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Geometry from general statistics

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We show how geometrical quantities as distance, topology and riemannian metric can be constructed from the correlation functions in general statistical systems.

1. Introduction

It is an old dream that a unified description of physics should explain the remaining free parameters of the standard model. String theories [1] are nowadays the best candidates for a unification of all forces. They suffer, however, from a proliferation of possible ground states [2]. Although these theories are very predictive for a given ground state (at least in principle), this predictivity is lost as a consequence of the absence of selection criteria for the ground state. One may still attempt to extract some general features of effective low-energy theories from general features of string theories. A computation of fermion masses or gauge couplings requires, however, the selection of the "true" ground state among the (infinitely many?) possible ground states of string theory. A similar problem appeared already in earlier attempts of unification in higher dimensions (Kaluza–Klein theories [3]): There are many possible classically stable compactifications of internal space *.

The proliferation of possible ground states may be viewed as an embedding problem: How should four-dimensional space-time be embedded into higher-dimensional space-time? Or, even more generally: How should the four-dimensional energy–momentum tensor be embedded into the (infinite-dimensional) space of all possible operators? We believe that the proliferation of ground states constitutes the central crisis for all attempts to unification where four-dimensional space-time is not given a priori. It seems to us very unlikely that this problem can be overcome

* The possible compactifications will even depend on continuous parameters [4] if we admit that internal space may be a noncompact manifold [5] or orbifold [6].

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by pure consistency requirements. (It is sometimes hoped that nonperturbative string field theory may select a unique ground state.) Indeed, different classically stable ground states will simply lead to different hamiltonians. Many of these hamiltonians will also have an acceptable quantum ground state which minimizes the energy (as defined by this given hamiltonian). There is then no consistency criterion which would exclude such a ground state. With present criteria it is impossible to say which ground state is "better" than another, since there is no way to compare energies defined with respect to different hamiltonians *. Even if we impose additional symmetry requirements like maximal four-dimensional symmetry or Poincaré symmetry, $SU(3) \times U(1)$ gauge invariance or the spontaneously broken symmetry $SU(3) \times SU(2) \times U(1)$ of the standard model, there is little reason why the ground state should be unique. In consequence, we will have to find new criteria why certain space-times give a better description of reality than others. "Which space-time to select?" – this seems to be the central question for the future of unification.

Before attacking this problem, one should first answer another question: How do concepts like space, time and geometry emerge at all if the space-time manifold is not given a priori? We are used to formulate a theory in a given space-time manifold. This is unsatisfactory since space-time should be understood as a property of matter rather than a preexisting category. Space-time manifests itself only through the motion of matter and seems to make no sense without matter. (Here matter includes gravitational fields like the graviton.) The framework for a discussion of this question is general statistics [7]. This deals with general statistical systems of infinitely many degrees of freedom without an a priori identification of operators with observables like energy. The identification of structures between operators with the observed properties of space-time should be done only after formulating criteria why certain operator structures give a better description of reality than others.

In this paper we establish a general framework how space and geometry arise as properties of correlation functions. Let us first ask how we could measure a distance in a statistical system without postulating Lorentz invariance and photons travelling with the speed of light. Intuitively, we would say that two points x_1, x_2 are far away from each other if the correlation between all operators defined at the point x_1 and those defined at x_2 is small. We also know that correlation functions necessarily reflect the properties of a given geometry (like symmetries, topology etc.). It seems therefore reasonable to attempt the inverse program, i.e. to extract all geometrical information from the properties of correlation functions. We demonstrate in this paper that these intuitive ideas indeed work. Geometrical

* The same problem appears in an embryonic way in four-dimensional gravity. There are different solutions with timelike Killing vectors which are not connected continuously to each other. Only the solution with maximal four-dimensional symmetry is unique.

concepts like distance, topology, metric or symmetry can be formulated in terms of the connected two-point function.

One of the basic tools in this investigation stems from the observation that the normalizable operators form a Hilbert space. The scalar product allows one to introduce a distance in operator space. This permits one to quantify the concept that for two points x_1 and x_2 close to each other the corresponding operators $v(x_1)$ and $v(x_2)$ should also be close to each other. The connected two-point function can be expressed in terms of the scalar product in operator space. The geometry of operator space is then used to define geometrical concepts relating to small distances as continuous space, local topology and metric. For the geometry at large distances we will strongly rely on an additional notion, namely that the (connected) correlation function should vanish for infinite distance and vice versa. This last requirement is not fulfilled for arbitrary operators inducing a geometry. In a sense it constitutes a first (still very weak) selection criterion for the choice of geometries which are well suited to describe the real world. We will briefly comment on additional possible selection criteria in the conclusions.

For simplicity we carry out all constructions for a finite number of degrees of freedom N. General statistics is defined in terms of sequences with $N \to \infty$ as discussed in detail in ref. [7].

2. Structures in the space of operators

Consider N continuous variables s^u (degrees of freedom) parametrizing $\mathbb{R}^N$ and a positive-definite probability density $p(s)$. We assume that $p(s)$ is continuous everywhere in $\mathbb{R}^N$ and has a finite integral $Z = \int p(s)\, \mathrm{d}s$. A map $v(s)$: $\mathbb{R}^N \to \mathbb{R}$ defines an *operator* if the expectation value

$$\langle v \rangle = Z^{-1} \int v(s) p(s)\, \mathrm{d}s \tag{2.1}$$

exists (in the sense of Lesbesques integrals). Two functions v_1 and v_2 which differ only on a subset of $\mathbb{R}^N$ with measure zero define the same operator, i.e.

$$\int \left(v_1(s) - v_2(s)\right)^2 p(s)\, \mathrm{d}s = 0 \Rightarrow v_1 = v_2. \tag{2.2}$$

Operators are therefore associated with representatives of the equivalence classes of maps differing only on a zero measure set. They form a (real) vectorspace V_p.

Since we are interested in correlations between operators we concentrate on *normalizable operators*, namely those for which

$$\| v \|^2 = \langle v^2 \rangle \tag{2.3}$$

is defined. For normalizable v the function $\hat{v} = p^{1/2} v$ is square integrable in $\mathbb{R}^N$. The normalizable operators form a vector space $\mathscr{K}^2 \subset \mathrm{V}_p$. The "correlation" between two normalizable operators v, w,

$$\langle v, w \rangle = \langle v \cdot w \rangle = Z^{-1} \int v(s) w(s) p(s) \, \mathrm{d}s, \tag{2.4}$$

always exists and defines a scalar product $\langle v, w \rangle$ on $\mathscr{K}^2$. As is well known the space of square integrable functions is complete and separable with respect to the norm induced by the scalar product. The space $\mathscr{K}^2$ of normalizable operators is a Hilbert space *.

For higher correlations we may consider the space $\mathscr{K}^n$ of all operators for which $\langle | v |^n \rangle$ is defined ($n \in \mathbb{N}$, $n \geqslant 2$). $\mathscr{K}^n$ is again a vector space and $\mathscr{K}^m$ is a true subspace of $\mathscr{K}^n$ for $m > n$. The scalar product (2.4) and the induced norm (2.3) are therefore defined for all $\mathscr{K}^n$. With respect to this norm $\mathscr{K}^n$ is separable, but not complete for $n > 2$. There are Cauchy sequences in $\mathscr{K}^n$ whose limit belongs to $\mathscr{K}^2$ but is not an element of $\mathscr{K}^n$. For arbitrary operators v_1, $v_2 \in \mathscr{K}^n$ the expectation values of products are defined for

$$\langle v_1^p v_2^q \rangle, \qquad p, q \in \mathbb{N}, \qquad p + q \leqslant n. \tag{2.5}$$

We call $\mathscr{K}$ the space of all *correlatable operators*, i.e. those $v(s)$ for which $\langle v^k \rangle$ is defined for arbitrary $k \in \mathbb{N}$. ($\mathscr{K} \subset \mathscr{K}^n$) for all n.) Again the vectorspace $\mathscr{K}$ is separable but not complete with respect to the norm induced by the scalar product (2.4). The expectation values of products $\langle v_1^{p_1} v_2^{p_2} \ldots v_k^{p_k} \rangle$ are defined for all $p_i \in \mathbb{N}$, $v_i \in \mathscr{K}$. Arbitrarily high correlations can be evaluated for the correlatable operators. We can therefore introduce an abelian product between operators in $\mathscr{K}$. It is defined by the pointwise multiplication of functions

$$\begin{gathered} (v_1, v_2) \in \mathscr{K} \times \mathscr{K} \to v_1 \cdot v_2 = w \in \mathscr{K}, \\ w(s) = v_1(s) \cdot v_2(s). \end{gathered} \tag{2.6}$$

The constant function $v(s) = 1$ is the identity operator of this multiplication, $v \cdot 1 = v$. The vectorspace $\mathscr{K}$ with the multiplication operation (2.6) forms a commutative algebra (over $\mathbb{R}$) with an identity **.

* Since the identity function $w(s) \equiv 1$ is normalizable the existence of the norm (2.3) implies that v is an operator (2.1). We also observe that $\|v_1 - v_2\| = 0$ implies $\langle w, v_1 \rangle = \langle w, v_2 \rangle$ for $w \in \mathscr{K}^2$ such that our definition of operators (2.2) is consistent with the structure of $\mathscr{K}^2$.

** We note that $\mathscr{K}$ is everywhere dense in $\mathscr{K}^2$, $\overline{\mathscr{K}} = \mathscr{K}^2$. The continuous normalizable (correlatable) functions are everywhere dense in $\mathscr{K}^2(\mathscr{K})$ and similar for k times (infinitely often) differentiable functions. Without loss of generality we could restrict our discussion to infinitely often differentiable correlatable operators $v(s)$ (the algebra preserves this property).

Let us next consider variable transformations. They are characterized by invertible functions $f(s)$,

$$f: \tilde{\mathbb{R}}^N \to \tilde{\mathbb{R}}'^N, \qquad \hat{f} = \det\left|\frac{\partial f^u}{\partial s^u}\right| > 0, \tag{2.7}$$

which are defined on the $\mathbb{R}^N$ except for some zero measure set. The image of f is again $\mathbb{R}^N$ except for a zero measure set (we denote these by $\tilde{\mathbb{R}}^N$, $\tilde{\mathbb{R}}'^N$). The function f should be continuous and differentiable in $\tilde{\mathbb{R}}^N$. An operator $v(s)$ transforms as

$$v(s) \to v(f(s)) = (f(v))(s). \tag{2.8}$$

We can use this transformation to define an isomorphism of scalar products

$$\begin{aligned} \langle v'w'\rangle_f &= Z^{-1}\int \mathrm{d}s\, p_f(s) v'(s) w'(s), \\ p_f(s) &= \hat{f} p(f(s)), \\ Z &= \int \mathrm{d}s\, p(s) = \int \mathrm{d}s\, p_f(s), \end{aligned} \tag{2.9}$$

with

$$\langle f(v), f(w)\rangle_f = \langle v, w\rangle. \tag{2.10}$$

Here we note that operators v, w which are normalizable with the probability density p are not necessarily normalizable with respect to p_f. Only the transformed operators $f(v)$, $f(w)$ are guaranted to be normalizable with p_f.

Similarly, if v, w are correlatable with respect to p the transformed operators $f(v)$, $f(w)$ are correlatable with respect to p_f. For all functions $v(s)$ one has

$$\begin{aligned} f(v+w) &= f(v) + f(w), \\ f(\lambda v) &= \lambda f(v), \\ f(v \cdot w) &= f(v) \cdot f(w). \end{aligned} \tag{2.11}$$

The transformation f therefore conserves the algebraic structure of $\mathscr{K}$ (e.g. $f((v_1+v_2)v_3) = (f(v_1) + f(v_2))f(v_3)$ etc.). It maps the algebra $\mathscr{K}$ defined with p on the algebra $f(\mathscr{K})$ defined with p_f. Since f^{-1} is well defined for all operators in $f(\mathscr{K})$ a variable transformation establishes an isomorphism between $\mathscr{K}$ and $f(\mathscr{K})$. It was shown in ref. [7] that variable transformations can be used to map two arbitrary probability densities p and p' into each other. We conclude that there is a one-to-one correspondence between algebras defined with p or p'. The choice of p is irrelevant for the possible structures between operators which arise from the algebra $\mathscr{K}$. It is associated with a "coordinate choice" in variable space. With respect to the product

$$(f \circ g)(v(s)) = v(f(g(s))), \tag{2.12}$$

the variable transformations form a group. It is the group of general coordinate transformations in N dimensions, denoted by gen_N. The variable transformations which leave the scalar product between all operators invariant,

$$\langle s(v), s(w) \rangle = \langle v, w \rangle, \tag{2.13}$$

are called symmetries. All operators obtained from each other by symmetry transformations have the same expectation value. The symmetries from the subgroup sgen_N characterized by an invariant volume element

$$\hat{f} p(f(s)) = p(s). \tag{2.14}$$

They define automorphisms of $\mathscr{K}$ for a given probality density p. In addition there are variable transformations f acting within $\mathscr{K}$ which are not symmetries. As a necessary and sufficient condition this demands that the scalar product $\langle f(v), f(w) \rangle$ is defined for all $v, w \in \mathscr{K}$. One concludes from

$$\begin{aligned} \langle f(v), f(w) \rangle &= \left\langle v w p_{f^{-1}} p^{-1} \right\rangle, \\ p_{f^{-1}} &= \frac{1}{\hat{f}} p(f^{-1}(s)), \end{aligned} \tag{2.15}$$

that the ratio $p_{f^{-1}}/p$ should be a normalizable operator, i.e. the integral

$$\int \mathrm{d}s \, p^{-1}(s) p^2(f^{-1}(s)) \hat{f}^{-2}(s) \tag{2.16}$$

should exist.

