, \ldots, n\}$. The lagrangian $L(q, \dot{q})$ determines the dynamics through the equations of motion. Defining canonical momenta $p_a \equiv \partial L / \partial \dot{q}_a$ as usual, one may pass to the hamiltonian description via the Legendre transformation

$$H(q,p) \equiv p_a \dot{q}_a(q,p) - L(q, \dot{q}(q,p)), \tag{3.1}$$

unless $p = \partial L / \partial \dot{q}$ cannot be solved with respect to $\dot{q}$.

This is only a special case of the general phenomenon that some coordinates and momenta may not be independent. The motion in *phase space* Γ spanned by $\eta \equiv (q,p)$ is then restricted by constraints

$$\Psi_\alpha : T^\star M \equiv \Gamma \to \mathbb{R}, \qquad \alpha = 1, \ldots, r < 2n$$

to the $(2n - r)$-dimensional constraint hypersurface $\Gamma_{\mathrm{C}} \subset \Gamma$ defined through the conditions

$$\Psi_\alpha(q,p) = 0 \qquad \text{for all } \alpha. \tag{3.2}$$

Within the framework of variational calculus one then proves the existence of r so-called lagrangian multipliers $\lambda_\alpha(t)$ such that the equations of motion are obtained by varying the *extended* action

$$S_{\mathrm{E}}(q,p;\lambda) \equiv \int \big(p_a \dot{q}_a - H(q,p) - \lambda_\alpha \Psi_\alpha(q,p)\big)\, \mathrm{d}t, \tag{3.3}$$

independently with respect to q_a, p_a and λ_α. Defining the *extended* hamiltonian

$$H_{\mathrm{E}}(q,p;\lambda) \equiv H(q,p) + \lambda_\alpha \Psi_\alpha(q,p)\,, \tag{3.4}$$

they take on the canonical form

$$\dot{\eta}_a = \{\eta_a, H_E(\eta;\lambda)\} \tag{3.5}$$

supplemented by the constraint equations

$$\Psi_\alpha(\eta) = 0\,. \tag{3.6}$$

So in the presence of the constraints (3.2) it is still possible to use the canonical formalism if one passes from the initial hamiltonian (3.1) to the extended hamiltonian (3.4) which explicitly depends on these constraints.

For sake of consistency one must also require that the time development on Γ_{C} respects the constraints, i.e.

$$\{\Psi_\alpha(q,p), H_E(q,p)\}|_{\Gamma_{\mathrm{C}}} = 0 \qquad \text{for all } \alpha\,.$$

These relations may lead to a certain number of independent new constraints called *secondary* in contrast to the *primary* constraints (3.2). Also the secondary constraints must be preserved in time and may eventually imply further restrictions on the dynamics. One has to proceed in this fashion until all constraints are generated. As we will see below, however, in the cases considered by us *no* secondary constraints arise. We will therefore henceforth assume that the r functions Ψ_α comprise all constraints present in the theory.

We now give some useful *definitions*:

(1) Two sets of constraints Ψ and Φ are called *equivalent* if there is a matrix $\Lambda(\eta)$ such that

$$\Psi_\alpha = \Lambda_{\alpha\beta}(\eta)\,\Phi_\beta\,, \qquad \det \Lambda(\eta)|_{\Phi=0} \neq 0\,.$$

Equivalent constraints restrict the motion to the same hypersurface Γ_{C} and thereby have the same implications on the dynamics. Furthermore, the equations of motion (3.5) and (3.6) are invariant under a transformation to equivalent constraints (cf. ref. [17]).

(2) For two quantities $A(\eta)$ and $B(\eta)$ coinciding on the hypersurface Γ_{C} one often writes

$$A(\eta) \simeq B(\eta)\,.$$

The symbol $\simeq$ therefore means equality up to terms which vanish when $\Psi = 0$.

(3) A quantity $A(q,p)$ is called *first class* if

$$\{A, \Psi_\alpha\} \simeq 0 \qquad \text{for all } \alpha = 1,\ldots,r$$

and *second-class* otherwise.

This distinction in particular applies to the constraints themselves. Thus a set of constraints $\{\Psi_\alpha, \alpha = 1,\ldots,r\}$ is second class iff the matrix

$$M_{\alpha\beta}(\eta) \equiv \{\Psi_\alpha(\eta), \Psi_\beta(\eta)\} \tag{3.7}$$

built from the Poisson brackets of these constraints is non-singular on the hypersurface Γ_C:

$$\det(M(\eta)) \not\simeq 0\,.$$

Because a non-singular anti-symmetric matrix always has even rank the number of constraints in a set of second-class constraints is necessarily even.

On the contrary, if the matrix (3.7) is singular on Γ_C with defect

$$t \equiv r - \operatorname{rank}(M)|_{\Psi=0} \equiv r - s\,,$$

then the theory contains t independent linear combinations of the initial constraints which are first class. One can show constructively [17] that for any set of constraints Ψ there is an equivalent set consisting of t first-class constraints $\Psi^{(1)}$ and s second-class constraints $\Psi^{(2)}$.

Now let us come back to the extended hamiltonian system (3.4). Of course it would be desirable to eliminate the lagrangian multipliers $\{\lambda_\alpha, \alpha = 1,\dots,r\}$ from the equations of motion (3.5). To this end we consider the conditions resulting from the requirement that the constraints be conserved in time,

$$\dot{\Psi}_\alpha = \{\Psi_\alpha, H\} + \sum_{\beta=1}^{r}\{\Psi_\alpha, \Psi_\beta\}\lambda_\beta \simeq 0\,, \qquad \alpha = 1,\dots,r\,. \tag{3.8}$$

Out of the r λ_α's, there are s corresponding to the second-class constraints $\{\Psi^{(2)}_\sigma, \sigma = 1,\dots,s\}$. These can be determined as functions of η,

$$\lambda_{t+\sigma}(\eta) \simeq -\sum_{\mu=1}^{s}\Big(\{\Psi^{(2)}(\eta), \Psi^{(2)}(\eta)\}\Big)^{-1}_{\sigma\mu}\{\Psi^{(2)}_\mu(\eta), H(\eta)\}\,, \qquad \sigma = 1,\dots,s\,,$$

and may be substituted directly into H_E.

The remaining functions $\{\lambda_\tau, \tau = 1,\dots,t\}$, however, which correspond to the t first-class constraints $\Psi^{(1)}_\tau$ where $\{\Psi^{(1)}, \Psi^{(1)}\} \simeq 0$, enter the equations of motion as arbitrary parameters:

$$\begin{aligned} \dot{\eta}_a &= \{\eta_a, H^{(2)}_E(\eta;\lambda_\tau)\}\,, \\ \Psi_\alpha(\eta) &= 0\,, \qquad \alpha = 1,2,\dots,r\,, \end{aligned} \tag{3.9}$$

where

$$H^{(2)}_E(\eta;\lambda_\tau) \equiv H(\eta) + \sum_{\tau=1}^{t}\lambda_\tau\Psi^{(1)}_\tau - \sum_{\sigma,\mu=1}^{s}\Big(\{\Psi^{(2)}, \Psi^{(2)}\}\Big)^{-1}_{\sigma\mu}\{\Psi^{(2)}_\mu, H(\eta)\}\Psi^{(2)}_\sigma\,.$$

3.2. THE REDUCED PHASE SPACE

In this subsection we present two theorems and lemmas which clarify the dynamical situation in hamiltonian systems with constraints and allow the identification of the truly physical degrees of freedom.

Theorem 3.1. Let $\{\Psi_\alpha, \alpha = 1, \ldots, r\}$ be a set of independent second-class constraints,

$$\det \| \{\Psi_\alpha, \Psi_\beta\} \|_{\Psi=0} \neq 0 .$$

Then canonical variables (ω, Ω) consisting of two separate sets of pairs of canonically conjugate coordinates and momenta exist such that the variables Ω are equivalent to the constraints Ψ.

