Nuclear Physics B 387 (1992) 617–637
North-Holland

NUCLEAR
PHYSICS B

Solutions to the Gribov–Levin–Ryskin equation in the nonperturbative region

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Received 9 March 1992
Accepted for publication 27 July 1992

We discuss solutions to the nonlinear GLR evolution equations in the nonperturbative part of the low-x_B region. A new definition of the critical line is obtained, and we conclude that there is only one solution which matches the perturbative Gribov–Lipatov–Alterelli–Parisi evolution.

1. Introduction

It is well known that perturbative QCD leads to a parton distribution of a hadron which grows rapidly with $1/x$. This increase eventually will violate the unitarity constraints, even at large values of Q^2, and it is therefore unavoidable that the conventional perturbative QCD evolution scheme will break down and give way for nonperturbative physics. At large Q^2, however, the strong coupling constant is still small, and we are thus in the novel situation of entering the nonperturbative region with a small coupling constant. At the moment we cannot do better than studying just the edge of this kinematic domain, using a careful analysis of Feynman diagrams. This analysis shows that, compared to the conventional evolution framework, new partonic interactions becoming important and lead to large screening effects (shadowing) inside the structure functions. More than ten years ago Gribov, Levin, and Ryskin [1] suggested that these new partonic interactions could be represented by a nonlinear term in the evolution equation:

$$\frac{\partial\phi(x_B, q^2)}{\partial \ln(1/x_B)} = \frac{N_c\alpha_s}{\pi}\int K(q^2, q'^2)\left\{1 - \alpha_s\frac{\phi(x_B, q'^2)}{\phi_0}\right\}\phi(x_B, q'^2)\frac{d^2q'}{2\pi}, \quad (1.1)$$

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where

$$\phi(x_B, q^2) = \frac{\partial F(x_B, q^2)}{\partial q^2}, \tag{1.2}$$

$$F(x_B, q^2) = x_B G(x_B, q^2),$$

$$\phi_0 = \frac{32 N_c}{9 Q_0^2}, \tag{1.3}$$

and

$$K(q^2, q'^2)\phi(x_B, q'^2) = \frac{\phi(x_B, q'^2)}{(q-q')_t^2} - \frac{q_t'^2 \phi(x_B, q^2)}{(q-q')_t^2 (q_t'^2 + (q-q')_t^2)}. \tag{1.4}$$

The negative sign of the second (nonlinear) term in (1.1) is intimately connected with the Abramovsky–Gribov–Kanchelli [2] cutting rules, and it has the effect of damping the increase of the structure functions at small x. Q_0^2 is the new scale that is responsible for the value of the screening correction. It is worthwhile mentioning that R_h^2 derived in ref. [3] is equal to $R_h^2 = 12\ (1/Q_0^2)$.

It should, however, be stressed that the derivation of this equation from QCD holds only in that kinematic region where the second term is still small. From the theoretical viewpoint, on the other hand, the most interesting region is that of smaller x where the parton density is large while α_s is still small: a better understanding of the dynamics in this domain would be extremely valuable also in many other fields of high-energy physics, e.g. in high-energy heavy-ion collisions, the structure of typical inelastic events in hadron–hadron scattering at high energies, nuclear shadowing or baryon number violation in the electroweak theory [3,4].

In view of this interest it seems instructive to further investigate the nonlinear evolution equation (1), even in those kinematic domains where we cannot be sure how well it describes QCD. So far most of our understanding is based upon the semiclassical approach [1,5,6] and on numerical studies [5,8–10]; in this paper we attempt to use, as much as possible, analytic methods. Our main result is a novel derivation of the existence of a critical line. Furthermore, we shall conclude that in the nonperturbative region the equation has only one unique solution. In sect. 2 we describe our approach which will lead us to a nonlinear differential equation. This equation will be investigated in sect. 3, together with the main implications for the behavior of the structure function at low x_B. Corrections to this equation as well as the correct set up of initial conditions ("matching procedure") for the GLR equation are discussed in sect. 4. We conclude in sect. 5 with a short discussion of our results.

2. The method

Throughout our discussion we will make a crucial simplification, namely we shall ignore the running of α_s: it is only in this approximation that we have been able to find an analytic method of studying the nonlinear evolution equation. On the other hand, as far as the basic properties of the equation are concerned, there are no qualitative differences between the equations with running or with fixed coupling constant. This applies, in particular, to the existence of the critical line and to the behavior of the solution along this line.