Infinitesimal variable transformations act as linear operators in $\mathscr{K}$,

$$\begin{aligned} v &\to v + \delta v, \\ \delta v(s) &= -\xi^u(s) \frac{\partial}{\partial s^u} v(s). \end{aligned} \tag{2.17}$$

Multiplication with a given operator w can also be interpreted as an operation in $\mathscr{K}$. The associated operator $\hat{w}$ acts linearly in $\mathscr{K}$,

$$\hat{w}(v) = wv. \tag{2.18}$$

It does, however, not conserve the product structure ($\hat{w}(vv') \neq \hat{w}(v)\hat{w}(v')$). The operations (2.17) and (2.18) do in general not commute.

3. Continuous and differentiable operators

Let us consider a family of normalizable operators $v(x^{\mu}; s) \equiv v(x)$ which depend on parameters x^{μ}. The parameters are cartesian coordinates of some open region of $\mathbb{R}^d (\mu = 1, \dots, d)$. An operator is continuous at x if

$$\lim_{y \to x} \|v(y) - v(x)\| = 0. \tag{3.1}$$

Here the limit $y \to x$ is induced by the topology of $\mathbb{R}^d$ and (3.1) must hold independently of the limiting procedure for all y in a local neighbourhood of x.

An operator family $v(x)$ is differentiable in a point x if there exist continuous operators $(\partial_{\mu} v)(x) \in \mathscr{K}^2$ such that for all $\mu = 1, \dots, d$,

$$\lim_{\epsilon \to 0} \left\| (v(x + \epsilon \Delta x^{\mu}) - v(x)) / \epsilon \Delta x^{\mu} - \partial_{\mu} v(x) \right\| = 0. \tag{3.2}$$

Differentiation commutes with the operations of forming expectation values and scalar products

$$\partial_{\mu} \langle v(x) \rangle = \langle \partial_{\mu} v(x) \rangle, \tag{3.3}$$

$$\partial_{\mu} \langle v_1(x), v_2(x) \rangle = \langle v_1(x), \partial_{\mu} v_2(x) \rangle + \langle v_2(x), \partial_{\mu} v_1(x) \rangle \tag{3.4}$$

(We assume that both v_1 and v_2 are differentiable. Continuity and differentiability of $v_i(x)$ implies continuity and differentiability of the corresponding expectation values.) The product of two differentiable operators v_1, v_2 is differentiable if $v_1(x) v_2(x) \in \mathscr{K}^2$,

$$\partial_{\mu} (v_1(x) v_2(x)) = v_1(x) \partial_{\mu} v_2(x) + v_2(x) \partial_{\mu} v_1(x). \tag{3.5}$$

An operator family $v(x)$ is twice differentiable if all $\partial_{\mu} v(x)$ are differentiable, with an obvious generalization to higher derivatives. For k times differentiable $v(x)$ with positive norm $\|v(x)\| > 0$ the norm $\|v(x)\|$ is also k times differentiable,

$$\partial_{\mu} \|v(x)\| = \frac{\langle v, \partial_{\mu} v \rangle}{\|v\|}. \tag{3.6}$$

4. Distance

Let us consider a family of normalizable operators $v(x)$ which are continuous in an open region $\tilde{\mathrm{R}} \subseteq \mathbb{R}^d$, dim $\tilde{\mathrm{R}} = d$. We also demand that $v(x^\mu, s)$ is different from the constant operator for all $x \in \tilde{\mathrm{R}}$,

$$\|v(x) - \langle v(x)\rangle\| > 0. \tag{4.1}$$

This allows us to introduce a normalized operator family, $\varphi(x) \in \mathscr{H}^2$,

$$\varphi(x) = \frac{v(x)}{\|v(x) - \langle v(x)\rangle\|}, \tag{4.2}$$

$$\tilde{\varphi}(x) = \varphi(x) - \langle\varphi(x)\rangle = \frac{v(x) - \langle v(x)\rangle}{\|v(x) - \langle v(x)\rangle\|}, \tag{4.3}$$

$$\|\tilde{\varphi}(x)\| = \|\varphi(x) - \langle\varphi(x)\rangle\| = 1. \tag{4.4}$$

We finally require that there is a open region $\mathrm{R} \subseteq \tilde{\mathrm{R}}$ (dim $\mathrm{R} = d$) where two operators $\tilde{\varphi}(x)$ and $\tilde{\varphi}(y)$ are different for $x \neq y$,

$$\|\tilde{\varphi}(x) - \tilde{\varphi}(y)\| = 0 \Leftrightarrow x^\mu = y^\mu, \tag{4.5}$$

and never orthogonal

$$\langle\tilde{\varphi}(x), \tilde{\varphi}(y)\rangle \neq 0. \tag{4.6}$$

for all $x, y \in \mathrm{R}$. We call an operator family $v(x)$ which fulfills the conditions (3.1) (4.1) (4.5) (4.6) a *prefield* defined in R. Continuity (3.1) implies that (4.6) is always fulfilled for some region $\mathrm{R} \subseteq \tilde{\mathrm{R}}$. The operator family $\varphi(x)$ is the corresponding *normalized prefield*. We want to use the connected two-point function,

$$\begin{aligned} G(x, y) &= \langle\varphi(x), \varphi(y)\rangle_c = \langle\tilde{\varphi}(x), \tilde{\varphi}(y)\rangle \\ &= \langle\varphi(x)\varphi(y)\rangle - \langle\varphi(x)\rangle\langle\varphi(y)\rangle, \end{aligned} \tag{4.7}$$

in order to introduce a distance in R in terms of this correlation. Within R the correlation $G(x, y)$ is continuous with respect to x and y, strictly positive and bounded by one,

$$0 < G(x, y) \leqslant 1 \tag{4.8}$$

with

$$G(x, y) = 1 \Leftrightarrow x^\mu = y^\mu. \tag{4.9}$$

If the prefield $v(x)$ is k times differentiable within R, the correlation $G(x, y)$ is k times differentiable with respect to x or y within R. We introduce the symmetric function $D(x, y)=D(y, x)$ by

$$D(x, y)\mu(D(x, y)) = \sqrt{2}\left(G^{-1}(x, y) - 1\right)^{1/2}. \tag{4.10}$$

Here we define for $D \geqslant 0$ the continuous positive scaling function $\mu(D)>0$. This guarantees

$$D(x, y) \geqslant 0, \tag{4.11}$$
$$D(x, y) = 0 \Leftrightarrow x = y. \tag{4.12}$$

We require $\mu(D)$ to remain finite for all D which occur within R and demand that $D\mu(D)$ is monotonically increasing such that D is uniquely defined in terms of G. This still allows that μ diverges for some finite $D_{\max}$ which measures the maximal extension of R and occurs at the boundary for $G(x, y) \to 0$. On the other hand an arbitrarily large extension of R ($D_{\max} \to \infty$) requires that $\mu(D)$ remains finite for all finite D and $D(x, y) \to \infty$ always implies $G(x, y) \to 0$.

We next want to use the norm inequality

$$\|\tilde{\varphi}(x) - \tilde{\varphi}(y)\| \leqslant \|\tilde{\varphi}(x) - \tilde{\varphi}(z)\| + \|\tilde{\varphi}(y) - \tilde{\varphi}(z)\|,$$

$$\left(1 - G(x, y)\right)^{1/2} \leqslant \left(1 - G(x, z)\right)^{1/2} + \left(1 - G(y, z)\right)^{1/2}, \tag{4.13}$$

in order to establish that $D(x, y)$ defines a distance

$$D(x, y) \leqslant D(x, z) + D(y, z) \tag{4.14}$$

at least within a region $\overline{\mathrm{R}} \subseteq \mathrm{R}$. Indeed, the distance inequality follows from (4.13) provided $\mu(D)$ has the property

$$F(D_1) + F(D_2) \geqslant F(D_3) \Rightarrow D_1 + D_2 \geqslant D_3 \tag{4.15}$$

with

$$F(D) = \frac{1}{\sqrt{2}} D\mu(D)\left(1 + \tfrac{1}{2}D^2\mu^2(D)\right)^{-1/2}. \tag{4.16}$$

The property (4.15) holds for infinitesimally small D if $\mu(0)=\mu_0>0$ and $\mu'(0)>0$. On the other hand, it cannot be implemented for $D \to \infty$ where $F(D)$ approaches one. Condition (4.15) is a sufficient, but not a necessary condition for D to define a distance.

As an illustration, one may consider a correlation function which ressembles at large distances the two-point function of a four-dimensional scalar field theory in flat space,

$$G(x, y) = \left(1 + r^2 \exp(mr)\right)^{-1},$$
$$r^2 = \sum_\mu (x^\mu - y^\mu)^2. \tag{4.17}$$

Choosing

$$\mu(D) = \sqrt{2}\exp(\tfrac{1}{2}mD), \tag{4.18}$$

one simply finds

$$D^2 = r^2. \tag{4.19}$$

The function $\mu(D)$ fulfills (4.15) only if D_1 and D_2 are smaller than a "boundary distance" $\overline{D}(m)$ which is determined by

$$\overline{D}^2 = \tfrac{1}{3}\left[\exp(-m\overline{D}) - \exp(-2m\overline{D})\right] \tag{4.20}$$

and obeys (for all m) $\overline{D}^2 \leqslant \frac{1}{12}$. The norm inequality (4.13) assures the distance inequality (4.14) only for this range of D. Nevertheless, (4.14) is obviously fulfilled for arbitrary D.

We conclude that every prefield defines a distance for some finite open region $\overline{\mathrm{R}} \subseteq \mathrm{R}$. The extension of $\overline{\mathrm{R}}$ depends on the choice of $\mu(D)$. For

$$\mu = (1 - D^2)^{-1/2}, \tag{4.21}$$

one has

$$D = \sqrt{2}(1 - G)^{1/2} \tag{4.22}$$

and $\overline{\mathrm{R}} = \mathrm{R}$. This particular choice implies a maximal distance

$$\mathrm{D}_{\max} = \sqrt{2}. \tag{4.23}$$

The units of D are, of course arbitrary. We can always rescale D by an appropriate rescaling of μ such that μD remains fixed. The function μ has the dimension of an inverse length and we may fix units by setting $\mu_0 = 1$. In fact, μ_0^{-1} plays the role of a fundamental length scale. Its significance will become more apparent in the following sections.

The possibility to implement arbitrarily large distances within R ($\mu_0 D \to \infty$) is in general not given. In this case $\mu(D)$ must be defined for $0 \leqslant D < \infty$ such that a nonvanishing correlation $G(x, y) > 0$ implies a finite distance $D(x, y)$ and vice versa. If the distance inequality (4.14) should hold for $D(x, y) \to \infty$ the prefield $v(x)$ must have additional properties which guarantee, for example, the necessary relation

$$G(x, y) = 0 \Rightarrow G(x, z) = 0 \quad \text{or} \quad G(y, z) = 0 \tag{4.24}$$

for all $z < R$. We will come back to such structures in sect. 9.