For a detailed proof see ref. [17]. In general, theorem 3.1 only holds locally, in the vicinity of a generic point $\eta_0 \in \Gamma_C$. Of course, this does not rule out the existence of an extension to the whole phase space in special cases.

In the new variables the constraint hypersurface Γ_C is determined by the equations $\Omega = 0$ and the pairs ω can be viewed as canonical coordinates and momenta on Γ_C. Making use of the invariance of the equations of motion (3.5) and (3.6) under the replacement of the constraints Ψ by equivalent ones, they read after the canonical transformation $\eta \mapsto (\omega, \Omega)$

$$\dot{\omega} = \{\omega, H_R\}, \tag{3.10}$$

$$\Omega = 0, \tag{3.11}$$

where $H_R(\omega) \equiv \tilde{H}(\omega, \Omega)|_{\Omega=0}$ and $\tilde{H}(\omega, \Omega) \equiv H(\eta)$.

As the time derivative of the constraints (3.11) also vanishes on Γ_C, it follows that only the variables ω are dynamical. Consequently, those represent the truly physical coordinates and momenta and parametrize the *reduced phase space* $\Gamma_R = \Gamma_C$. Their dynamics is described by ordinary hamiltonian equations of motion (3.10) with the *reduced hamiltonian* $H_R(\omega)$.

In general, the problem of actually finding a canonical transformation to the distinguished variables (ω, Ω) described in theorem 3.1 is highly non-trivial. Fortunately, there is a

Lemma 3.1. Consider a classical hamiltonian system subject to a set of r second-class constraints. If one can carry out a canonical transformation $\eta \mapsto (\eta^\star, \eta_\star) = (q^\star, p^\star; q_\star, p_\star)$ such that the initial constraint equations $\Psi = 0$ assume the form

$$q_\star = 0, p_\star = f(\eta^\star) \tag{3.12}$$

with a corresponding function f, then the dynamics of the variables $\eta^\star$ is determined by the ordinary hamiltonian equations of motion

$$\dot{\eta}^\star = \{\eta^\star, H_R(\eta^\star)\} \qquad \text{where} \quad H_R(\eta^\star) \equiv H(\eta^\star, \eta_\star)|_{q_\star=0, p_\star=f(\eta^\star)} .$$

Therefore on the constraint hypersurface the variables $\eta^\star$ coincide with some variables ω from theorem 3.1, i.e. on $\Gamma_R = \Gamma_C$ there exists a canonical transformation from $\eta^\star$ to ω, and the reduced hamiltonian can be constructed by simply eliminating the unphysical degrees of freedom $\eta_\star$ from H using the constraints in the special form (3.12).

Generalizing theorem 3.1 one can also prove the (local) existence of distinguished variables for theories with first-class constraints.

Theorem 3.2. Let $\{\Psi_\alpha, \alpha = 1, \dots, r\}$ be a set of independent constraints where

$$\text{rank} \parallel \{\Psi_\alpha, \Psi_\beta\} \parallel_{\Psi=0} = s < r .$$

Then there are canonical variables $(\omega, (Q, P), \varphi)$ such that

(1) ω is a set of pairs of canonically conjugate coordinates and momenta;

(2) P are momenta canonically conjugate to the coordinates Q and φ is a set of pairs of canonically conjugate variables;

(3) $\Omega \equiv (P, \varphi)$ is equivalent to the constraints Ψ, the number of momenta in P equals $t = r - s$ and the pairs φ constitute s coordinates and momenta.

Obviously, the momenta $\{P_\tau, \tau = 1, \dots, t\}$ represent the t first-class constraints of the theory and the s second-class constraints are transformed to canonical pairs $\{\varphi_{t+\sigma}, \sigma = 1, \dots, s/2\}$ as in theorem 3.1 (recall that s is always even as was explained just after eq. (3.7)).

One finds that the equations of motion (3.5) and (3.6) read in the new variables

$$\dot{\omega} = \{\omega, \tilde{H}_{\mathrm{E}}\}, \tag{3.13}$$

$$\dot{Q} = \{Q, \tilde{H}_{\mathrm{E}}\}, \tag{3.14}$$

$$\Omega = 0,$$

where

$$\tilde{H}_{\mathrm{E}} \equiv H_{\mathrm{R}} + \sum_{\tau=1}^{t} \lambda_\tau P_\tau + \sum_{\sigma=1}^{s} \lambda_{t+\sigma} \varphi_\sigma , \tag{3.15}$$

$$H_{\mathrm{R}}(\omega, Q) \equiv \tilde{H}(\omega, Q, \Omega)|_{\Omega=0}, \quad \tilde{H}(\omega, Q, \Omega) \equiv H(\eta) .$$

By construction H_{R} depends on ω and Q, but the conservation of the constraints P in time implies

$$\dot{P} = \{P, \tilde{H}_{\mathrm{E}}\} = -\frac{\partial H_{\mathrm{R}}}{\partial Q} \simeq 0$$

and thus

$$H_{\mathrm{R}} = H_{\mathrm{R}}(\omega) . \tag{3.16}$$

Hence, the dynamical equations (3.13) for the variables ω are canonical with the hamiltonian H_{R} and independent of the other coordinates and momenta:

$$\dot{\omega} = \{\omega, H_{\mathrm{R}}(\omega)\} . \tag{3.17}$$

Furthermore, recalling (3.15) as well as (3.16), it follows from eq. (3.14) that

$$\dot{Q}_\tau(t) = \frac{\partial \tilde{H}_{\mathrm{E}}}{\partial P_\tau} = \lambda_\tau(t) \qquad \text{for all } \tau = 1, 2, \dots, t . \tag{3.18}$$

As a consequence, the equations for the variables Q_τ have solutions for any functions λ_τ, i.e. the lagrangian multipliers $\{\lambda_\tau, \tau = 1,\dots,t\}$ corresponding to the t first-class constraints of the theory are not determined by the complete set of equations of motion either (cf. the discussion following eq. (3.8)).

Due to this arbitrariness, the motion on the constraint hypersurface Γ_C is *not* completely determined in theories with first class constraints. Depending on the choice of the t arbitrary functions λ_τ, equal initial conditions η^0 lead to different trajectories in Γ_C.

Such a degeneracy, whose extent is given by the number of the first class constraints $\Psi_\tau^{(1)}$, is unphysical. One therefore identifies these trajectories, which in turn leads to a division of Γ_C into equivalence classes . By choosing exactly one representative out of each equivalence class the $(2n-2t-s)$-dimensional reduced phase space Γ_R is obtained.

After having achieved a formulation of the theory in terms of the distinguished variables (ω, Q, Ω), equivalent points on the constraint hypersurface Γ_C due to (3.18) only differ in the coordinates Q conjugate to the first-class constraints P. A choice of representatives is achieved by t independent conditions

$$Q_\tau = Y_\tau(\omega), \qquad \tau = 1,\dots,t. \tag{3.19}$$

One usually demands continuity for the functions $Y_\nu(\omega)$ which are arbitrary otherwise.

Since the dynamical equations for the variables ω (3.17) as well as the constraint equations $\Omega = 0$ remain unaffected by the above choice (3.19), all physical information is contained in the canonical equations (3.17). Therefore the reduced phase space Γ_R is parametrized by the canonical pairs ω and we identify $H_R(\omega)$ as the reduced hamiltonian.