The nonlinear evolution equation we wish to investigate is given in eq. (1.1). Away from the critical line we make the following ansatz for the moments ($f(\omega, Q^2)$) of the structure function $F(x_B, Q^2)$:

$$f(\omega, Q^2) = f(\omega)\, e^{\gamma(\omega)(r-r_0)}, \tag{2.1}$$

where $r = \ln(Q^2/\Lambda^2)$, $\omega = n-1$, $r_0 = \ln(Q_0^2/\Lambda^2)$ (here Q_0^2 is the starting point for our evolution equation) and

$$f(\omega, Q^2) = \int_0^1 dx_B\, x_B^{n-2} F(x_B, Q^2) = \int d\ln\frac{1}{x_B} e^{-\omega \ln(1/x_B)} F(x_B, Q^2). \tag{2.2}$$

It will be more convenient to use the function $\phi(\omega, Q^2)$ rather than the structure function F. From (1.2) it follows that

$$\begin{aligned}\phi(\omega, Q^2) &= \int dy\, e^{-\omega(y-y_0)}\, \phi(y, Q^2) = e^{\omega y_0} \frac{df(\omega, Q^2)}{dQ^2} \\ &= \gamma(\omega) f(\omega)\, e^{\omega y_0} \frac{1}{Q_0^2} e^{-\bar{\gamma}(\omega)(r-r_0)}, \\ &= \phi(\omega)\, e^{-\bar{\gamma}(\omega)(r-r_0)},\end{aligned} \tag{2.3}$$

where $y = \ln(1/x_B)$, $y_0 = \ln(1/x_0)$ and

$$\bar{\gamma}(\omega) = 1 - \gamma(\omega). \tag{2.4}$$

Substituting this ansatz into the evolution equation (1.1), we have the following conditions on $\phi(\omega)$ and the function $\bar{\gamma}(\omega)$ which determines the evolution in Q^2:

$$\begin{aligned}&\left\{\omega - \frac{\alpha_s N_c}{\pi} \chi(1-\bar{\gamma}(\omega))\right\} \phi(\omega) \exp(-\bar{\gamma}(\omega)(r-r_0)) \\ &= -\frac{\alpha_s^2 N_c}{\pi} \frac{1}{\phi_0} \int \frac{d\omega}{2\pi i} \phi(\omega-\omega')\phi(\omega')\chi(1-\bar{\gamma}(\omega-\omega')-\bar{\gamma}(\omega')) \\ &\quad \times \exp(-[\bar{\gamma}(\omega-\omega')+\bar{\gamma}(\omega')](r-r_0)).\end{aligned} \tag{2.5}$$

Here we have made use of the eigenvalue condition [11]

$$\int \frac{d^2 q'}{2\pi} K(q^2, q'^2)\, e^{\gamma r'} = \chi(1-\gamma)\, e^{\gamma r} \tag{2.6}$$

with

$$\chi(1-\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1-\gamma), \tag{2.7}$$

where $\psi(t) = d \ln \Gamma(t)/dt$ denotes the derivative of the logarithm of Euler's gamma function. The integral on the r.h.s. of eq. (2.5) is evaluated by means of the saddle-point method:

$$\left\{\omega - \frac{\alpha_s N_c}{\pi}\chi(1-\bar{\gamma}(\omega))\right\}\phi(\omega)\, \exp(-\bar{\gamma}(\omega)(r-r_0))$$
$$= -\frac{\alpha_s^2 N_c}{\pi}\frac{1}{\phi_0}\phi^2\left(\frac{\omega}{2}\right)\frac{1}{2\pi^{1/2}\sqrt{2\bar{\gamma}''(\omega/2)(r-r_0)}}\exp\left(-2\bar{\gamma}\left(\frac{\omega}{2}\right)(r-r_0)\right). \tag{2.8}$$

Eq. (2.8) is the starting point of our study. First one notices that for large enough ω we can neglect the r.h.s., and $\bar{\gamma}$ can be calculated from the linear equation

$$\omega - \frac{\alpha_s N_c}{\pi}\chi\left(1 - \bar{\gamma}_{\text{pert}}(\omega)\right) = 0. \tag{2.9}$$

It should be noted that in the so-called double-leading-log approximation [12]:

$$\chi(1-\bar{\gamma}) = \frac{1}{1-\bar{\gamma}}, \tag{2.10}$$

which leads to

$$\gamma(\omega) = \frac{\alpha_s N_c}{\pi\omega}. \tag{2.11}$$

This is just the singular part of the anomalous dimension of the two-gluon operator. However, in the critical region where the r.h.s. of eq. (2.8) is of the same order as the l.h.s., we find another condition on $\bar{\gamma}$:

$$\bar{\gamma}_{\text{nonpert}}(\omega) = 2\bar{\gamma}_{\text{nonpert}}\left(\frac{\omega}{2}\right). \tag{2.12}$$

It has the simple solution

$$\bar{\gamma}_{\text{nonpert}}(\omega) = C\omega. \tag{2.13}$$

Both solutions should match at a certain ω_{crit}:

$$\gamma_{\text{pert}}(\omega_{\text{crit}}) = \gamma_{\text{nonpert}}(\omega_{\text{crit}}), \tag{2.14}$$

and its value can be determined from the matching conditions (2.12) and

$$\frac{\partial \bar{\gamma}_{\text{nonpert}}}{\partial \omega} = \frac{\bar{\gamma}_{\text{nonpert}}}{\omega}, \tag{2.15}$$

which leads to the equation

$$\frac{\partial \bar{\gamma}_{\text{pert}}}{\partial \omega} = \frac{\bar{\gamma}_{\text{pert}}}{\omega}. \tag{2.16}$$