5. Metric

In the following we assume that $v(x)$ is differentiable. If $\tilde{\varphi}$ is defined (4.1), (4.3) it is also differentiable by virtue of (2.6). The square of an infinitesimal distance can then be used to define a metric

$$g_{\mu\nu}(x) = -\frac{1}{2}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial y^{\nu}} D^2(x, y)\bigg|_{y=x} \tag{5.1}$$

(We assume in the following that $\mu(D)$ is at least twice differentiable at $D = 0$.) The first derivative of D^2 at $x = y$ vanishes,

$$\begin{aligned}\frac{\partial}{\partial y^{\mu}} D^2(x, y)\bigg|_{y=x} &= -2\mu_0^{-2}\frac{\partial}{\partial y^{\mu}} G(x, y)\bigg|_{y=x} \\ &= -2\mu_0^{-2}\left\langle\tilde{\varphi}(x), \frac{\partial\tilde{\varphi}}{\partial x^{\mu}}(x)\right\rangle \\ &= -\mu_0^{-2}\partial_{\mu}\langle\tilde{\varphi}^2(x)\rangle = 0.\end{aligned} \tag{5.2}$$

In fact, D is differentiable at $D = 0$,

$$\begin{aligned}\frac{\partial}{\partial y^{\mu}} D\bigg|_{D=0} &= \sqrt{2}\,\mu_0^{-1}\frac{\partial}{\partial y^{\mu}}\left(G^{-1} - 1\right)^{1/2}\bigg|_{G=1} \\ &= \sqrt{2}\,\mu_0^{-1}\frac{\partial}{\partial y^{\mu}}\sqrt{1 - \langle\tilde{\varphi}(x), \tilde{\varphi}(y)\rangle}\bigg|_{x=y} \\ &= \mu_0^{-1}\frac{\partial}{\partial y^{\mu}}\|\tilde{\varphi}(y) - \tilde{\varphi}(x)\|\bigg|_{x=y} \\ &= \mu_0^{-1}\left\|\frac{\partial}{\partial y^{\mu}}\tilde{\varphi}(y)\right\|.\end{aligned} \tag{5.3}$$

Small distances take the form.

$$D^2(x, y) = g_{\mu\nu}(y^\mu - x^\mu)(y^\nu - x^\nu) + \text{higher-order terms.} \tag{5.4}$$

The metric (5.1) can be expressed as a correlation between operators $\partial_\mu \tilde{\varphi}(x)$,

$$g_{\mu\nu}(x) = \mu_0^{-2} \langle \partial_\mu \tilde{\varphi}(x), \partial_\nu \tilde{\varphi}(x) \rangle. \tag{5.5}$$

It is defined in terms of the correlation function $G(x, y)$ up to a free constant μ_0. Otherwise it is independent of the choice of $\mu(D)$. We finally establish that $g_{\mu\nu}$ is a riemannian metric with positive signature,

$$\det g_{\mu\nu} > 0, \tag{5.6}$$

if $\tilde{\varphi}$ fulfills the condition

$$\begin{aligned} &\|\partial_i \tilde{\varphi}\| > 0, \\ &\partial_i \tilde{\varphi} = O_i^\mu \partial_\mu \tilde{\varphi}, \end{aligned} \tag{5.7}$$

for an arbitrary orthogonal matrix O_i^μ. Indeed, $g_{\mu\nu}$ is a symmetric matrix and can be diagonalized by an orthogonal transformation ($\tilde{g} = \text{diag}(\lambda_1, \ldots, \lambda_d)$)

$$\tilde{g}_{ij} = O_i^\mu O_j^\nu g_{\mu\nu}. \tag{5.8}$$

Every eigenvalue

$$\lambda_i = \tilde{g}_{ii} = \mu_0^{-2} \|\partial_i \tilde{\varphi}\|^2 \tag{5.9}$$

is positive semidefinite and becomes positive definite by virtue of (5.7). Condition (5.7) is also necessary since all diagonal elements $g_{\mu\mu}$ must be positive in an arbitrarily rotated basis.

The condition (5.7) plays a crucial role for the existence of a metric in terms of correlation functions. It means that there is no direction for which the expectation value of the "kinetic operator" $\partial_i \tilde{\varphi} \partial_i \tilde{\varphi}$ associated with $\tilde{\varphi}$ vanishes. In a more algebraic language it is equivalent to the condition that the partial derivatives $\partial_\mu \tilde{\varphi}$ form a set of d linearily independent operators in $\mathscr{H}^2$. We emphasize that (5.7) implies the condition (4.5) but is actually stronger.

An operator family $v(x) \in \mathscr{H}^2$ is called a *field* defined in a (d-dimensional) region $\mathrm{R_M} \subseteq \mathbb{R}^d$ if, for all $x \in \mathrm{R_M}$, it is

(i) continuous (3.1) and infinitely often * differentiable (3.2);

(ii) different from the constant operator (4.1) such that $\tilde{\varphi}(x)$ is defined and differentiable

* We could require only differentiability of $v(x)$. Our stronger condition ensures that all derivatives of the metric exist and are continuous.

(iii) fluctuating in d directions in the sense that the d derivative operators $\partial_\mu \tilde{\varphi}$ are all linearily independent (5.7).

Every field defines a riemannian metric $g_{\mu\nu}$ in $\mathrm{R_M}$ which is unique up to an overall scale factor μ_0. We can use this metric to define a geodesic distance D_g in the standard way. In general D_g will be different from D. Since G cannot always be expressed as a function of D_g it is not even possible to establish a functional relation between D_g and D in the most general case *. Nevertheless, the distance inequality (4.14) implies by construction for arbitrary $\mu(D)$,

$$D_g(x, y) \geqslant D(x, y). \tag{5.10}$$

In particular, $D \to \infty$ implies $D_g \to \infty$.

It is instructive to consider a few simple examples. Let 0_i be orthonormal operators with vanishing expectation values

$$\langle 0_i \rangle = 0, \qquad \langle 0_i 0_j \rangle = \delta_{ij}. \tag{5.11}$$

We start with one-dimensional examples, where

$$\tilde{\varphi} = a_1(x) 0_1 + a_2(x) 0_2,$$

$$a_1^2 + a_2^2 = 1. \tag{5.12}$$

The correlation and the metric read ($\mu_0 = 1$)

$$G(x, y) = a_1(x) a_1(y) + a_2(x) a_2(y), \tag{5.13}$$

$$g_{11} = a_1'(x)^2 + a_2'(x)^2 = a_1'(x)^2 \left(1 - a_1(x)^2\right)^{-1}. \tag{5.14}$$

We may compare the distance

$$D = \sqrt{2}\,(1 - G)^{1/2} = \left(2 - 2a_1(x) a_1(y) - 2a_2(x) a_2(y)\right)^{1/2} \tag{5.15}$$

with the geodesic distance

$$D_g = \int_x^y \mathrm{d}x' |a_1'| \left(1 - a_1^2\right)^{-1/2} = \arcsin a_1(y) - \arcsin a_1(x) \tag{5.16}$$

(Eq. (5.16) holds for $a_1' > 0$ everywhere between x and y. For $a_1' < 0$ the sign of the r.h.s. has to be inverted and the contributions of regions with different sign of a_1' have to be added.)

* In field theory the correlation function may not only depend on the geodesic distance but also on properties of the curvature tensor [8].

The first example describes a circle in $\mathscr{K}^2$:

$$
\begin{aligned}
&a_1 = \sin x, \qquad a_2 = \cos x, \\
&G = \cos(y - x), \\
&D_g = |y - x|, \\
&D = (2 - 2\cos(y - x))^{1/2}.
\end{aligned} \tag{5.17}
$$

We may obtain $D = D_g$ by a different choice of μ, namely for

$$
\mu(D) = \frac{\sqrt{2}}{D}\left(\frac{1 - \cos D}{\cos D}\right)^{1/2}. \tag{5.18}
$$

The boundaries of R are given by (4.6), namely $\pi/4 < x < \pi/4$ such that R covers only one quarter of the circle. In contrast, R_M extends to the whole real axis. An infinite geodesic distance would require, however, an infinite number of windings around the circle. These features can be easily generalized: If G can be expressed as a function of D_g we can always obtain within a certain range $D_g = D$ by a suitable choice of μ. Within R all distances obtained from (5.12) are bounded since the r.h.s. of (5.16) is bounded and at most two values of a_2 can correspond to a given value of a_1.

The second example has a singularity of the metric at $x = 0$ (with R defined by $|x| < \frac{1}{2}$),

$$
a_1 = \begin{cases} \frac{1}{2} + \sqrt{x(1-x)} & \text{for } x > 0 \\ \frac{1}{2} - \sqrt{-x(1+x)} & \text{for } x < 0, \end{cases} \tag{5.19}
$$

$$
g_{11} = \begin{cases} \frac{1}{4}(1-2x)^2\left\{x(1-x)\left[\frac{3}{4} - x(1-x) - \sqrt{x(1-x)}\right]\right\}^{-1} & \text{for } x > 0 \\ \frac{1}{4}(1+2x)^2\left\{-x(1+x)\left[\frac{3}{4} + x(1+x) - \sqrt{-x(1+x)}\right]\right\}^{-1} & \text{for } x < 0. \end{cases} \tag{5.20}
$$

For small $|x|$ one has

$$
\mathrm{d}s \approx (3|x|)^{-1/2}|\mathrm{d}x| \tag{5.21}
$$

and D_g remains finite at $x = 0$. In fact, the "coordinate singularity" at $x = 0$ can be removed by a redefinition of the coordinate $x = F(x)$. The correlation is continuous in the whole interval $-\frac{1}{2} < x < \frac{1}{2}$ with $G(-\frac{1}{2}, \frac{1}{2}) = 0$. All distances in R remain bounded despite the divergence of the metric at $x = 0$.

Our third example is two dimensional. We consider for $x_1^2+x_2^2<(1+c^2)^{-1}$,

$$\tilde{\varphi}(x_1x_2)=x_1 0_1+x_2 0_2+c\sqrt{x_1^2+x_2^2}\,0_3 + \sqrt{1-(1+c^2)(x_1^2+x_2^2)}\,0_4, \tag{5.22}$$

where $\tilde{\varphi}$ is not differentiable for $x_1=x_2=0$ (except for $c=0$) and at the boundary $x_1^2+x_2^2=(1+c^2)^{-1}$. The metric reads ($\mu_0=1$)

$$g_{11}=1+c^2\frac{x_1^2}{x_1^2+x_2^2}+(1+c^2)^2\frac{x_1^2}{1-(1+c^2)(x_1^2+x_2^2)},$$

$$g_{22}=1+c^2\frac{x_2^2}{x_1^2+x_2^2}+(1+c^2)^2\frac{x_2^2}{1-(1+c^2)(x_1^2+x_2^2)},$$

$$g_{12}=c^2\frac{x_1x_2}{x_1^2+x_2^2}+(1+c^2)^2\frac{x_1x_2}{1-(1+c^2)(x_1^2+x_2^2)}. \tag{5.23}$$

For small $x_1^2+x_2^2$ we can neglect the last term and find, with standard polar coordinates around the origin at $x_1=x_2=0$,

$$\mathrm{d}s^2=(1+c^2)\left(\mathrm{d}r^2+\frac{1}{1+c^2}r^2\,\mathrm{d}\varphi^2\right). \tag{5.24}$$

For $c^2>0$ this represents the geometry of a cone. There is a true singularity at $r=0$ which cannot be removed by a change of coordinates and R is not a riemannian manifold. The geodesic distance is, nevertheless, finite everywhere in R (including the boundary). The correlation $G(-x_1,\ -x_2\,;x_1,\ x_2)=1-2(x_1^2+x_2^2)$ vanishes for $x_1^2+x_2^2=\frac{1}{2}$. This determines the boundary of R for $c^2\leqslant 1$. In summary, various types of singularities can arise from very simple $\tilde{\varphi}$ with only a few operators. On the other hand, it seems impossible to obtain arbitrarily large distances if $\tilde{\varphi}$ involves only a finite number of operators.

6. Topology

We can use the correlation $G(x,\ y)$ in order to define a topology. The notion of a local neighbourhood of a point x can be induced from the distance D (4.10). For this purpose the different possible choices of the function $\mu(D)$ are equivalent. If $v(x)$ is a field the definition of a local neighbourhood in terms of the geodesic distance D_g is also equivalent. The region R is a manifold and $\mathrm{R}'_\mathrm{M}=\mathrm{R}\cap\mathrm{R}_\mathrm{M}$ is a riemannian manifold. (We remind that R'_M is the subset of R_M for which $G(x,\ y)>0$ everywhere (4.6).)

It may happen that R'_M covers R only except for some sets of points $\hat{x}$ with dimension smaller than or equal to $d-1$. In this case R is not guaranteed to be a riemannian manifold. It may have cusps or edges at the "singular" points $\hat{x}$. (Compare example (5.22).) Such singularities are an obstruction to find a metric everywhere in R, even if we cover R by different coordinate patches *. A violation of (5.7) typically leads to such an obstruction. On the other hand, "coordinate singularities" (compare example (5.19)) can be removed by choosing appropriate patches and coordinates. If R/R'_M contains only coordinate singularities we can find an "atlas" such that R is a riemannian manifold.

Within $\mathbb{R}^d$ the open region R may have $(d-1)$-dimensional hypersurfaces as boundaries. (These boundaries may have boundaries themselves. Also R needs not to be connected.) Within these bounded regions (or within $\mathbb{R}^d$ if there is no $d-1$ dimensional boundary) there may in addition be sets of points with dimension smaller than $d-1$ which do not belong to R – for example isolated points or lines. We will denote by S the set of points in $\mathbb{R}^d$ which are arbitrarily close ** to R but not within R. If appropriate, we also include in S the "points at infinity" (for $x \to \infty$) on the boundary S_∞^{d-1} of the compactified version of $\mathbb{R}^d$. For all points $\bar{x}$ in S least one of the following properties must hold:

(i) $\bar{x} \in S_\infty^{d-1}$;

(ii) the operator $v(\bar{x})$ is not defined or v is not continuous in $\bar{x}$;

(iii) the point $\bar{x}$ belongs to the set S_0 of points where $v(\bar{x})$ is a constant operator. Then $\|v(\bar{x}) - \langle v(\bar{x})\rangle\| = 0$ such that $\tilde{\varphi}(\bar{x})$ may not be defined ((4.1) violated);

(iv) the point $\bar{x}$ belongs to the set $S_=$ of "identifiable points". This means that there exists a point $y \in R \cup S_=$, $y \neq \bar{x}$, such that $\|\tilde{\varphi}(\bar{x}) - \tilde{\varphi}(y)\| = 0$ ((4.5) violated);

(v) the point $\bar{x}$ belongs to the set S_∞ for which $G(\bar{x}, y) = 0$ for some $y \in R \cup S_\infty$ ((4.6) violated).