On the other hand, if a transformation to the preferred set of variables described in theorem 3.2 cannot be achieved, the reduced phase space cannot be isolated. To formulate the theory consistently it is nevertheless necessary to lift its unphysical degeneracy. This may be achieved by choosing an element from each gauge class implicitly introducing additional constraints on the phase space variables, a procedure called *gauge fixing.*

Lemma 3.2. A classical hamiltonian system subject to a set of t first-class and s second-class constraints $\Psi = (\Psi^{(1)}, \Psi^{(2)})$ is consistently gauge fixed by t supplementary conditions on the phase space variables $\chi(\eta) = 0$ with the property that for $\Phi \equiv (\Psi, \chi)$

(1) rank $(\partial\Phi/\partial\eta)|_{\Phi=0} = r$,

(2) det $\| \{\Phi_\alpha, \Phi_\beta\} \|_{\Phi=0} \neq 0$ or equivalently det $\| \{\Psi_\mu^{(1)}, \chi_\tau\} \|_{\Phi=0} \neq 0$.

So the gauge-fixing constraints χ have to be chosen in such a way that the whole set of constraints Φ is second class. Then the undetermined parameters still inherent in the equations of motion (3.9) are fixed through the consistency requirements for the supplementary conditions and the degeneracy of the theory is lifted. Because all the constraints are second class no further restrictions on the phase space variables are generated.

A consistent gauge fixing means to pass from the description of a physical system in terms of a t-fold degenerate theory to a description in terms of a physically equivalent non-degenerate theory with $(2t+s)$ second-class constraints. In general, such a formulation does not provide a representation for the reduced phase space either. However, making use of theorem 3.1 or lemma 3.1, respectively, one can again try to isolate the unphysical variables and construct the reduced hamiltonian H_{R}.

4. The reduced hamiltonian of SU(2) YMCM

The phase space variables of SU(2) YMCM are the canonical pairs A_{ai} and $E_{ai} \equiv \dot{A}_{ai}$ which will be regarded as components of 3×3 matrices $A=(A_{ai})$ and $E=(E_{ai})$, respectively. The hamiltonian reads

$$\begin{aligned} H &= \tfrac{1}{2}E_{ai}E_{ai} + \tfrac{1}{4}\varepsilon_{abe}\varepsilon_{ecd}\,A_{ai}A_{bj}A_{ci}A_{dj} \\ &= \tfrac{1}{2}\mathrm{Tr}\,(EE^{\mathrm{T}}) + \tfrac{1}{4}\left\{\left[\mathrm{Tr}(AA^{\mathrm{T}})\right]^2 - \mathrm{Tr}\left[(AA^{\mathrm{T}})^2\right]\right\}, \qquad (4.1)\end{aligned}$$

and as phase space analogs of the completely non-holonomic constraints (2.16) and (2.17) for arbitrary spatial angular momenta we get

$$n_a = \varepsilon_{abc}A_{bi}E_{ci} = 0, \qquad (4.2)$$

$$\tilde{m}_i = m_i - k_i = \varepsilon_{ijk}A_{aj}E_{ak} - k_i = 0. \qquad (4.3)$$

Without loss of generality we choose $\boldsymbol{k}$ to point into the x_1 direction, i.e. $\boldsymbol{k}=(k,0,0)$, which simplifies the further calculations considerably.

Because the constraints arise from constants of motion, they are invariant under the time evolution. Therefore no secondary constraints emerge and the hypersurface $\Gamma_{\mathrm{C}} \subset \Gamma$ is completely determined by (4.2) and (4.3). As

$$\begin{aligned} \{n_a, n_b\} &= \varepsilon_{abc}n_c \simeq 0, \\ \{n_a, \tilde{m}_i\} &= 0, \\ \{\tilde{m}_1, \tilde{m}_2\} &= -\tilde{m}_3 \simeq 0, \\ \{\tilde{m}_1, \tilde{m}_3\} &= \tilde{m}_2 \simeq 0, \\ \{\tilde{m}_2, \tilde{m}_3\} &= -m_1 = \tilde{m}_1 - k \simeq -k, \end{aligned}$$

there are four first-class constraints $(n_a, \tilde{m}_1)$ and two second-class constraints $(\tilde{m}_2, \tilde{m}_3)$.

Our aim in the following will be to replace (4.1) by a reduced hamiltonian (eq. (4.24) below) in the truly dynamical degrees of freedom through a transformation to a preferred set of canonical variables as described in theorem

3.2 of sect. 3. Such a description of SU(2) YMCM without any constraints is attained in two steps.

First of all we eliminate the gauge degrees of freedom using once more the point transformation (2.12):

$$A(\boldsymbol{\theta}^I, \boldsymbol{x}, \boldsymbol{\theta}^J) = O(\boldsymbol{\theta}^I)\hat{A}(\boldsymbol{x})R^{\mathrm{T}}(\boldsymbol{\theta}^J). \tag{4.4}$$

The canonical momenta $v \equiv (\boldsymbol{l}^I, \boldsymbol{p}, \boldsymbol{l}^J)$ conjugate to the coordinates $y \equiv (\boldsymbol{\theta}^I, \boldsymbol{x}, \boldsymbol{\theta}^J)$ can be constructed using the generating function

$$F_3(y;E) \equiv -\sum_{a,i} A_{ai}(y)E_{ai} = -\sum_{a,i} O_{ab}(\boldsymbol{\theta}^I)\hat{A}_{bj}(\boldsymbol{x})\,R_{ij}(\boldsymbol{\theta}^J)E_{ai}.$$

After some algebra we find the expression for the old momenta E as functions of y and v,

$$E(y,v) = O(\boldsymbol{\theta}^I)\hat{E}(y,v)R^{\mathrm{T}}(\boldsymbol{\theta}^J), \tag{4.5}$$

where $\hat{E}$ is given by

$$\hat{E}_{ai} = \delta_{ai}p_{[i]} + X^n_{ai}(M^I)^{-1}_{nm}l^I_m + Y^n_{ai}(M^J)^{-1}_{nm}l^J_m \tag{4.6}$$

and the remaining quantities are defined as in ref. [18]:

$$X^n_{ai}(\boldsymbol{x}) \equiv S^n_{ai}x_{[i]}, \qquad Y^n_{ai}(\boldsymbol{x}) \equiv S^n_{ai}x_{[a]},$$
$$S^n_{ai}(\boldsymbol{x}) \equiv \varepsilon_{nai}\left(-x^2_{[a]} + x^2_{[i]}\right)^{-1},$$

i.e. the only non-vanishing components of S are

$$S^1_{23} = S^1_{32} = \frac{1}{-x_2^2 + x_3^2},$$
$$S^2_{13} = S^2_{31} = \frac{1}{-x_3^2 + x_1^2},$$
$$S^3_{12} = S^3_{21} = \frac{1}{-x_1^2 + x_2^2},$$
$$M^I_{nm}(\boldsymbol{\theta}^I) \equiv \tfrac{1}{2}\varepsilon_{mcd}O^{\mathrm{T}}_{ca}\frac{\partial O_{ad}}{\partial \theta^I_n},$$
$$M^J_{nm}(\boldsymbol{\theta}^J) \equiv \tfrac{1}{2}\varepsilon_{mkl}R^{\mathrm{T}}_{ks}\frac{\partial R_{sl}}{\partial \theta^J_n}.$$

Inserting (4.4) as well as (4.5) into (4.1) and using the orthogonality of the matrices O and R, respectively, we get the hamiltonian in the new variables,

$$H(y,v) = T + \tfrac{1}{2}\left(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2\right), \tag{4.7}$$

where the kinetic term T explicitly reads

$$\begin{aligned} T &\equiv \tfrac{1}{2}\hat{E}_{ai}\,\hat{E}_{ai} \\ &= \tfrac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \tfrac{1}{2}\big(X^n_{ai}X^r_{ai}(M^I)^{-1}_{nm}(M^I)^{-1}_{rs}l^I_m l^I_s \\ &\qquad + 2X^n_{ai}Y^r_{ai}(M^I)^{-1}_{nm}(M^J)^{-1}_{rs}l^I_m l^J_s \\ &\qquad + Y^n_{ai}Y^r_{ai}(M^J)^{-1}_{nm}(M^J)^{-1}_{rs}l^J_m l^J_s\big)\,. \end{aligned}$$

Furthermore, making use of $\varepsilon_{abc}O_{a\alpha}O_{b\beta}O_{c\gamma} = \varepsilon_{\alpha\beta\gamma}$ for $O \in \mathrm{O}(3)$ and $\varepsilon_{\mu\alpha\beta}S^n_{\alpha\beta}x^2_{[\alpha]} = \delta_{\mu n}$, the conserved angular momenta n_a and m_i can now be written as

$$n_a(\boldsymbol{\theta}^I, \boldsymbol{l}^I) = O_{an}(M^I)^{-1}_{nm}l^I_m\,, \tag{4.8}$$

$$m_i(\boldsymbol{\theta}^J, \boldsymbol{l}^J) = R_{in}(M^J)^{-1}_{nm}l^J_m\,. \tag{4.9}$$

The identity (4.8) shows that the Gauss' law constraints $n_a = 0$ are equivalent to $l^I_a = 0$. According to theorem 3.2 the canonical variables $\boldsymbol{\theta}^I$ and $\boldsymbol{l}^I$ are therefore unphysical and gauge-invariant quantities are independent of them.