Case (i) can always be eliminated by a change of coordinates $x = F(x')$ which preserves the topology of $\mathbb{R}^d$ but maps S_∞^{d-1} into a hypersphere with finite radius. For simplicity we work in this section with coordinates where S has no points at infinity. The formulation in terms of the original coordinates can be recovered at the end by inverting the coordinate change. We will also omit possible boundaries of the type (ii) and assume that v is continuous everywhere in $R \cup S$.

For purposes of topology we only need the existence of $G(x, y)$ but not necessarily (4.1). It is sufficient that $\lim_{x \to \bar{x}} G(x, y)$ and $\lim_{x \to \bar{x}} \lim_{y \to \bar{y}} G(x, y)$ are uniquely defined for all $\bar{x}, \bar{y} \in S_0$ and $x, y \in R$. This allows one to extend the

* These singularities may also be an obstruction to find an "atlas" for R consistent with an embedding in a higher-dimensional space. (The two-dimensional cone cannot be smoothly embedded in $\mathbb{R}^3$.) Then R is a manifold only with the "atlas" resulting from its embedding in $\mathbb{R}^d$.

** We use here the canonical distance in $\mathbb{R}^d$. This also applies to the dimensionality of subsets of S which are treated as subsets of $\mathbb{R}^d$. From the point of view of the manifold R (without embedding in $\mathbb{R}^d$) the boundary has, of course, always dimension $d-1$.

definition of G to include S_0. Indeed, the conditions (4.5) and (4.6) can be formulated in terms of the correlation $G(x, y)$,

$$G(x, y) = 1 \Leftrightarrow x = y, \tag{6.1}$$
$$G(x, y) > 0. \tag{6.2}$$

Let us consider two curves $\mathscr{C}_1$, $\mathscr{C}_2$ in R which intersect at the boundary in $\bar{x} \in S_0$ We choose $x \in \mathscr{C}_1$, $y \in \mathscr{C}_2$ and observe that along each curve the limits $\lim_{x \to \bar{x}} G(x, z)$ and $\lim_{y \to \bar{x}} G(y, z)$ exist since G is bounded between zero and one and $v(x)$ is continuous in $\bar{x}$. The question is only if $G(\bar{x}, z)$ can be defined independently of the choice of curve. A necessary and sufficient condition is the existence of

$$\lim_{x \to \bar{x}} \lim_{y \to \bar{x}} G(x, y) = G(\bar{x}, \bar{x}) \tag{6.3}$$

independent of the curves $\mathscr{C}_1$, $\mathscr{C}_2$ and the order of limits. In this case one has $G(\bar{x}, \bar{x}) = 1$. The inequality

$$\lim_{x \to \bar{x}} (1 - G(x, z))^{1/2} \leqslant \lim_{y \to \bar{x}} (1 - G(y, z))^{1/2} + \lim_{x \to \bar{x}} \lim_{y \to \bar{x}} (1 - G(x, y))^{1/2} \tag{6.4}$$

implies that $G(\bar{x}, z)$ is defined uniquely and continuous at $\bar{x}$. For two points $\bar{x}$, $\bar{y} \in S_0$ where $G(\bar{x}, \bar{x})$ and $G(\bar{y}, \bar{y})$ are uniquely defined, the limit

$$G(\bar{x}, \bar{y}) = \lim_{x \to \bar{x}} \lim_{y \to \bar{y}} G(x, y) \tag{6.5}$$

is also uniquely defined. Furthermore, the existence of the limit (6.3) is equivalent to the existence of

$$\lim_{x \to \bar{x}} \tilde{\varphi}(x) = \tilde{\varphi}(\bar{x}). \tag{6.6}$$

The independence of (6.6) from the limiting procedure follows from

$$\lim_{x \to \bar{x}} \lim_{y \to \bar{x}} \|\tilde{\varphi}(x) - \tilde{\varphi}(y)\| = 0. \tag{6.7}$$

In consequence, we can weaken the condition (4.1) and include in R all points where (6.3) is defined. Possible remaining points where the limit (6.3) is not defined uniquely form the boundary set $\hat{S}_0$ and will not be considered further.

We next turn to the "natural boundaries" $S_=$ and S_∞. Unless $S_=$ is empty there will be at least two distinct points in $\hat{R} = R \cup S_=$ between which the distance vanishes. We will identify two points in R or S whenever $G(x, \bar{y}) = 1$ or $G(\bar{x}, \bar{y}) = 1$. Indeed, $G(x, y)$ remains well defined after this identification for all $x, y \in \hat{\mathrm{R}} \cup \mathrm{S}_\infty$. Inequality (4.13) implies

$$G(\bar{x}, \bar{y}) = 1 \Rightarrow G(x, \bar{x}) = G(x, \bar{y}) \tag{6.8}$$

for all x where G is defined. Similarly, the continuity of G at the identified points follows from the continuity of G in $\mathrm{R} \cup \mathrm{S}$ without identification and (6.8).

As a consequence of the identification the set $\hat{\mathrm{R}}$ can, in general, intersect itself. We will concentrate on the case where $\hat{\mathrm{R}}$ is not self-intersecting. This requires, in particular, that no point in R can be identified with a point in $\mathrm{S}_=$. As a necessary condition we therefore demand that for all $x \in \mathrm{R}$, $y \in \mathrm{R} \cup \mathrm{S}$ there is a constant $\bar{\epsilon}$ such that for all $\epsilon < \bar{\epsilon}$ there exists a continuous function $\delta(\epsilon)$ which tends to zero with ϵ and has the property

$$G(x, y) \geqslant 1 - \epsilon \Rightarrow (x^\mu - y^\mu)(x_\mu - y_\mu) \leqslant \delta^2(\epsilon). \tag{6.9}$$

This condition is stronger than (6.1). It guarantees that there is no point of intersection within R but it is not yet sufficient to exclude such points in S. For a generalization of condition (6.9) we consider in $\mathscr{H}^2$ all operators $\tilde{\varphi}(x)$ whose distance from a given operator $\tilde{\varphi}(\bar{x})$ is smaller than ϵ, $x, \bar{x} \in \hat{\mathrm{R}} \cup \mathrm{S}_\infty$,

$$\mathrm{U}_\epsilon(\tilde{\varphi}(\bar{x})) = \{\tilde{\varphi}(x), \|\tilde{\varphi}(x) - \tilde{\varphi}(\bar{x})\| < \epsilon\}. \tag{6.10}$$

The set $\hat{\mathrm{R}}$ is non-selfintersecting if for all $\bar{x}$ there is a value ϵ such that there exists an invertible map from U_ϵ to some local neighbourhood $\hat{\mathrm{U}}$ of a point in $\mathbb{R}^d$. In this case we consider $\hat{\mathrm{U}}$ as a coordinate patch which overlaps one or several parts of R which are "close to $\bar{x}$" in the sense of (6.10). If $\hat{\mathrm{R}}$ is not selfintersecting it can be made a manifold M by use of an atlas consisting of patches covering R plus patches $\hat{\mathrm{U}}$ covering the identified points.

We conclude that $\hat{\mathrm{R}}$ has S_∞ as the only possible natural boundary. On this boundary the distance D tends to infinity or to $D_{\max}$, depending on the choice of $\mu(D)$. The correlation $G(x, y)$ is continuous everywhere in $\hat{\mathrm{R}} \cup \mathrm{S}_\infty$. It defines completely the topology of the set $\hat{\mathrm{R}}$. We emphasize that the topology of a manifold M cannot be reduced to either the topology of the parameter space which is a submanifold of $\mathbb{R}^d$ or the topology of the operator space $\mathscr{H}^2$. It is a property of the mapping $\tilde{\varphi}$ from parameter space to operator space which selects a d-dimensional submanifold of $\mathscr{H}^2$.

We finally mention that $\hat{\mathrm{R}}$ is a riemannian manifold if

(i) R is a riemannian manifold;

(ii) $\hat{\mathrm{R}}$ is not self-intersecting;

(iii) The identification of points in $S_{\equiv}$ does not induce "true singularities" (cusps, edges etc) in $\hat{R}$. (Coordinate singularities are tolerated.)

In this context we observe that the condition (4.1) for the definition of a field may be weakened without affecting the metric structure. We may include in R_M all points $\bar{x}$ where $g_{\mu\nu}(\bar{x})$ is uniquely defined as $\lim_{x \to \bar{x}} g_{\mu\nu}(x)$ even though $\|v(\bar{x}) - \langle v(\bar{x})\rangle\|$ may vanish. (This implies that the limit (6.3) also exists at $\bar{x}$.) If the metric is defined everywhere in $S_=$ the identification is always consistent with the metric structure. (This follows from the definition of the metric in terms of G (5.1) and (6.8).) A new singularity at $\bar{x} \in S_=$ can only arise if $\lim_{x \to \bar{x}} g_{\mu\nu}(x)$ is not defined uniquely or if $g_{\mu\nu}$ is not infinitely often differentiable in $\bar{x}$.

In summary, riemannian manifolds with arbitrary dimension and arbitrary topology can arise in our formalism. Every d-dimensional manifold can be mapped on an open region R within $\mathbb{R}^d$ after cutting out appropriate sets of points with dimension smaller than d (cf. the discussion in ref. [7]). The definition of topology in terms of the correlation G "repairs" these cuts by gluing appropriate pieces together as a result of the identification of points between which the distance vanishes. Moreover, our formalism can describe sets or manifolds which are not riemannian manifolds because of singularities of the curvature tensor. Such spaces have been discussed as internal spaces in higher-dimensional gravity ("noncompact spaces" [5]) and string theories ("orbifolds" [6]). Beyond that, the formalism is even flexible enough to accommodate intersecting sets of points. We will, however, restrict the following discussion to manifolds.

So far we have chosen to embed R in $\mathbb{R}^d$. This is not necessary. We could replace $\mathbb{R}^d$ by any other d-dimensional parameter manifold as, for example, the torus T^d. Since $\tilde{\varphi}$(or G, $g_{\mu\nu}$) need not to be continuous everywhere on T^d there is no additional requirement like periodicity which would restrict the choice of prefields. Discontinuities or the vanishing of G act as "cuts" on the parameter manifold. For every prefield $\tilde{\varphi}$ defined in an open region $R \subseteq \mathbb{R}^d$ there exists an equivalent prefield $\tilde{\varphi}_T$ defined in $R_T \subset T^d$ which leads to the same topology in terms of G (and vice versa). The parameter space is a purely auxiliary construction. Its only relevant geometrical content is its dimension. If we deal with a manifold M it is often convenient for practical purposes to choose a parameter space with the same topology as induced by the correlation. The identification of points in $S_=$ is then in exact correspondence to the identification of parameters in different coordinate patches of the parameter manifold.

We finally note that topology and geometry are entirely determined by the behaviour of v in R. (The properties of the boundaries can be infered from it.) In the following discussion we will simply omit the boundaries of R with the understanding that one should deal with S as described in this section. If appropriate, disconnected parts of R may be viewed as different coordinate patches describing a d-dimensional manifold. Instead of overlapping patches we use here the equivalent concept of identification of boundary points, with the understanding that overlapping patches are provided as discussed above.

7. Fields as embeddings in operator space

As we have seen the coordinates x play only an auxiliary role. We should be able to describe the geometrical concepts in a coordinate free setting. From a more abstract point of view our attempt to describe topology, distance and metric in terms of the correlation G leads to the following general construction: We deal with (continuous) embeddings of a d-dimensional manifold M into a subspace of the infinite-dimensional space of operators with unit norm $\bar{\mathscr{H}}^2$. This subspace, which we denote by $\bar{\mathscr{H}}^2_+$, is orthogonal to the constant operator

$$\langle \tilde{\varphi} \rangle = 0, \tag{7.1}$$

and has the additional property that two operators $\tilde{\varphi}$, $\tilde{\varphi}'$ are never orthogonal with

$$\langle \tilde{\varphi}, \tilde{\varphi}' \rangle > 0. \tag{7.2}$$

Since $\mathscr{H}^2$ is a Hilbert space we can choose an orthonormal basis 0_i (5.11),

$$\tilde{\varphi} = \sum_i a_i 0_i, \tag{7.3}$$

such that $\bar{\mathscr{H}}^2_+$ is defined by the conditions

$$\sum_i a_i^2 = 1, \qquad \sum_i a_i a_i' > 0. \tag{7.4}$$

The conditions (7.4) are invariant under (infinite-dimensional) orthogonal transformations in $\mathscr{H}^2$. Using appropriate rotations we can always find a basis where the second condition in (7.4) reads

$$\begin{aligned} &0 < a_1 \leqslant 1, \\ &0 \leqslant a_j < 1 \quad \text{for } j > 1. \end{aligned} \tag{7.5}$$

On the boundary of $\bar{\mathscr{H}}^2_+$ at least one of the a_i vanishes. An arbitrary finite-dimensional manifold can be embedded in $\bar{\mathscr{H}}^2_+$. For the given manifold there exist infinitely many possible embeddings. A normalized prefield $\tilde{\varphi}$ specifies a particular embedding $\mathrm{M} \to \bar{\mathscr{H}}^2_+$. The metric structure (5.5) is induced by the natural metric in $\bar{\mathscr{H}}^2_+$ which in turn is induced by the natural metric in $\mathscr{H}^2$,

$$\mathrm{d}s^2 = \delta^{ij}\,\mathrm{d}a_i\,\mathrm{d}a_j. \tag{7.6}$$

Indeed the embedding

$$\tilde{\varphi}(x) = \sum a_i(x) 0_i \tag{7.7}$$

induces

$$ds^2 = \sum_i \frac{\partial a_i}{\partial x^\mu} \frac{\partial a_i}{\partial x^\nu} dx^\mu\, dx^\nu. \tag{7.8}$$

The metric

$$g_{\mu\nu} = \sum_i \frac{\partial a_i}{\partial x^\mu} \frac{\partial a_i}{\partial x^\nu} = \langle \partial_\mu \tilde{\varphi}(x) \partial_\nu \tilde{\varphi}(x) \rangle \tag{7.9}$$

coincides with (5.5) for $\mu_0 = 1$. A riemannian manifold corresponds to an infinitely often differentiable embedding. In particular, all partial derivatives $\partial_\mu a_i$ are continuous in appropriate coordinate patches. In addition, the embedding must fulfill the condition (5.7) for the linear independence of the derivative operators,

$$b^\mu b^\nu \sum_i \partial_\mu a_i \partial_\nu a_i = 0 \Rightarrow b^\mu = 0. \tag{7.10}$$

A normalized field $\tilde{\varphi}$ defined in a riemannian manifold M specifies a particular embedding of this type.

We note that only the particular choice of $\mu(D)$ (4.21) defines the distance D in M as the canonical distance in $\overline{\mathscr{H}}{}^2_+$. This choice of $\mu(D)$ is obviously not suitable for the description of "large distances". Although the "geometry at small distances" and the topology can be infered from the scalar product (2.4) in $\mathscr{H}^2$ this will not hold for the "geometry at large distances". The large-distance geometry is related to particular properties of almost orthogonal operators which will be discussed in sect. 9.

We finally emphasize that our simple geometrical construction only arises on the level of the *normalized* (pre)field $\tilde{\varphi}$. The manifold M is not given a priori when we define the prefield v which is an embedding in $\mathscr{H}^2$. The structure of M can only be determined a posteriori once we have mapped the prefield v into $\tilde{\varphi}$ according to (4.3). The map $v \to \tilde{\varphi}$ from $\mathscr{H}^2$ to $\overline{\mathscr{H}}{}^2_+$ may lead to singularities (for $\|v(\bar{x}) - \langle v(\bar{x})\rangle\| = 0$ and $\tilde{\varphi}(\bar{x})$ not defined) or induce additional identifications (if $\|\tilde{v}(x) - c\tilde{v}(y)\| = 0$ for some constant c, $\tilde{v} = v - \langle v \rangle$). It can also smoothen out singularities of the original embedding v.

8. General coordinate transformations and symmetries of the metric

We are free to change the coordinates of the parameter space R,

$$x^\mu = F^\mu(x'), \tag{8.1}$$

$$v(x) \to v'(x') = v(F(x')). \tag{8.2}$$

The map F: $\mathrm{R}' \to \mathrm{R}$ is in general a map between two different regions in $\mathbb{R}^d$. (We require F to be continuous, differentiable and invertible everywhere in R′ and R except for zero measure sets.) If F is a one-to-one map $\mathrm{R} \to \mathrm{R}$ it defines a general coordinate transformation. Its action on the operators $v(x)$ obtains by replacing x' by x on the r.h.s. of (8.2),

$$v(x) \to v'(x) = v(F(x)). \tag{8.3}$$

Under infinitesimal transformations,

$$F^{\mu}(x) = x^{\mu} - \xi^{\mu}(x), \tag{8.4}$$

a field $v(x)$ or $\tilde{\varphi}(x)$ transforms as a scalar and their derivatives as vectors,

$$\begin{aligned} v'(x) &= v(x) + \delta v(x), \\ \delta v &= -\xi^{\mu}\partial_{\mu}v, \\ \delta\partial_{\mu}v &= -\partial_{\mu}\xi^{\nu}\partial_{\nu}v - \xi^{\nu}\partial_{\nu}\partial_{\mu}v. \end{aligned} \tag{8.5}$$

This implies immediately that the metric has the standard transformation property under general coordinate transformations (gen_d),

$$\delta g_{\mu\nu} = -\partial_{\mu}\xi^{\rho}g_{\rho\nu} - \partial_{\nu}\xi^{\rho}g_{\mu\rho} - \xi^{\rho}\partial_{\rho}g_{\mu\nu}. \tag{8.6}$$

Two fields $v(x)$ and $v'(x)$ related by (8.3) lead to equivalent geometries. The induced metrics $g_{\mu\nu}$ and $g'_{\mu\nu}$ differ only by a general coordinate transformation.

Symmetries of the metric $g_{\mu\nu}$ arise if there exist Killing vectors $\xi^{\rho}(x) = \theta^{z}K_{z}^{\rho}(x)$ such that the associated infinitesimal transformations leave $g_{\mu\nu}$ invariant,

$$\partial_{\mu}K_{z}^{\rho}g_{\rho\nu} + \partial_{\nu}K_{z}^{\rho}g_{\mu\rho} + K_{z}^{\rho}\partial_{\rho}g_{\mu\nu} = 0. \tag{8.7}$$

A symmetry of the correlation function,

$$G'(x, y) = G(x, y), \tag{8.8}$$

can also be realized by an appropriate general coordinate transformation (8.3). By the construction (5.1) every symmetry of the correlation function induces a corresponding symmetry of the metric. For example, if G depends only on $(x^{\mu} - y^{\mu}) \times (x_{\mu} - y_{\mu})$ with $\mathrm{R} = \mathbb{R}^d$ the metric is invariant under d-dimensional translations and rotations.

A convenient way to realize a symmetry of G is the association of a coordinate transformation F with a variable transformation s which leaves the scalar product invariant (2.13). As a simple example we consider, with $N=d$,

$$p(s)=\exp\left\{-\pi\sum_{\mu=1}^{d}(s^\mu)^2\right\}=\exp(-\pi s_\mu s^\mu), \tag{8.9}$$

$$v(x)=\exp\left\{-\frac{\pi}{2\eta}(s^\mu-x^\mu)(s_\mu-x_\mu)\right\}, \tag{8.10}$$

where v is a field in the whole $\mathbb{R}^d$. Obviously, p is invariant under rotations of the s^μ and the effect ot this rotation on v corresponds to a rotation of x^μ. The resulting correlation function G and metric $g_{\mu\nu}$ must have the symmetry SO(d). Every (variable-) symmetry s which acts nontrivially within the d-dimensional subset of $\mathscr{H}_+^2$ defined by the embedding $\tilde{\varphi}$ leads to a symmetry of G. The inverse is not true: A symmetry of G is not necessarily associated with a variable transformation. We finally mention that gen_d is not a symmetry of the metric. Although gen_d is a subgroup of sgen_N for $N>d$ this subgroup should not act as a transformation $\tilde{\varphi}\to\tilde{\varphi}'$. Otherwise the expectation value (5.5) would have to vanish. Our formulation corresponds to the metric in a fixed gauge.

The variable transformations s which leave the probability density invariant form the group sgen_N. In general, only a small subgroup $\mathrm{G}\subset\mathrm{sgen}_N$ acts among the operators $\tilde{\varphi}(x)$ ($g(\tilde{\varphi}(x))=\tilde{\varphi}'(x)=\tilde{\varphi}(x')$; $x, x'\in\mathrm{R}$, $g\in\mathrm{G}$) whereas most transformations change $\tilde{\varphi}(x)$ into a new field $\tilde{\varphi}'(x)$ ($\hat{s}(\tilde{\varphi}(x))=\tilde{\varphi}'(x)\neq\tilde{\varphi}(x')$, $\hat{s}\in\mathrm{sgen}_N/\mathrm{G}$). Since the scalar product

$$\langle\tilde{\varphi}'(x),\tilde{\varphi}'(y)\rangle=\langle\tilde{\varphi}(x),\tilde{\varphi}(y)\rangle \tag{8.11}$$

as well as all other geometrical constructions used so far remain unchanged, the (pre)fields $\tilde{\varphi}(x)$ and $\tilde{\varphi}'(x)$ (or $v(x)$ and $v'(x)$) define the same geometrical structure. For a given probability density (action) there are infinitely many different fields $\tilde{\varphi}(x)$ which describe the same geometry. The transformations in the quotient $\mathrm{sgen}_N/\mathrm{G}$ act as isomorphisms between equivalent geometrical structures.

9. Long-distance behaviour

The construction of geometry in terms of the correlation function introduces a natural length scale ("fundamental scale") μ_0^{-1}. We fix units by the choice $\mu_0=1$ such that an infinitesimally small distance coincides with the distance induced by the scalar product in $\mathscr{H}^2$. As we have seen in the examples at the end of sect. 5 it is not always possible to reach large distances. We will now turn our attention to fields which permit a long-distance behaviour.

Consider $G(x, y)$ as a function of y for a fixed point x. For y in the vicinity of x the correlation decreases in all directions from the maximum at $y=x$. The decrease continues either until G reaches zero or y hits the boundary of R for $G(x, y)=G_{\mathrm{cr}}$ or else a saddle point or minimum appears for some critical value G_{cr}. We denote by $\mathrm{R}_{\mathrm{mon}}(x)$ the region in R around x for which G is larger than this critical value,

$$R_{\mathrm{mon}}(x)=\{y, G(x, y)>G_{\mathrm{cr}}\},$$

$$G_{\mathrm{cr}} \geqslant 0. \tag{9.1}$$

We may represent $\mathrm{R}_{\mathrm{mon}}$ as the sum of "equicorrelation surfaces",

$$\mathrm{R}_{\bar{G}}(x)=\{y, G(x, y)=\bar{G}\}, \tag{9.2}$$

such that the boundary of $\mathrm{R}_{\mathrm{mon}}$ corresponds to $\mathrm{R}_{G_{\mathrm{cr}}}$. We assume that the geodesic distance D_{g} is finite everywhere in $\mathrm{R}_{\mathrm{mon}}$. Let us define the function $\bar{D}_{\mathrm{g}}(\bar{G})$ as the minimum value of the geodesic distance between x and a point y within $\mathrm{R}_{\bar{G}}(x)$. This establishes $\bar{D}_{\mathrm{g}}(x, y)$ as a function of $G(x, y)$ with

$$\bar{D}_{\mathrm{g}}(x, y) \leqslant D_{\mathrm{g}}(x, y). \tag{9.3}$$

Since the inequality in (9.3) is saturated for a particular point y for every value of $\bar{D}_{\mathrm{g}}$ we conclude from (5.10) that $\bar{D}_{\mathrm{g}}$ corresponds to the maximal value that the distance D could possibly taken for a given value of $G>G_{\mathrm{cr}}$.

As a first requirement for a field to admit a long-distance behaviour we demand that $\bar{D}_{\mathrm{g}}(G_{\mathrm{cr}})$ becomes much larger than one for at least one point $x \in \mathrm{R}$,

$$\bar{D}_{\mathrm{g}}(G_{\mathrm{cr}})>D_{\mathrm{l.d.}} \gg 1 \tag{9.4}$$

(For definiteness we may take for $D_{\mathrm{l.d.}}$ at least 10.) For a finite-dimensional sphere with unit radius there is a maximal value of $\bar{D}_{\mathrm{g}}$. This explains the impossibility to construct arbitrarily large distances with a finite number of orthonormal operators (our examples in sect. 5). On the other hand the *infinite-dimensional* unit sphere admits lines with infinite length between two points x and y on the boundary of $\mathscr{K}_{+}^{2}$, such that $G(x, z)$ decreases continuously between one and zero for z moving from x to y.