After the identification of these non-dynamical degrees of freedom, the partially reduced phase space is parametrized by the canonical pairs $(\boldsymbol{x},\boldsymbol{p};\boldsymbol{\theta}^J \equiv \boldsymbol{\theta}, \boldsymbol{l}^J \equiv \boldsymbol{l})$. The dynamics is determined by the partially reduced hamiltonian

$$\begin{aligned} H'_{\mathrm{R}}(\boldsymbol{x},\boldsymbol{p};\boldsymbol{\theta},\boldsymbol{l}) &= H|_{\boldsymbol{l}^I=0} \\ &= \tfrac{1}{2}\left(p_1^2 + p_2^2 + p_3^2 + Y^n_{ai}Y^r_{ai}M^{-1}_{nm}M^{-1}_{rs}l_m l_s\right) \\ &\quad + \tfrac{1}{2}\left(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2\right), \end{aligned} \tag{4.10}$$

and still restricted by the three remaining constraints

$$\tilde{m}_i(\boldsymbol{\theta},\boldsymbol{l}) = R_{in}M^{-1}_{nm}l_m - k\delta_{i1} = 0\,. \tag{4.11}$$

Having eliminated the gauge degrees of freedom one would now like to carry out a canonical transformation such that the two second-class spatial angular momentum constraints $\tilde{m}_2$ and $\tilde{m}_3$ are simply related to a canonical coordinate and momentum pair, whereas the first-class spatial angular momentum constraint $\tilde{m}_1$ is equivalent to another canonical momentum. This task also has to be split in several parts.

First we map two of the constraints onto canonical momenta. Even this is not immediately possible because the $\tilde{m}_i$ do not have vanishing Poisson brackets with each other. It is therefore necessary to replace them by equivalent quantities

$$\bar{m}_i \equiv A_{ij}(\boldsymbol{\theta},\boldsymbol{l})\tilde{m}_j\,,$$

where

$$\Lambda \equiv \begin{pmatrix} \tilde{m}_1 + 2k & \tilde{m}_2 & \tilde{m}_3 \\ 0 & k\sin^2\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \det(\Lambda)|_{\tilde{m}_j=0} = 2k^2\sin^2\theta_2 \neq 0 .$$

The factor $k\sin^2\theta_2$ has been inserted for later convenience. Parametrizing the elements $R \in \mathrm{SO}(3)$ by three Euler angles $0 \leqslant \theta_1, \theta_3 \leqslant 2\pi$ and $0 < \theta_2 < \pi$ as usual, the new constraints read explicitly

$$\begin{aligned} \bar{m}_1 &= m_1^2 + m_2^2 + m_3^2 - k^2 \\ &= \frac{1}{\sin^2\theta_2}\left(l_3^2 - 2l_1l_3\cos\theta_2 + l_2^2\sin^2\theta_2 + l_1^2\right) - k^2 , \end{aligned} \tag{4.12}$$

$$\bar{m}_2 = k\sin\theta_2\,(l_3\cos\theta_1 - l_2\sin\theta_1\sin\theta_2 - l_1\cos\theta_1\cos\theta_2) , \tag{4.13}$$

$$\bar{m}_3 = l_1 , \tag{4.14}$$

leading to $\{\bar{m}_1, \bar{m}_2\} = \{\bar{m}_1, \bar{m}_3\} = 0$. Now, performing the canonical transformation $(\boldsymbol{\theta}, \boldsymbol{l}) \mapsto (\boldsymbol{\varphi}, \boldsymbol{\pi})$ generated by

$$\begin{aligned} F_2(\boldsymbol{\theta}, \boldsymbol{\pi}) &\equiv \pi_1\theta_1 + \pi_3\theta_3 \\ &\quad + \int_{\pi/2}^{\theta_2} \frac{1}{\sin\xi}\left(\pi_2^2\sin^2\xi - \pi_3^2 + 2\pi_1\pi_3\cos\xi - \pi_1^2\right)^{1/2} \mathrm{d}\xi , \end{aligned}$$

the momenta π_1 and π_2 in fact correspond to the constraints $\bar{m}_3$ and $\bar{m}_1$, respectively (for more explicit formulae see ref. [19]).

To complete our reduction program we will still have to find the canonical coordinate equivalent to $\bar{m}_2$. Instead of computing the $\varphi_i = \partial F_2/\partial\pi_i$ we immediately proceed with a second transformation defined by

$$\begin{aligned} \Phi_1 &\equiv \varphi_1 - \arcsin\left(\frac{\pi_3}{\sqrt{\pi_2^2 - \pi_1^2}}\right) , \\ \Phi_2 &\equiv \varphi_2 + \arcsin\left(\frac{\pi_1\pi_3}{\sqrt{(\pi_2^2 - \pi_1^2)(\pi_2^2 - \pi_3^2)}}\right) , \\ \Phi_3 &\equiv \varphi_3 - \arcsin\left(\frac{\pi_1}{\sqrt{\pi_2^2 - \pi_3^2}}\right) , \\ \Pi_i &\equiv \pi_i \qquad \text{for } i = 1, 2, 3 . \end{aligned}$$

Obviously

$$\{\Pi_i, \Pi_j\} = 0 , \qquad \{\Phi_i, \Pi_j\} = \delta_{ij} \quad \text{for all } i, j$$

and it can also be verified that (cf. ref. [19])

$$\{\Phi_i, \Phi_j\} = 0 \qquad \text{for all } i, j .$$

Hence $(\boldsymbol{\varphi},\boldsymbol{\pi}) \mapsto (\boldsymbol{\Phi},\boldsymbol{\Pi})$ is indeed canonical and the constraints (4.12)–(4.14) lead to the equations

$$\begin{aligned}
\bar{m}_1 &= \Pi_2^2 - k^2 = 0,\\
\bar{m}_2 &= -\left(\Pi_3^2 \sin^2\Phi_2 + k^2\cos^2\Phi_2\right)\sin\Phi_1 = 0,\\
\bar{m}_3 &= \Pi_1 = 0.
\end{aligned}$$

As a result we have shown that the initial constraints (4.11) after the canonical transformations from $(\boldsymbol{\theta},\boldsymbol{l})$ to $(\boldsymbol{\Phi},\boldsymbol{\Pi})$ can be cast into the form

$$\Pi_1 = 0, \tag{4.15}$$

$$\Phi_1 = 0, \tag{4.16}$$

$$\Pi_2 = k. \tag{4.17}$$

As we pointed out already above, the first two of them, (4.15) and (4.16), which are equivalent to $\bar{m}_3$ and $\bar{m}_2$, are second class and the last constraint (4.17), equivalent to $\bar{m}_1$, is first class. We have thereby explicitly completed the transformation of variables known to exist (locally) according to theorem 3.2.