We may illustrate the long-distance behaviour of fields with two examples. For the first one we consider infinitely many orthonormal operators (4.11) which we order with an index n^{μ}, $\mu=1, \ldots, d$; $-\infty<n_{\mu}<\infty$. We define

$$\tilde{\varphi}(x)=c_{k}(x) \sum_{\{n^{\mu}\}} f_{k}(x^{\mu}-n^{\mu}) 0_{n^{\mu}}. \tag{9.5}$$

Here the function f_k depends only on $(x^\mu - n^\mu)(x_\mu - n_\mu)$ and decreases rapidly if this argument becomes much larger than k^2, for example

$$f_k(x^\mu - n^\mu) = (k^2/\pi)^{d/2} \exp\{-k^2(x^\mu - n^\mu)(x_\mu - n_\mu)\}. \tag{9.6}$$

The normalization

$$c_k(x) = (\pi/k^2)^{d/2}\left(\sum_{\{n_\mu\}} \exp\{-2k^2(x^\mu - n^\mu)(x_\mu - n_\mu)\}\right)^{-1/2} \tag{9.7}$$

is periodic in all x^μ. The correlation function

$$\begin{aligned} G(x, y) &= \sum_{\{n_\mu\}} f_k(x^\mu - n^\mu) f_k(y^\mu - n^\mu) c_k(x) x_k(y) \\ &= \frac{c_k(x) c_k(y)}{c_k^2(\frac{1}{2}(x+y))} \exp\{-\tfrac{1}{2}k^2(x^\mu - y^\mu)(x_\mu - y_\mu)\} \end{aligned} \tag{9.8}$$

vanishes only for $(x-y)^2 \to \infty$. For $k^2 \ll 1$ the normalization c_k becomes almost independent of x ($c_k \approx (2\pi/k^2)^{d/4}$). In this limit the metric becomes flat,

$$\begin{aligned} g_{\mu\nu}(x) &= \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} G(x, y)\bigg|_{x=y} \\ &= k^2 \delta_{\mu\nu} - \tfrac{1}{2}\partial_\mu \partial_\nu \ln c_k(x). \end{aligned} \tag{9.9}$$

On the other hand, for k^2 in the vicinity of one the metric reflects strongly the discreteness of the operators O_n^μ since the periodic second term in (9.9) becomes important. For fields with a long-distance behaviour we can often find related "average fields" [9],

$$\tilde{\phi}_{k'}(x) = C_{k'}(x) \int \mathrm{d}y\, f_{k'}(x-y) \tilde{\varphi}(y), \tag{9.10}$$

which lead to a smoothening of the metric for $k' \ll 1$. Average fields are useful in order to have a smooth metric in a discrete setting as in lattice theories.

For $k^2 \ll 1$ we finally note that we can choose a distance $D(G)$ in close analogy to (4.17) and (4.18),

$$\begin{aligned} &D \approx D_g \approx kr, \\ &\mu(D) = \left\{\frac{2}{D^2}\left(\exp \tfrac{1}{2}D^2 - 1\right)\right\}^{1/2}. \end{aligned} \tag{9.11}$$

The function $\mu(D)$ is defined for all positive D and condition (4.24) is fulfilled.

Secondly, we continue the example of the last section (8.9) (8.10), which leads to

$$\tilde{\varphi}(x) = c(x)\left\{\exp\left[-\frac{\pi}{2\eta}(x^\mu - s^\mu)(x_\mu - s_\mu)\right]\right.$$
$$\left. -\left(\frac{2\eta}{1+2\eta}\right)^{d/2} \exp\left[-\frac{\pi}{1+2\eta}x^\mu x_\mu\right]\right\}, \tag{9.12}$$

with

$$c(x) = \left\{\left(\frac{\eta}{1+\eta}\right)^{d/2} \exp\left(-\frac{\pi}{1+\eta}x^\mu x_\mu\right)\right.$$
$$\left. -\left(\frac{2\eta}{1+2\eta}\right)^{d} \exp\left(-\frac{2\pi}{1+2\eta}x^\mu x_\mu\right)\right\}. \tag{9.13}$$

The correlation function

$$G(x, y) = c(x)c(y)\left\{\left(\frac{\eta}{1+\eta}\right)^{d/2} \exp\left(-\frac{\pi}{1+\eta}\Sigma^2\right) \exp\left(-\frac{\pi}{\eta}\Delta^2\right)\right.$$
$$\left. -\left(\frac{2\eta}{1+2\eta}\right)^{d} \exp\left(-\frac{2\pi}{1+2\eta}(\Sigma^2 + \Delta^2)\right)\right\}, \tag{9.14}$$

$$\Sigma^2 = \tfrac{1}{4}(x^\mu + y^\mu)(x_\mu + y_\mu), \qquad \Delta^2 = \tfrac{1}{4}(x^\mu - y^\mu)(x_\mu - y_\mu), \tag{9.15}$$

vanishes for

$$\Delta^2 = \frac{\eta}{1+\eta}\Sigma^2 + \eta(1+2\eta)\frac{d}{2\pi}\ln\left(1 + \frac{1}{4\eta(1+\eta)}\right). \tag{9.16}$$

This correlation is positive everywhere within the region

$$\mathrm{R} = \left\{x;\ x^\mu x_\mu < x_c^2\right\},$$
$$x_c^2 = \frac{\eta(1+2\eta)d}{2\pi}\ln\left(1 + \frac{1}{4\eta(1+\eta)}\right), \tag{9.17}$$

and vanishes only on the boundary at "opposite points" $G(x, -x) = 0$ for $x^\mu x_\mu =$

x_c^2. Carrying out the construction of $R_{\mathrm{mon}}(0)$ one finds a nonvanishing critical value for G_{cr}. We concentrate on $\eta \ll 1$ where

$$x_c^2 \approx \frac{\eta d}{2\pi} \ln \frac{1}{4\eta} \ll \frac{1}{\pi}. \tag{9.18}$$

In this limit one has

$$\tilde{\varphi}(x) \approx \eta^{-d/4} \exp\left\{ -\frac{\pi}{2\eta}(x^\mu - s^\mu)(x_\mu - s_\mu) \right\} - (4\eta)^{d/4}, \tag{9.19}$$

$$G(x, y) \approx \exp\left\{ -\frac{\pi}{4\eta}(x^\mu - y^\mu)(x_\mu - y_\mu) \right\} - (4\eta)^{d/2}, \tag{9.20}$$

$$G_{\mathrm{cr}} \approx (4\eta)^{d/8}. \tag{9.21}$$

The metric is given by

$$g_{\mu\nu} = \frac{\pi}{2\eta} \delta_{\mu\nu}, \tag{9.22}$$

and the geodesic distance between $x = 0$ and $y = x_c$,

$$D_g = \overline{D}_g = \left(\frac{d}{4} \ln \frac{1}{4\eta} \right)^{1/2}, \tag{9.23}$$

can be large only for exponentially small values of η. We conclude that every function $\mu(D)$ (4.10) for which D fulfills the distance inequality must diverge for $D_{\max} \leqslant (d \ln 1/4\eta)^{1/2}$. In this example the impossibility to define μ for all D is closely related to the fact that $\langle \varphi(x)\varphi(y) \rangle$ decreases faster than $\langle \varphi(x) \rangle \langle \varphi(y) \rangle$ for large values of Δ^2.

These problems lead us to a second requirement for fields admitting a long-distance behaviour: We demand that there exists a function $\mu(D)$ which remains finite for all positive D such that D fulfills the distance inequality everywhere in R. Then $D(G)$ is an invertible function for all positive G with the property

$$\lim_{G \to 0} D(G) \to \infty. \tag{9.24}$$

The example (9.5) fulfills this criterion for sufficiently small k (9.11). The further development of the concept of space within general statistics will involve a much more detailed investigation of the properties of fields which admit a long-distance behaviour.

10. Metric from correlation function of free scalar field theory

In the preceding sections we have shown that geometrical concepts can be formulated under very general circumstances. So far we have chosen very simple examples in order to demonstrate this generality. Nevertheless, our concepts are directly applicable to realistic physical theories as field theory or string theory. In this section we construct explicitly the metric $g_{\mu\nu}$ from the correlation function of a free scalar field.

In flat d-dimensional euclidean space the action for a scalar field,

$$\chi(x) = \sum_q \exp(-iq_\mu x^\mu)\chi(q),$$

$$\chi(-q) = \chi^*(q), \tag{10.1}$$

reads

$$S = \tfrac{1}{2}\Omega \sum_q z_k(q)(q^2 + m^2)\chi^*(q)\chi(q),$$

$$p = \exp(-S). \tag{10.2}$$

We work on a torus with finite volume Ω. (One may take $\Omega \to \infty$ at the end.) As long as we impose a sharp cutoff Λ with $q^2 \leqslant \Lambda^2$ ($q^2 = \delta^{\mu\nu} q_\mu q_\nu$), the number of degrees of freedom $\chi(q)$ (which correspond to s^u in sect. 2) remains finite. For a finite number of degrees of freedom the scalar field $\chi(x)$ is a normalizable operator.

$$\langle \chi^2(x) \rangle = Z^{-1} \int \mathscr{D}\chi\, \chi^2(x) \exp(-S),$$

$$Z = \int \mathscr{D}\chi \exp(-S) \tag{10.3}$$

(Functional integration stands here as a shorthand for integration over all modes $\chi(q)$.) One finds

$$\begin{aligned}\langle \chi^2(x) \rangle &= \sum_q \sum_{q'} \exp\left[-i(q_\mu - q'_\mu)x^\mu\right] \langle \chi(q)\chi^*(q') \rangle \\ &= \sum_{\substack{q \\ q^2 \leqslant \Lambda^2}} G(q)\Omega^{-1} \\ &= (2\pi)^{-d} \int_{q^2 < \Lambda^2} \mathrm{d}^d q\, G(q),\end{aligned} \tag{10.4}$$

with

$$\langle \chi(q)\chi^*(q')\rangle = G(q)\delta_{qq'}\Omega^{-1},$$

$$G^{-1}(q) = z_k(q)(q^2+m^2) \tag{10.5}$$

(In the last expression in (10.4) we have taken $\Omega \to \infty$.)

In the limit $\Lambda \to \infty$ the number of degrees of freedom becomes infinite. An operator $v(x) \in \mathscr{K}^2$ in general statistics must remain normalizable in this limit. For $z_k(q) = 1$, however, $\chi(x)$ ceases to be normalizable for $\Lambda \to \infty$. For a definition of a normalizable operator we may either introduce a smooth momentum cutoff by choosing $z_k(q) = f_k^{-2}(q)$ or keep $z_k = 1$ and introduce averaged field operators

$$\phi_k(x) = \int \mathrm{d}^d y\, f_k(y-x)\chi(y),$$

$$\phi_k(q) = f_k(q)\chi(q). \tag{10.6}$$

Here $f_k(x-y)$ is given by (9.6) and $f_k(q)$ corresponds to the Fourier transform of $f_k(x)$,

$$f_k(q) = \exp\left(-\frac{q^2}{4k^2}\right). \tag{10.7}$$

In both cases one finds

$$\begin{aligned} c_k^{-2} &= \left\langle \phi_k^2(x) \right\rangle_{z_k=1} = \langle \chi^2(x)\rangle_{z_k=f_k^{-2}} \\ &= (2\pi)^{-d}\int \mathrm{d}^d q\ (q^2+m^2)^{-1} \exp\left(-\frac{1}{2}\frac{q^2}{k^2}\right) \\ &= 2v_d k^{d-2}\int_0^\infty \mathrm{d}y\, y^{d/2-1}\left(y+\frac{m^2}{k^2}\right)^{-1} \exp(-\tfrac{1}{2}y) \\ &= 2v_d k^{d-2} I_d\left(\frac{m^2}{k^2}\right), \end{aligned} \tag{10.8}$$

with

$$v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(\tfrac{1}{2}d),$$

$$I_3(0) = \sqrt{2\pi}, \qquad I_4(0) = 2,$$

$$
\begin{aligned}
&I_{d+2}(0)=(d-2)I_d(0) \qquad \text{for } d>2,\\
&I_d\left(\frac{m^2}{k^2}\right)=I_d(0)-\frac{m^2}{k^2}I_{d-2}\left(\frac{m^2}{k^2}\right) \qquad \text{for } d>2,\\
&I_2\left(\frac{m^2}{k^2}\right)=\ln\frac{k^2}{m^2} \qquad \text{for } k^2\gg m^2,\\
&I_1\left(\frac{m^2}{k^2}\right)=\pi k/m \qquad \text{for } k^2\gg m^2.
\end{aligned} \tag{10.9}
$$

The operator family $\phi_k(x)$ (or $\chi(x)$ with $z_k=f_k^{-2}$) obeys all criteria for a field. The metric therefore follows directly from (5.5),

$$
\begin{aligned}
g_{\mu\nu}&=\mu_0^{-2}c_k^2\Omega^{-1}\sum_q q_\mu q_\nu\left(q^2+m^2\right)^{-1}f_k^2(q)\\
&=A_d\delta_{\mu\nu},\\
A_d&=\frac{k^2}{d\mu_0^2}\left(\frac{I_{d+2}(0)}{I_d(m^2/k^2)}-\frac{m^2}{k^2}\right).
\end{aligned} \tag{10.10}
$$

If we choose $k^2\gg m^2$ we may expand

$$
\begin{aligned}
A_3&=\frac{k^2}{3\mu_0^2}\left(1+\sqrt{\frac{\pi}{2}}\,\frac{m}{k}+\dots\right),\\
A_4&=\frac{k^2}{2\mu_0^2}\left(1+\frac{m^2}{2k^2}\ln\frac{k^2}{m^2}+\dots\right).
\end{aligned} \tag{10.11}
$$

This metric is flat and we can obtain $g_{\mu\nu}=\delta_{\mu\nu}$ by a suitable choice of k^2/μ_0^2. The integrals have been performed for $\Omega\to\infty$ but all formulae can easily be evaluated on a torus with finite volume as well. The topology is then recovered from the periodicity of $\langle\phi_k(x)\phi_k(y)\rangle$ following the prescription of sect 6.

The derivation of the metric from the Green function in coordinate space requires some care. As is well known, the Green function $G_0(x-y)=\langle\chi(x)\chi(y)\rangle$ diverges for $y\to x$ ($z_k=1$, $\Lambda\to\infty$). The leading terms for $r^2\ll m^{-2}$ ($r^2=\delta_{\mu\nu}(x^\mu-y^\mu)(x^\nu-y^\nu)$) read

$$
G_0(y\to x)=\begin{cases}\dfrac{1}{4\pi|r|}-\dfrac{m}{4\pi}+\dfrac{1}{8\pi}m^2|r| & \text{for } d=3\\[2ex] \dfrac{1}{4\pi^2r^2}+\dfrac{1}{16\pi^2}m^2\ln\dfrac{r^2}{r_0^2} & \text{for } d=4.\end{cases} \tag{10.12}
$$

Using the average field $\phi_k(x)$ (10.6) for a definition of the field $v(x)$ we may naively try to obtain the two-point function by averaging over G_0,

$$\hat{G}(x-y) = c_k^2 \int \mathrm{d}^d z\ \mathrm{d}^d z'\ f_k(x-z) f_k(y-z') G_0(z-z')$$

$$= c_k^2 \int \mathrm{d}^d z\, f_{\tilde{k}}(z) G_0(x-y+z) \tag{10.13}$$

$$c_k^{-2} = \int \mathrm{d}^d z\, f_{\tilde{k}}(z) G_0(z). \tag{10.14}$$

Here we have used the identity

$$\int \mathrm{d}^d z\, f_k(x-z) f_k(z-y) = f_{\tilde{k}}(x-y),$$

$$\tilde{k} = k/\sqrt{2}\,. \tag{10.15}$$

The function $\hat{G}$ is finite and normalized according to (4.9). The metric should follow from differentiation (5.5),

$$g_{\mu\nu} = \hat{A}_d \delta_{\mu\nu},$$

$$\hat{A}_d = -2\mu_0^{-2} c_k^2 \int \mathrm{d}^d r\, f_{\tilde{k}}(r) \left[G_0'(r) + \frac{2}{d} r^2 G_0''(r) \right]. \tag{10.16}$$

(Here we use the fact that G_0 is a function of r^2 and define $G_0' = \partial/\partial r^2 G_0$ etc.) We observe, however, that the leading terms $G_0 \sim r^{2-d}$ cancel in (10.16), whereas the terms $\sim m^2$ give a negative contribution to $\hat{A}_d$! Obviously, something must have gone wrong, since the averaging according to (10.13) leads to $\hat{G} > 1$ for small values of $r^2 \ll m^{-2}$ ($m^2 \ll k^2$). This is impossible for the correlation function of a normalizable operator ϕ_k.

Indeed, the cancellation of the contribution from the leading term in G_0 is very sensitive to the precise behaviour of G_0 for $r \to 0$. We should remember that operators are defined as the limit of infinitely many degrees of freedom ($\Lambda \to \infty$) of a system with a finite number of degrees of freedom (finite Λ). For finite Λ the Green function for χ approaches a constant for $r \to 0$. We regularize the leading term (for $d = 4$),

$$G_0^{(\Lambda)} = \frac{1}{4\pi^2} \frac{1}{r^2 + \epsilon}, \tag{10.17}$$

and take $\epsilon \sim \Lambda^{-2}$ to zero at the end. The normalization (10.14) is not altered by this procedure, but the metric differs from the unregularized expression (10.16) if we use $G_0^{(\Lambda)}$ and take $\epsilon \to 0$ after evaluating $\hat{A}_d$. The regularized procedure reproduces now correctly the leading term for A_4 in (10.11). At this point one may wonder if the necessity of a regularization introduces some ambiguity in the derivation of the metric from the correlation function of ϕ_k. This is not the case, as can be seen easily in momentum space where (10.10) is independent of the precise form how the sum over momenta is cut off for $q^2 \approx \Lambda^2$, $\Lambda^2 \gg k^2$. For any regularization based on an effective momentum cutoff the metric is uniquely defined and, of course, independent of the formulation in coordinate or momentum space.

Infinitesimal distances are invariant under rescalings of the coordinates $x^\mu \to \alpha x^\mu$ since this is a particular coordinate transformation (compare sect. 8). The metric scales $g_{\mu\nu} \to \alpha^{-2} g_{\mu\nu}$, whereas it is invariant under field rescalings $\chi \to \beta\chi$ if μ_0 is kept fixed. On the other hand, the metric depends on the "average scale" k if we take a fixed μ_0. We may wish to define a distance from the Green function of a free scalar field (with standard normalization of the kinetic term) which is independent of the details of the averaging. In this case it is more appropriate to relate μ_0 to the averaging procedure and we choose (for $d > 2$) *

$$\mu_0^2 = b_0 \left[\Omega^{-1} \int \mathrm{d}^d x \, \langle \phi_k^2(x) \rangle \right]^\gamma , \tag{10.18}$$

$$\gamma = \frac{2}{d-2} . \tag{10.19}$$

Here the power γ is fixed by the dimension of ϕ_k. This guarantees that the metric is now invariant under a simultaneous scaling of coordinates and fields which leaves the kinetic term invariant, i.e. $\beta^2 = \alpha^{2-d}$. One obtains

$$\mu_0^2 = b_0 c_k^{-2\gamma} \sim k^2 \tag{10.20}$$

and the metric becomes independent of the scale k. The k-independent constant b_0 may finally be chosen such that $A_d = 1$ for a free massless field with standard kinetic term, i.e.

$$\mu_0^2 = \frac{d-2}{2} k^2 . \tag{10.21}$$

We observe that the choice of scale (10.18) guarantees that we can "measure distances" with the two-point functions of arbitrary scalar fields, as long as their mass is much smaller than k! We expect that this property generalizes to interacting fields, provided the normalization of the kinetic term is fixed properly

* If we want to measure distances in units of the canonical distance in the space of unit norm operators, we have, of course, to keep $\mu_0 = 1$.

at length scales of the order k^{-1} and all relevant additional length scales which may be introduced by the interaction are much smaller than k. The relation $g_{\mu\nu} = A_d \delta_{\mu\nu}$ follows the symmetries of the action for an interacting field theory and the question reduces to the possibility of a convenient and universal choice of the scale factor μ_0. We also believe that our construction of the metric from correlation functions can be generalized to fields carrying spinor indices or Lorentz indices. In string theories the corresponding construction should use suitable vertex operators for $v(x)$. In particular, if string theories are finite, the vertex operators representing $v(x)$ should be normalizable and an averaging involving the scale k could be avoided.

At this point we should mention a particularity of the long-distance behaviour for four-dimensional massless fields. For $r^2 \gg k^{-2}$ the Green function (10.13) is simply proportional to G_0 (10.20),

$$G(r) = c_k^2 G_0(r). \tag{10.22}$$

(The use of a regularized $G_0^{(\Lambda)}$ is here irrelevant.) For a massless field one finds for the distance (4.10) in the limit $r^2 k^2 \gg 1$

$$\frac{D^2 \mu^2(D)}{\mu_0^2} = \frac{2}{\mu_0^2}(G^{-1} - 1) = 4 v_d \frac{d}{d-2} I_d(0) k^{d-4} G_0^{-1}, \tag{10.23}$$

which yields, for $d = 4$,

$$\frac{D^2 \mu^2(D)}{\mu_0^2} = 2r^2. \tag{10.24}$$

In four dimensions D coincides with the geodesic distance r provided $\mu^2(D)/\mu_0^2$ approaches two for $D^2 k^2 \gg 1$. The possibility to define a distance D with a scaling function $\mu(D)/\mu_0$ which becomes independent of all length scales in the limit $D \to \infty$ seems to be particular for four-dimensional massless field theories. We do not know if there is a relation between this property and the description of our world with four-dimensional space-time using a field theory containing the photon as a massless field.

So far we have always worked in this section with field theories in flat space. The reader may suspect that the whole discussion can be reduced to symmetry arguments: "flat space in – flat space out". The main point of this discussion is, however, that we do not need to know that the Green functions come from a field theory which already is formulated using geometrical concepts. The knowledge of the Green function $G_0^{(\Lambda)}$ (10.12),(10.17) is sufficient to construct the geometry!

Furthermore, our construction equally applies to nonflat geometries. For example, we may keep the flat space action (10.2) but introduce modified operators

$$\phi_k(x) = f_k(-D^\mu D_\mu)\chi(x), \tag{10.25}$$

where the function f_k (10.7) now depends on the covariant laplacian with a curved background metric $G_{\mu\nu}$ instead of $f_k(-\partial^\mu\partial_\mu)$ used previously. This "distortion" in the definition of the basic field reflects itself in a "distortion" of the Green function and results in a deviation of $g_{\mu\nu}$ from a flat space metric.

As a distinct possibility we could also use the action of a free field propagating in a curved background characterized by $G_{\mu\nu}$. The modifications of the singular behaviour of the two-point function G_0 in a curved background are well known [8]. Using a regularized expression of the type (10.17) (with r the geodesic distance, best expressed in normal coordinates) and a definition of ϕ_k (10.25) adapted to the same background metric, one can carry out all constructions similar to flat space. For $k^2 \gg m^2$, $k^2 \gg R(G)$, $k^4 \gg R_{\mu\nu\rho\sigma}(G)R^{\mu\nu\rho\sigma}(G)$ etc. one finds from symmetry and dimension considerations (with a suitable normalization of μ_0)

$$\begin{aligned} g_{\mu\nu} = G_{\mu\nu} &+ \frac{c_1}{k^2}R(G)G_{\mu\nu} + \frac{c_2}{k^2}R_{\mu\nu}(G) \\ &+ \frac{c_3}{k^4}R_{\mu\rho\sigma\lambda}(G)R_\nu^{\rho\sigma\lambda}(G) + \dots . \end{aligned} \tag{10.26}$$

Here $R_{\mu\nu\rho\sigma}(G)$ denotes the curvature tensor formed with $G_{\mu\nu}$ (similar for $R_{\mu\nu}(G)$, $R(G)$) and c_i are constants. In principle, $g_{\mu\nu}$ can be used as well as $G_{\mu\nu}$ in order to describe a geometry. If we identify k with the Planck mass M_p the differences are minor as long as the curvature remains small on the Planck scale. On the other hand, the averaging (10.25) may actually smoothen out sufficiently mild singularities of the geometry defined by $G_{\mu\nu}$. In other words, a mild singularity in invariants formed from the curvature tensor $R_{\mu\nu\rho\sigma}(G)$ may not appear in corresponding invariants formed from the curvature tensor $R_{\mu\nu\rho\sigma}(g)$. The nonlinear transformation (10.26) may map a singular background metric $G_{\mu\nu}$ into a regular metric $g_{\mu\nu}$. One may even argue that $g_{\mu\nu}$ corresponds to the "true" metric since in practice one needs correlation functions for a measurement of distances and only a finite resolution can be attained.

11. Conclusions

We have demonstrated how to construct geometry from the two-point correlation function. The concepts of distance, topology, and Riemannian metric are introduced in a very general context. We only use the existence of a field, i.e. a family of operators with norm one which depend in a differentiable way on one or several parameters. Our approach should constitute the fundamental framework for the emergence of space in general probabilistic systems. Besides this, our approach reveals geometrical structures in a multitude of situations in statistical mechanics. As an example, we discuss the Ising model in a constant magnetic field in the appendix.

Our construction exhibits a fundamental length scale $l_0 = \mu_0^{-1}$. After the fundamental constants c and h which are related to the unification of space and time and the duality between location and momentum the fundamental length should be the last fundamental constant in nature. The fundamental length l_0 arises from the "emergence of space". More precisely, it is related to the fact that $\tilde{\varphi}$ has no dimension due to the normalization condition $\|\tilde{\varphi}\| = 1$. This implies that the correlation functions of $\tilde{\varphi}$ are dimensionless. Any relation between a distance and correlation functions must therefore introduce a proportionality constant with the dimension of a length. Putting $\mu_0 = 1$ measures distances in units of the canonical distance in the space of operators with unit norm. We expect that the concept of space looses its universal meaning at distances of the order of l_0. Indeed, two fields with a similar behaviour of their correlation functions at distances sufficiently large compared to l_0 may well have very different correlation functions at distances of the order l_0.