The residual unphysical degrees of freedom in the only *partially* reduced hamiltonian H'_{R} (4.10) could thus also be isolated and the *completely reduced phase space* Γ_R of SU(2) YMCM is parametrized by the canonical pairs $(\boldsymbol{x},\boldsymbol{p};\Phi_3,\Pi_3)$.

To construct the *completely* reduced hamiltonian H_{R} in these variables we collect the preceding results as follows:

$$\sin\theta_1 = \frac{\Pi_3}{k\sin\theta_2}, \tag{4.18}$$

$$\cos\theta_2 = -\frac{\sqrt{k^2-\Pi_3^2}}{k}\sin\Phi_2, \tag{4.19}$$

$$\theta_3 = \Phi_3 + \arctan\left(\frac{\Pi_3}{k}\tan\Phi_2\right), \tag{4.20}$$

$$l_1 = 0, \tag{4.21}$$

$$l_2 = \frac{\sqrt{k^2-\Pi_3^2}}{\sin\theta_2}\cos\Phi_2, \tag{4.22}$$

$$l_3 = \Pi_3. \tag{4.23}$$

Making use of well-known identities for the trigonometric functions we get the necessary expressions to insert into H'_{R} (4.10). Using REDUCE [12] for the algebraic manipulations and after a change of notation ($\Phi_3 \equiv \varphi, \Pi_3 \equiv p_\varphi$) we finally end up with the completely reduced hamiltonian of SU(2)–YMCM

$$H_{\mathrm{R}}(\boldsymbol{x},\boldsymbol{p};\varphi,p_\varphi) = \tfrac{1}{2}\left(p_1^2+p_2^2+p_3^2+\left(\frac{x_1^2+x_2^2}{(x_1^2-x_2^2)^2}-h(x_1,x_2,x_3,\varphi)\right)p_\varphi^2\right) + \tfrac{1}{2}\left(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2+h(x_1,x_2,x_3,\varphi)k^2\right), \quad (4.24)$$

where

$$h(x_1,x_2,x_3,\varphi) \equiv \frac{x_1^2+x_3^2}{(x_1^2-x_3^2)^2}\sin^2\varphi + \frac{x_2^2+x_3^2}{(x_2^2-x_3^2)^2}\cos^2\varphi\,.$$

The singular points in H_{R} correspond to the singularities of the transformation (4.4) at

$$x_n = \pm x_m \qquad \text{for } n,m=1,2,3\,. \quad (4.25)$$

This is due to the fact that the singular value decomposition (2.12) is one to one only if we restrict the values of the variables x_n to a fundamental domain $\{x_1 \geqslant x_2 \geqslant x_3 \geqslant 0\} \subset \mathbb{R}^3$. *After* such a restriction all the singular points lie on the boundary of this domain. Presumably they are of no direct dynamical significance and the corresponding terms in H_{R} represent a kind of centrifugal barrier preventing the dynamical system to reach the degenerate configurations (4.25) except for special values of φ and p_φ. The analogy to the three-dimensional potential problem in spherical coordinates is conspicuous where the origin constitutes a kinematical singularity (cf. ref. [18]).

In summary, we have achieved the reduction of the nine-dimensional system (4.1) restricted by six constraints (4.2) and (4.3) to an effective unconstrained four-dimensional system. Hence we present a formulation of the *full* SU(2) YMCM in the truly physical degrees of freedom.

The additional terms in (4.24) compared to the lagrangian (2.15) describing the subspace of vanishing spatial angular momenta ($k=0 \Rightarrow \varphi = p_\varphi = 0$) contain the dynamics in the remaining rotational degree of freedom and its influence on the variables x_n. For an investigation of these effects it is natural to start with an analysis of dynamically invariant subsystems. Since all coordinates and momenta enter the hamiltonian (4.24) at least quadratically it follows that

$$\varphi = p_\varphi = 0 \qquad \text{or} \qquad x_i = p_i = 0 \qquad \text{for } i=1,2 \text{ or } 3$$

constitute such invariant subsystems.

In the first case the lagrangian reads

$$L = \tfrac{1}{2}\left(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2\right) - \tfrac{1}{2}\left(x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2+\frac{x_2^2+x_3^2}{(x_2^2-x_3^2)^2}k^2\right).$$

Additionally demanding $x_1 = \dot{x}_1 = 0$ and performing the coordinate transformation

$$x_2 \mapsto \frac{1}{\sqrt{2}}(r_1 + r_2),$$
$$x_3 \mapsto \frac{1}{\sqrt{2}}(r_1 - r_2),$$

we obtain the system

$$L = \tfrac{1}{2}(\dot{r}_1^2 + \dot{r}_2^2) - \left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)k^2 - \tfrac{1}{8}(r_1^2 - r_2^2)^2,$$

already investigated by Frøyland [20] for vanishing colour momenta $L_1 = L_2 = \sqrt{2}k$ (in his terminology). His results – he finds dynamically stable as well as unstable orbits and a system of multifurcations – therefore fit in the context described by eq. (4.24) and hence are of general validity.

For an investigation of the role of the nontrivial part of the metric in the kinetic term one might study the subsystem $x_3 = \dot{x}_3 = 0$:

$$L = \tfrac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 - \frac{x_1^2 x_2^2 (x_1^2 - x_2^2)^2}{x_2^4(3x_1^2 - x_2^2)\sin^2\varphi + x_1^4(3x_2^2 - x_1^2)\cos^2\varphi}\dot{\varphi}^2\right)$$
$$-\tfrac{1}{2}\left(x_1^2 x_2^2 + \left(\frac{\sin^2\varphi}{x_1^2} + \frac{\cos^2\varphi}{x_2^2}\right)k^2\right).$$

All this makes clear that we are still far from understanding SU(2) YMCM outside the subspace of vanishing spatial angular momenta. Having identified the dynamical degrees of freedom and constructed the reduced hamiltonian (4.24) we nevertheless present the necessary prerequisites for a systematic study of this mechanical model of the simplest non-abelian gauge theory.

5. The reduced phase space of SU(3) YMCM

In this section we will try to find a formulation of SU(3) YMCM in the truly dynamical variables. Remembering the results of the constraint analysis in configuration space carried out in sect. 2 and taking into account the enormous complexity of the system, it seems reasonable to concentrate in a first step on the subspace of vanishing spatial angular momenta. The dynamics in the 48-dimensional phase space therefore is restricted by 11 non-holonomic constraints

$$n_a = f_{abc}A_{bi}E_{ci} = 0, \tag{5.1}$$
$$m_i = \varepsilon_{ijk}A_{aj}E_{ak} = 0, \tag{5.2}$$

which are exclusively first-class:

$$\begin{aligned}
\{n_a, n_b\} &= f_{abc} n_c \simeq 0, \\
\{n_a, m_i\} &= 0, \\
\{m_i, m_j\} &= \varepsilon_{ijk} m_k \simeq 0.
\end{aligned}$$

As they all represent constants of motion there are again no secondary constraints.

In contrast to the SU(2) case where the singular value decomposition effects a decomposition of the configuration space in gauge, rotational and physical degrees of freedom, here a comparable transformation to special coordinates reflecting the symmetries of the system could not be found. So there is only little hope to obtain a canonical transformation to distinguished variables described in theorem 3.2, which allow an immediate identification of the reduced phase space.

Therefore we have to lift the degeneracy of the theory by a consistent gauge fixing. In addition to the 11 initial first-class constraints (5.1) and (5.2) we have to find 11 independent supplementary conditions on the phase space variables in such a way that we end up with a set of 22 second-class constraints (cf. lemma 3.2).

Following an idea of Marnelius and Kihlberg [21] we define generators $\{T_a\}$ of the Lie algebra su(3) such that the first three basis elements represent the so-called *minimal embedding* of the subalgebra su(2) in su(3) (cf. refs. [22,23])

$$\begin{aligned}
T_1 &\equiv \tau_2, \quad T_2 \equiv \tau_5, \\
T_3 &\equiv \tau_7, \quad T_4 \equiv \tau_4, \\
T_5 &\equiv \tau_1, \quad T_6 \equiv \tau_6, \\
T_7 &\equiv \tau_3, \quad T_8 \equiv \tau_8,
\end{aligned}$$

where $\tau_a \equiv i\lambda_a/2$ and λ_a are the well-known Gell-Mann matrices. The non-trivial non-vanishing structure constants c_{abc} are listed in table 1. The canonical hamiltonian then reads

$$H = \tfrac{1}{2} E_{ai} E_{ai} + \tfrac{1}{4} c_{abe} c_{ecd} A_{ai} A_{bj} A_{ci} A_{dj} \tag{5.3}$$

and Gauss' law (5.1) takes the form

$$n_a = c_{abc} A_{bi} E_{ci} = 0. \tag{5.4}$$

Because no (non-vanishing) element out of the algebra su(3) commutes with *all* three generators $\{T_1, T_2, T_3\}$, the $8 \times (8 \times 3)$ matrix c_{abi} has maximum rank (8) and the eight conditions

$$\psi_a \equiv \sum_{b=1}^{8} \sum_{i=1}^{3} c_{abi} A_{bi} = 0 \tag{5.5}$$

TABLE 1
Non-vanishing structure constants w.r.t. the basis $\{T_a\}$

a	b	c	c_{abc}
1	2	3	$\frac{1}{2}$
1	4	6	$\frac{1}{2}$
1	5	7	-1
2	4	7	$-\frac{1}{2}$
2	4	8	$-\frac{1}{2}\sqrt{3}$
2	5	6	$\frac{1}{2}$
3	4	5	$-\frac{1}{2}$
3	6	7	$\frac{1}{2}$
3	6	8	$-\frac{1}{2}\sqrt{3}$

are linearly independent. As further supplementary conditions we choose

$$\phi_1 \equiv \tfrac{1}{2}(A_{23} + A_{32}) = 0, \tag{5.6}$$

$$\phi_2 \equiv \tfrac{1}{2}(A_{13} + A_{31}) = 0, \tag{5.7}$$

$$\phi_3 \equiv \tfrac{1}{2}(A_{12} + A_{21}) = 0. \tag{5.8}$$

The Poisson brackets between the gauge-fixing conditions (5.5)–(5.8) and the constraints (5.2) as well as (5.4) form the 11×11 block matrix

$$M \equiv \begin{pmatrix} \{m_i, \phi_j\} & \{m_i, \psi_a\} \\ \{n_a, \phi_i\} & \{n_a, \psi_b\} \end{pmatrix}. \tag{5.9}$$

Just as $\{n_a, \psi_b\} = c_{a\mu\nu}\, c_{b\nu i}\, A_{\mu i}$ the other entries are linear in the coordinates A_{ai}, too, and we find

$$\det(M)|_{n=\psi=m=\phi=0} \neq 0$$

for infinitely many open sets of field configurations. Because the singular surfaces determined by $\det(M)|_{n=\psi=m=\phi=0} = 0$ are at most 10-dimensional hypersurfaces of the 24-dimensional configuration space $\{A_{ai}\}$, the matrix (5.9) is therefore invertible except for sets of measure zero.

Away from these singular surfaces — we will comment on them later — the 11 supplementary conditions $\chi \equiv (\phi, \psi) = 0$ consistently fix the 11 first-class constraints $G \equiv (m, n) = 0$. So we managed to pass from the description of SU(3) YMCM in the subspace of vanishing spatial angular momenta by a degenerate theory to a description by a physically equivalent non-degenerate theory with 22 second-class constraints $(\chi, G) = 0$.

The next step towards the elimination of the non-dynamical degrees of freedom is to find a parametrization of the reduced phase space Γ_R and to construct the effective hamiltonian on Γ_R. To this end we regard the independent quantities χ as new coordinates and extend them to a complete coordinate set $\{q_a, a = 1, \dots, 24\}$ in the following manner:

$$
\begin{aligned}
&q_1 \equiv A_{11}\,, && q_{14} \equiv \phi_1 = \tfrac{1}{2}(A_{23} + A_{32})\,,\\
&q_2 \equiv A_{22}\,, && q_{15} \equiv \phi_2 = \tfrac{1}{2}(A_{13} + A_{31})\,,\\
&q_3 \equiv A_{33}\,, && q_{16} \equiv \phi_3 = \tfrac{1}{2}(A_{12} + A_{21})\,,\\
&q_4 \equiv A_{41}\,, && q_{17} \equiv \psi_1 = \tfrac{1}{2}(A_{23} - A_{32})\,,\\
&q_5 \equiv A_{42}\,, && q_{18} \equiv \psi_2 = \tfrac{1}{2}(-A_{13} + A_{31})\,,\\
&q_6 \equiv A_{43}\,, && q_{19} \equiv \psi_3 = \tfrac{1}{2}(A_{12} - A_{21})\,,\\
&q_7 \equiv A_{52}\,, && q_{20} \equiv \psi_4 = \tfrac{1}{2}(-A_{53} + A_{61} - A_{72} - \sqrt{3}A_{82})\,,\\
&q_8 \equiv A_{53}\,, && q_{21} \equiv \psi_5 = \tfrac{1}{2}(A_{43} + A_{62} - 2A_{71})\,,\\
&q_9 \equiv A_{61}\,, && q_{22} \equiv \psi_6 = \tfrac{1}{2}(-A_{41} - A_{52} + A_{73} - \sqrt{3}A_{83})\,,\\
&q_{10} \equiv A_{62}\,, && q_{23} \equiv \psi_7 = \tfrac{1}{2}(A_{42} + 2A_{51} - A_{63})\,,\\
&q_{11} \equiv A_{72}\,, && q_{24} \equiv \psi_8 = \tfrac{1}{2}\sqrt{3}\,(A_{42} + A_{63})\,,\\
&q_{12} \equiv A_{73}\,, && \\
&q_{13} \equiv A_{81}\,. &&
\end{aligned}
\tag{5.10}
$$

The canonically conjugate momenta $\{p_a, a = 1, \dots, 24\}$ are

$$
\begin{aligned}
&p_1 = E_{11}\,, && p_{14} = E_{23} + E_{32}\,,\\
&p_2 = E_{22}\,, && p_{15} = E_{13} + E_{31}\\
&p_3 = E_{33}\,, && p_{16} = E_{12} + E_{21}\,,\\
&p_4 = E_{41} - \frac{1}{\sqrt{3}}E_{83}\,, && p_{17} = E_{23} - E_{32}\,,\\
&p_5 = E_{42} - E_{51} - E_{63}\,, && p_{18} = -E_{13} + E_{31}\,,\\
&p_6 = E_{43} + \tfrac{1}{2}E_{71}\,, && p_{19} = E_{12} - E_{21}\,,\\
&p_7 = E_{52} - \frac{1}{\sqrt{3}}E_{83}\,, && p_{20} = -\frac{2}{\sqrt{3}}E_{82}\,,\\
&p_8 = E_{53} - \frac{1}{\sqrt{3}}E_{82}\,, && p_{21} = -E_{71}\,,\\
&p_9 = E_{61} + \frac{1}{\sqrt{3}}E_{82}\,, && p_{22} = -\frac{2}{\sqrt{3}}E_{83}\,,\\
&p_{10} = E_{62} + \tfrac{1}{2}E_{71}\,, && p_{23} = E_{51}\,,
\end{aligned}
$$

$$p_{11} = E_{72} - \frac{1}{\sqrt{3}}E_{82}, \qquad p_{24} = \frac{2}{\sqrt{3}}(E_{51} + 2E_{63}),$$
$$p_{12} = E_{73} + \frac{1}{\sqrt{3}}E_{83},$$
$$p_{13} = E_{81}.$$