There are infinitely many ways of choosing fields which define a geometrical structure. Which one should be selected to describe "real space" – this brings us back to the question posed in the introduction. A first selection arises from the requirement that the geometry should not only describe distances of the order l_0 but also allow for distances large compared to l_0. In particular, a vanishing connected two-point function should be equivalent to infinite distance. Although these requirements are not realized for generic fields there still remain infinitely many possibilities to select fields which admit such a long-distance behaviour. In particular, the notion of infinitely many degrees of freedom is not yet needed since long distances can even be described with a single degree of freedom (9.5).

Geometry is a structure in the space of operators. As such it does not need the specification of an action. In particular, the field defining the geometry must not coincide with variables in the functional integral. This is not new. A similar situation arises in field theory in a momentum representation of the functional integral or in the representation of string theories as two-dimensional field theories. So far we have only used the Hilbert space structure of the space of all normalizable operators. All geometrical quantities have ultimately been constructed with the help of the scalar product from operators with unit norm in the subspace $\bar{\mathscr{H}}^2$. This construction is independent of the choice of the probability density or action as we have shown by establishing explicitly suitable equivalence transformations. It is also independent of the number of degrees of freedom N since all Hilbert spaces are isomorphic. At this stage, nothing particular happens in the limit $N \to \infty$ which is defined as a limit of sequences with increasing N (for details see ref. [7]) *. We have not yet used the notions of correlatable operators, operator multiplication or the requirement $\langle \tilde{\varphi}(x) \rangle = 0$. They will play a role in the further development of selection criteria for "real space".

If $v(x)$ is a field in R and is correlatable the squared operators $v^2(x)$ or, more generally, $v^{\mathrm{P}}(x)$ may also define fields. They will lead to the same topology but in general induce a different metric. It seems to be reasonable to demand for "real space" that all $v^{\mathrm{P}}(x)$ should lead to the same geometry at least at long distances and for suitable averages of these operators (9.10). This may be generalized to other composite operators like $v\partial_\mu v$ etc. We also may consider several independent fields. We have restricted so far our discussion to the (connected) two-point function. Additional requirements for "real space" should arise from the consideration of higher correlations (cluster properties ...). In addition, one has to introduce a time structure and require the compatibility of space and time.

All this is presumably not sufficient to select "real space" uniquely. In particular, many local field theories will remain as candidates to describe "real space". Intuitively real space should not depend very sensitively on the precise choice of the field. Assume that there are many fields $\tilde{\varphi}_\alpha(x)$ all meeting the criteria for "real space" mentioned above. (The precise formulation of these criteria still needs to be worked out.) If α is a continuous parameter (or set of parameters) the "best" space should be the one which is least sensitive to small changes of α. This concept may finally lead to a very strong selection criterion. This becomes apparent if one realizes that a change in the action $S = -\ln\, p$ is equivalent to a change of operators according to (2.8), (2.9). For example, a small change of the parameters of the standard model is equivalent to a small change of the fields. If this change occurs within the fields $\tilde{\varphi}_\alpha(x)$ which are acceptable for a definition of space one may be led to a new extremum condition for $g_{\mu\nu}$ as a function of α. One may even speculate that this fixes the "best parameters". In this respect local field theories have the enormous advantage that their long-distance behaviour is very insensitive to the addition of irrelevant operators in the action. The universality of the long-distance behaviour reduces the selection problem to the selection of the universality class and the selection of the "best parameters" within a given universality class.

In summary, this paper should only be considered as the beginning of the development of geometrical concepts in general statistics. We hope, nevertheless, that our discussion draws some attention to the problem of *selection* of the space-time structure which corresponds to reality. This problem is at the basis of the crisis of unification arising from the proliferation of possible ground states.

* We note that only $\tilde{\varphi}$ must be defined in the limit $N \to \infty$ whereas $v(x)$ must not necessarily remain a normalizable operator.

Appendix A

ENLARGED GEOMETRY IN ISING MODELS WITH CONSTANT MAGNETIC FIELD

The construction of geometry from the correlation function should not only be considered as the basic geometrical setting for a unified field theory. It can also be applied to a multitude of situations in statistical mechanics. As a simple example for a geometric structure we consider the Ising model with a constant magnetic field B. The action is related to the probability density $p(s)$ by

$$\exp(-S_B) = p \exp(-\Delta S(B)), \tag{A.1}$$

$$p = \exp\left(-\beta \sum_{\{n^\mu\}} \sum_{e^\mu} s_{n^\mu} s_{n^\mu + e^\mu}\right), \tag{A.2}$$

$$\Delta S(B) = B \sum_{\{n^\mu\}} s_{n^\mu}, \tag{A.3}$$

Here the sum over n^μ extends over the sites of a d-dimensional lattice and $\sum_{e^\mu}$ denotes a sum over unit vectors in the different lattice directions. The number N of variables $s^u \equiv s_{n^\mu}$ is given by the number of lattice sites and may be taken to infinity at the end. For the Ising model we take discrete variables $s_{n^\mu} = \pm 1$. This is related to a formulation with continuous s_{n^μ} as the limit where p approaches the distribution

$$p = \prod_{\{n^\mu\}} \{\delta(s_{n^\mu} - 1) + \delta(s_{n^\mu} + 1)\} p_0, \tag{A.4}$$

and p_0 given by (A.2). For finite N we use a periodic lattice such that invariance under the discrete lattice translations is guaranteed.

We now introduce a $(d+1)$-dimensional geometry $(x^{\hat{\mu}} \equiv (x^\mu, B))$ by the field

$$v(x^\mu, B) = s(x^\mu) \exp(-\Delta S(B)), \tag{A.5}$$

with

$$s(x^\mu) = \sum_{\{n^\mu\}} f_k(x^\mu - n^\mu) s_{n^\mu} \tag{A.6}$$

defined in analogy with the example (9.5) and (9.6). (We use μ for d-dimensional and $\hat{\mu}$ for $(d+1)$-dimensional indices.) All relevant quantities can be expressed in terms of the expectation value of s with a magnetic field and the corresponding two-point function

$$\langle s \rangle_B = \langle s_{n^\mu} \rangle_B = \int \mathrm{D}s\, s_{n^\mu} \exp(-S_B), \tag{A.7}$$

$$G_B(d^\mu) = \langle s_{n^\mu} s_{n^\mu + d^\mu} \rangle_B = \int \mathrm{D}s\, s_{n^\mu} s_{n^\mu + d^\mu} \exp(-S_B), \tag{A.8}$$

$$G_B^{\mathrm{c}}(d^\mu) = G_B(d^\mu) - \langle s \rangle_B^2, \tag{A.9}$$

$$\int \mathrm{D}s = Z^{-1} \prod_{\{m^\mu\}} \sum_{s_{m^\mu} = \pm 1}, \qquad \int \mathrm{D}s\, p(s) = 1. \tag{A.10}$$

One finds

$$\langle v(x, B) \rangle = \sum_{\{n^\mu\}} f_k(x^\mu - n^\mu) \langle s \rangle_B, \tag{A.11}$$

$$\langle v(x, B), v(y, B') \rangle = \sum_{\{n^\mu\}} \sum_{\{d^\mu\}} f_k(x^\mu - n^\mu) f_k(y^\mu - n^\mu - d^\mu) G_{B+B'}(d^\mu), \tag{A.12}$$

$$\langle v(x, B) v(y, B') \rangle - \langle v(x, B) \rangle \langle v(y, B') \rangle$$
$$= \sum_{\{n^\mu\}} \sum_{\{d^\mu\}} f_k(x^\mu - n^\mu) f_k(y^\mu - n^\mu - d^\mu) \left(G_{B+B'}(d^\mu) - \langle s \rangle_B \langle s \rangle_{B'} \right), \tag{A.13}$$

and defines the normalized field $\tilde{\varphi}$ as

$$\tilde{\varphi}(x, B) = c_k(x, B) \left(v(x, B) - \langle v(x, B) \rangle \right), \tag{A.14}$$

with

$$c_k^{-2}(x, B) = \sum_{\{n^\mu\}} \sum_{\{d^\mu\}} f_k(x^\mu - n^\mu) f_k(x^\mu - n^\mu - d^\mu) \tag{A.15}$$

$$\left(G_{2B}^{\mathrm{c}}(d^\mu) + \langle s \rangle_{2B}^2 - \langle s \rangle_B^2 \right). \tag{A.16}$$

We note that $\tilde{\varphi}$ is defined for all x and B since $G_B^{\mathrm{c}}(d^\mu) \geqslant 0$ and $\langle s \rangle_B$ is a monotonically increasing function of B. The correlation $G(x, B; y, B')$ is positive for all x, $y \in \mathbb{R}^d$ and B, $B' \in \mathbb{R}$ since

$$\langle s \rangle_{B+B'}^2 \geqslant \langle s \rangle_B \langle s \rangle_{B'}. \tag{A.17}$$

For the symmetric phase, $\tilde{\varphi}$ is continuous and differentiable for all values of B such that $\mathrm{R} \equiv \mathbb{R}^{d+1}$. In the spontaneously broken phase $\langle s \rangle_B$ develops a discontinuity at $B=0$ for $N\to\infty$. (All quantities remain continuous for finite N.) We expect the appearance of geometrical singularities at $B=0$ for $N\to\infty$.

Let us consider the limit $k^2 \ll 1$. (The fundamental length scale μ_0^{-1} is here proportional to the lattice spacing.) We can then approximate sums by integrals,

$$\sum_{\{n^\mu\}} f_k(x^\mu - n^\mu) \approx \int \mathrm{d}^d z\, f_k(x-z) = 1,$$

$$\sum_{\{n^\mu\}} f_k(x^\mu - n^\mu) f_k(y^\mu - n^\mu) \approx \int \mathrm{d}^d z\, f_k(x-z) f_k(y-z)$$

$$= f_{k/\sqrt{2}}(x-y), \tag{A.18}$$

such that $\langle v \rangle$ and c_k become independent of x^μ,

$$\langle v(x, B)\rangle = \langle s \rangle_B = s(B),$$

$$c_k^{-2}(x, B) = c^{-2}(B, k) = g(B, k) - s^2(B),$$

$$g(B, k) = \sum_{\{d^\mu\}} f_{k/\sqrt{2}}(d^\mu) G_{2B}(d^\mu). \tag{A.19}$$

The metric

$$g_{\hat\mu\hat\nu} = c_k^2 \left\{ \langle \partial_{\hat\mu} v \partial_{\hat\nu} v \rangle - \partial_{\hat\mu}\langle v \rangle \partial_{\hat\nu}\langle v \rangle - \tfrac{1}{4} c_k^2 \partial_{\hat\mu} c_k^{-2} \partial_{\hat\nu} c_k^{-2} \right\} \tag{A.20}$$

can again be expressed in terms of correlation functions in a constant magnetic field. The d-dimensional submetric

$$g_{\mu\nu} \approx 4k^4 c_k^2(B) \sum_{\{d^\mu\}} \int \mathrm{d}^d z\, z_\mu (z_\nu - d_\nu) f_k(z) f_k(z-d) G_{2B}(d)$$

$$= k^2\left(1 + \langle s \rangle_B^2 c_k^2(B)\right)\delta_{\mu\nu} - k^4 c_k^2(B) \sum_{\{d^\mu\}} d_\mu d_\nu f_{k/\sqrt{2}}(d) G_{2B}(d)$$

$$= \frac{2}{d} k^2 \frac{\partial \ln c^{-2}(B, k)}{\partial \ln k^2} \delta_{\mu\nu} \tag{A.21}$$

becomes flat in leading order. The off-diagonal elements vanish,

$$g_{\mu B} \approx -k^2 c_k^2(B) \frac{\partial}{\partial B} \sum_{\{d^\mu\}} \int \mathrm{d}^d z\, z_\mu f_k(z) f_k(z-d) G_{2B}(d)$$

$$= -\tfrac{1}{2} k^2 c_k^2(B) \frac{\partial}{\partial B} \sum_{\{d^\mu\}} d_\mu f_{k/\sqrt{2}}(d) G_{2B}(d) = 0, \tag{A.22}$$

since $G_B(-d) = G_B(d)$. Finally, one obtains

$$g_{BB} = \frac{1}{4}\frac{\partial^2}{\partial B^2}\ln c^{-2}(B, k) + \tfrac{1}{4}c^2(B, k)s^2(B)\frac{\partial^2}{\partial B^2}\ln s^2(B). \qquad \text{(A.23)}$$

We see that $g_{\hat{\mu}\hat{\nu}}$ can be expressed in terms of the quantities $s(B)$ and $g(B, k)$ (A.18) and their derivatives with respect to B and k. It remains to be seen if this or similar geometrical structures in statistical mechanics are of some practical use.

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