Inserting the inverse relations $A_{ai}(q)$ and $E_{ai}(p)$ into (5.2) and (5.4) we get the constraint equations $G_a(q,p) = 0$ in these new variables. They are linear in p and thus may be written as $(a = 1,\dots,11)$

$$G_a(q,p) = \sum_{b=1}^{11} N_{ab}(q)p_{b+13} + \sum_{b=1}^{13} R_{ab}(q)p_b = 0, \tag{5.11}$$

with a corresponding 11×11 matrix N and an 11×13 matrix R, respectively. Furthermore, due to the invariance of the Poisson brackets under canonical transformations we get for the block matrix (5.9)

$$M_{ab}(q,p) = \{G_a(q,p),\chi_b\} = \{G_a(q,p),q_{b+13}\} = -\frac{\partial G_a(q,p)}{\partial p_{b+13}}, \tag{5.12}$$

hence

$$M_{ab}(q) = -N_{ab}(q).$$

Therefore the 22 second-class constraints $(\chi,G) = 0$ can be solved for the variables $(a = 1,\dots,11)$

$$q_{a+13} = 0,$$
$$p_{a+13} = \sum_{b=1}^{13} \left(M^{-1}R\right)_{ab} p_b \equiv f_a(q_1,\dots,q_{13};p_1,\dots,p_{13}). \tag{5.13}$$

Following lemma 3.1 we can now identify and isolate $\{(q_a,p_a), a = 14,\dots,24\}$ as the *unphysical* coordinates and momenta of the theory. On the other hand, the remaining canonical pairs $\{(q_a,p_a), a = 1,\dots,13\}$ represent the truly dynamical degrees of freedom and parametrize the 26-dimensional *reduced phase space* Γ_R. By simply inserting the constraints (5.13) into the canonical hamiltonian (5.3) we finally get the *reduced hamiltonian* on Γ_R

$$\begin{aligned} H_R(q_1,\dots,q_{13};p_1,\dots,p_{13}) &= H(A_{ai}(q),E_{ai}(p))|_{q_{a+13}=0,p_{a+13}=f_a} \\ &= \frac{1}{2}\left(\sum_{a=1}^{13} p_a^2 + \sum_{a,b=1}^{13} \sigma_{ab}(q_1,\dots,q_{13})\, p_a p_b\right) \\ &\quad + V(q_1,\dots,q_{13}), \end{aligned} \tag{5.14}$$

where the non-trivial part of the kinetic term is given by

$$\sum_{a,b=1}^{13} \sigma_{ab} p_a p_b \equiv \frac{1}{2}\sum_{a=1}^{6} f_a^2 + \frac{3}{2}\sum_{a=7}^{11} f_a^2 \\ -(p_8 - p_9 + p_{11}) f_7 + (p_6 + p_{10}) f_8 \\ -(p_4 + p_7 - p_{12}) f_9 + p_5 f_{10} + \sqrt{3} p_5 f_{11} \,,$$

and where the potential V is a fourth-order polynomial in the coordinates $\{q_1, \dots, q_{13}\}$.

In summary, after a simple coordinate transformation (5.10) we managed to eliminate the unphysical degrees of freedom corresponding to the constraints (5.2) and (5.4) as well as (5.5) and (5.6)–(5.8) thereby reducing SU(3) YMCM in the subspace of vanishing spatial angular momenta to an *effective unconstrained* dynamical system. All physical information is contained in the reduced hamiltonian (5.14) and the dynamics is completely described by ordinary hamiltonian equations of motion in the canonical variables $\{(q_a, p_a), a = 1, \dots, 13\}$.

For a discussion of the role of the singular points inherent in the reduced hamiltonian (5.14) which coincide with the singularities of the 11×11 matrix $M(q)|_{q_{a+13}=0}$ (5.12), it may be helpful to go back for a moment to the case of SU(2). Let us assume we would fix Gauss' law (4.2) implicitly using as supplementary conditions the SU(2) analogon of (5.5) (the minimal embedding is then trivial)

$$\psi_a = \sum_{b=1}^{3}\sum_{i=1}^{3} \varepsilon_{abi} A_{bi} = 0 \,, \qquad a = 1, 2, 3 \,,$$

thus demanding the antisymmetric part of the 3×3 matrix (A_{ai}) to vanish.

Employing the reduction procedure described above for this system we finally get a parametrization of the 12-dimensional reduced phase space Γ_{R} by similarly defined coordinates and momenta $\{(q_b, p_b), b = 1, \dots, 6\}$ as well as a reduced hamiltonian $H_{\mathrm{R}}(q_b, p_b)$. Diagonalizing the symmetric matrix

$$(A_{ai})|_{\Gamma_{\mathrm{R}}} = \begin{pmatrix} q_1 & q_6 & q_5 \\ q_6 & q_2 & q_4 \\ q_5 & q_4 & q_3 \end{pmatrix} = O(\boldsymbol{\theta}) \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} O^{\mathrm{T}}(\boldsymbol{\theta}) \qquad (5.15)$$

then defines a canonical transformation $(q_1, \dots, q_6; p_1, \dots, p_6) \mapsto (\boldsymbol{x}, \boldsymbol{p}; \boldsymbol{\theta}, \boldsymbol{l})$ on Γ_{R} and one easily checks that in the new variables H_{R} is equal to the expression (4.10) directly deduced in sect. 4. The analogy can be completed when in the special case $k_i = 0$ the constraints (4.3) are fixed by the supplementary conditions (5.6)–(5.8).

So the gauge-fixing conditions (5.5) and (5.6)–(5.8) employed for SU(3) YMCM are natural extensions of supplementary conditions which have been successfully tried and tested in the simpler SU(2) case. We have therefore good reason to assume that the singularities of the reduced SU(3) hamiltonian (5.14) as the singular points of the reduced SU(2) hamiltonian (4.24) do not have any direct dynamical significance, but rather arise from the fact that the coordinates and momenta resulting from our reduction procedure cannot be defined on the entire phase space without singularities (cf. ref. [24]). In Yang–Mills field theories such singularities are related to the problem discovered by Gribov [25] that a unique representative in each gauge class may not be fixed globally due to the non-trivial topology carried by the potentials.

After having succeeded in presenting a formulation of SU(3) YMCM for vanishing spatial angular momenta in the physically relevant variables, one might start the analysis of (5.14) by looking at dynamically invariant subsystems. This, first of all, requires a systematic identification of such systems. For a given number of degrees of freedom we checked for each possible combination of pairs of canonical variables if the thereby defined subspace of Γ_{R} dynamically decouples, i.e. whether also the time derivatives of the other coordinates and momenta vanish when restricted to this subspace. Because we confined ourselves to low-dimensional cases (up to five degrees of freedom only) we did not have to invert the whole matrix (5.12) explicitly.

We end up with the following classification: In two and three dimensions the dynamics remains restricted to the SU(2) system parametrized by $(q_1, q_2, q_3; p_1, p_2, p_3)$,

$$H_{\mathrm{R}}^{\mathrm{SU}(2)} = \tfrac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \tfrac{1}{8}\left(q_1^2 q_2^2 + q_1^2 q_3^2 + q_2^2 q_3^2\right), \tag{5.16}$$

where the deviation from (2.15) by a factor 1/4 in the potential is due to the modified structure constants ($c_{123} = 1/2$).

Genuine SU(3) dynamics sets in only when leaving this subspace and exciting (q_5, p_5) as the next degree of freedom. The corresponding reduced hamiltonian reads

$$\begin{aligned} H_{\mathrm{R}}^{\mathrm{inv}} = {} & \tfrac{1}{2}\left(p_1^2 + p_2^2 + p_3^2 + p_5^2\right) + \frac{1}{(q_1 q_2 + q_1 q_3 + q_2 q_3)^2} \sum_{a,b=1,2,3,5} \sigma_{ab}^{\mathrm{inv}} p_a p_b \\ & + \tfrac{1}{8}\left(q_1^2 q_2^2 + q_1^2 q_3^2 + q_2^2 q_3^2\right) + \tfrac{1}{8} q_5^2 \left(2q_1^2 + 2q_2^2 + 2q_3^2 + 3q_5^2\right), \end{aligned} \tag{5.17}$$

and the non-trivial part of the metric is given by

$$\begin{aligned} \sigma_{11}^{\mathrm{inv}} &= q_5^2 \left[q_2^2 + q_2 q_3 + q_3^2\right], \\ \sigma_{12}^{\mathrm{inv}} &= q_5^2 \left[q_1(q_2 - q_3) - q_3(q_2 + 2q_3)\right], \\ \sigma_{13}^{\mathrm{inv}} &= q_5^2 \left[q_1(q_3 - q_2) - q_2(q_3 + 2q_2)\right], \\ \sigma_{15}^{\mathrm{inv}} &= q_5 \left[-q_1(q_2^2 + q_3^2) + q_2 q_3(q_2 + q_3)\right], \end{aligned}$$

$$\sigma_{22}^{\text{inv}} = q_5^2 \left[q_1^2 + q_1 q_3 + q_3^2\right] ,$$
$$\sigma_{23}^{\text{inv}} = q_5^2 \left[q_2(q_3 - q_1) - q_1(q_3 + 2q_1)\right] ,$$
$$\sigma_{25}^{\text{inv}} = q_5 \left[-q_2(q_1^2 + q_3^2) + q_1 q_3(q_1 + q_3)\right] ,$$
$$\sigma_{33}^{\text{inv}} = q_5^2 \left[q_1^2 + q_1 q_2 + q_2^2\right] ,$$
$$\sigma_{35}^{\text{inv}} = q_5 \left[-q_3(q_1^2 + q_2^2) + q_1 q_2(q_1 + q_2)\right] ,$$
$$\sigma_{55}^{\text{inv}} = -q_1 q_2 q_3 \left(q_1 + q_2 + q_3\right) .$$

There are no other dynamically invariant subsystems neither in four nor in five dimensions within the chosen parametrization of the reduced phase space. As a result, (5.17) represents the *simplest case of genuine SU(3) dynamics* and allows to study realistic dynamical effects of SU(3) YMCM beyond the well-known SU(2) case (5.16). Such an analysis may lead to interesting new phenomena and should be the starting point to a better understanding of the Yang–Mills theory of spatially constant fields for the gauge group SU(3).

6. Summary

The aim of this paper was to gain a deeper insight into the dynamical structure of classical SU(2) and SU(3) Yang–Mills theory for spatially constant fields. The systematic study of the constraints led to a complete elimination of the unphysical gauge and rotational degrees of freedom. By isolating the truly dynamical variables we obtained an effective unconstrained formulation as a necessary prerequisite for further, more explicit investigations.

We showed that the three-dimensional system (2.15) represents the subspace of vanishing spatial angular momenta of SU(2) YMCM. For the general case we also managed to identify the four physically relevant degrees of freedom whose dynamics is determined by the reduced hamiltonian (4.24) and can now be analysed directly.

The holonomy properties of the constraints already pointed to a more complicated structure of SU(3) YMCM. Actually we did not find a transformation to preferred canonical variables which perfectly suit the symmetries of the system. The 11-fold degeneracy of the theory in the subspace of vanishing spatial angular momenta could nevertheless be lifted by a consistent gauge fixing and we subsequently found a coordinate transformation providing us with a convenient parametrization of the reduced phase space.

In spite of the rather complex effective hamiltonian (5.14), the low-dimensional invariant subsystems could be classified and the four-dimensional system (5.17) allows a first study of genuine SU(3) dynamics.

We thank J. Bartels for helpful discussions.

References

[1] S.G. Matinyan, G.K. Savvidi and N.G. Ter-Arutyunyan-Savvidi, Zh. Eksp. Teor. Fiz. 80 (1981) 830 [Sov. Phys. JETP 53 (1981) 421]
[2] B.V. Chirikov and D.L. Shepelyanskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. 34 (1981) 171 [JETP Lett. 34 (1981) 163];
E.S. Nikolaevskiĭ and L.N. Shur, Pis'ma Zh. Eksp. Teor. Fiz. 36 (1982) 176 [JETP Lett. 36 (1982) 218];
E.S. Nikolaevskiĭ and L.N. Shur, Zh. Eksp. Teor. Fiz. 85 (1983) 3 [Sov. Phys. JETP 58 (1982) 1];
G.K. Savvidy, Phys. Lett. B130 (1983) 303;
A. Carnegie and I.C. Percival, J. Phys. A17 (1984) 801;
S.-J. Chang, Phys. Rev. D29 (1984) 259;
W.-H. Steeb, C.M. Villet and A. Kunick, J. Phys. A18 (1985) 3269;
G. Sohos, T. Bountis and H. Polymilis, Nuovo Cimento B104 (1989) 339
[3] G.K. Savvidy, Nucl. Phys. B246 (1984) 302; Phys. Lett. B159 (1985) 325;
B.V. Medvedev, Teor. Mat. Fiz. 60 (1984) 224 [Theor. Math. Phys. 60 (1984) 782];
C.C. Martens, R.L. Waterland and W.P. Reinhardt, J. Chem. Phys. 90 (1989) 355
[4] P. Dahlqvist and G. Russberg, Phys. Rev. Lett. 65 (1990) 2837
[5] J. Villarroel, Phys. Lett. B181 (1986) 321
[6] R. Abraham, J.E. Marsden and T. Ratiu, Manifolds, tensor analysis, and applications, second edition (Springer, Berlin, 1988)
[7] H.M. Asatryan and G.K. Savvidy, Phys. Lett. A99 (1983) 290
[8] P.A.M. Dirac, Can. J. Math. 2 (1950) 129; Proc. Roy. Soc. (London) A246 (1958) 326; Lectures on Quantum Mechanics (Belfer Graduate School of Science, Yeshiva University, New York, 1964)
[9] K.E. Atkinson, An introduction to numerical analysis (Wiley, New York, 1978)
[10] E. Nelson, Tensor analysis (Princeton Univ. Press, Princeton, 1967)
[11] V.I. Arnold, Mathematical methods of classical mechanics (Springer, New York, 1978)
[12] A.C. Hearn, REDUCE User's Manual, Version 3.2 (Rand Publication, Santa Monica, 1985)
[13] The NAG Fortran Library Manual, Mark 13 (NAG Ltd., Oxford, 1988)
[14] A.J. MacFarlane, Commun. Math. Phys. 11 (1968) 91
[15] A.J. Hanson, T. Regge and C. Teitelboim, Constrained hamiltonian systems (Accademia Nazionale dei Lincei, Roma, 1976)
[16] K. Sundermeyer, Constrained dynamics, Springer Lecture Notes in Physics 169 (Springer, Berlin, 1982)
[17] D.M. Gitman and I.V. Tyutin, Quantization of fields with constraints (Springer, Berlin, 1990)
[18] J. Goldstone and R. Jackiw, Phys. Lett. B74 (1978) 81
[19] B. Dahmen, Diplomarbeit, Univ. Hamburg (1991)
[20] J. Frøyland, Phys. Rev. D27 (1983) 943
[21] A. Kihlberg and R. Marnelius, Phys. Rev. D26 (1982) 2003
[22] J.F. Cornwell, Group theory in physics, Vol. II (Academic Press, London, 1984)
[23] D. Olive, Proc. Int. Summer Institute on Theoretical Physics, Bad Honnef 1980 (Plenum Press, New York, 1981) p. 207
[24] R. Jackiw, Rev. Mod. Phys. 52 (1980) 661
[25] V. Gribov, Nucl. Phys. B139 (1978) 1