Eqs. (2.15), (2.16) and (2.9) give an equation for $\bar{\gamma}_{\text{crit}} = \bar{\gamma}(\omega_{\text{crit}})$:

$$\frac{\chi(1-\bar{\gamma}_{\text{crit}})}{\bar{\gamma}_{\text{crit}}} = -\frac{\mathrm{d}\chi(1-\bar{\gamma}_{\text{crit}})}{\mathrm{d}(1-\bar{\gamma}_{\text{crit}})}. \tag{2.17}$$

The above equation is just the same that was received in ref. [1] in the semiclassical approach. For ω_{crit} the result is

$$\omega_{\text{crit}} = \frac{\alpha_s N_c}{\pi}\chi(1-\bar{\gamma}_{\text{crit}}). \tag{2.18}$$

Within the semiclassical approximation one defines evolution paths in the ($y = \ln(1/x)$, $r = \ln(Q^2/\Lambda^2)$)-plane. Starting from the integral representation of our solution:

$$\phi(x_B, Q^2) = \int \frac{\mathrm{d}\omega}{2\pi i}\phi(\omega)\exp\big(\omega(y-y_0) - \bar{\gamma}_{\text{pert}}(\omega)(r-r_0)\big), \tag{2.19}$$

we follow ref. [1] and use the saddle-point approximation. For large y and r the saddle point is determined by

$$y - y_0 = \frac{\mathrm{d}\bar{\gamma}}{\mathrm{d}\omega}\cdot(r-r_0). \tag{2.20}$$

This equation describes the family of curves $y = y(r)$ which pass through the point $y = y_0$, $r = r_0$, and ω is a constant of motion. Any such curve can be considered [1,5,13] as an evolution path of the parton distribution, and the semiclassical approximation of (2.19) along this path provides an approximate estimate of the structure function. For $\omega < \omega_{\text{crit}}$ a case of particular interest is the critical trajectory. It is the "last" trajectory for which we can use the linear equation (since the nonlinear term in the GLR equation is still small). Its equation is obtained from (2.19):

$$y - y_0 = \left.\frac{\mathrm{d}\bar{\gamma}}{\mathrm{d}\omega}\right|_{\omega_{\mathrm{crit}}} \cdot (r - r_0) = \frac{\pi}{\alpha_s N_c} \frac{\bar{\gamma}_{\mathrm{crit}}}{\chi(1 - \bar{\gamma}_{\mathrm{crit}})} (r - r_0). \tag{2.21}$$

Using the solution of (2.17), namely $\bar{\gamma}_{\mathrm{crit}} = 0.63$ (see ref. [1] for details), we rewrite (2.21):

$$y - y_0 = \frac{\pi}{\alpha_s N_c} \frac{0.63}{\chi(0.37)} (r - r_0) = \frac{0.21\pi}{\alpha_s N_c} (r - r_0). \tag{2.22}$$

The structure function along the critical line follows from eq. (2.3):

$$F(x_{\mathrm{B}}, Q^2) = \mathrm{e}^{(r - r_0)} \int \frac{\mathrm{d}\omega}{2\pi i} f(\omega) \exp\left(\omega \left.\frac{\mathrm{d}\bar{\gamma}}{\mathrm{d}\omega}\right|_{\omega_{\mathrm{crit}}} - \bar{\gamma}_{\mathrm{pert}}(\omega) \right) (r - r_0), \tag{2.23}$$

where $f(\omega)$ has to be determined from initial conditions. As an example, the condition $xG(x, Q_0^2) = \delta(y - y_0)$ would lead to $f(\omega) = \mathrm{e}^{\omega y_0}$. We believe, however, that the correct way of calculating $f(\omega)$ proceeds via matching the solution (2.19) to the solution of the nonlinear equation along the critical line. We will return to this point below.

Let us now further study the region $\omega < \omega_{\mathrm{crit}}$. We return to the function γ in the nonperturbative region (2.13) and determine the constant C:

$$\bar{\gamma}_{\mathrm{nonpert}}(\omega_{\mathrm{crit}}) = \bar{\gamma}_{\mathrm{pert}}(\omega_{\mathrm{crit}}) = \bar{\gamma}_{\mathrm{crit}}, \tag{2.24}$$

$$C = \frac{\bar{\gamma}_{\mathrm{crit}}}{\omega_{\mathrm{crit}}}. \tag{2.25}$$

Using eqs. (2.13) and (2.25) we cast eq. (2.5) into the form

$$\left\{ \omega - \frac{\alpha_s N_c}{\pi} \chi\left(1 - \frac{\bar{\gamma}_{\mathrm{crit}}}{\omega_{\mathrm{crit}}} \omega\right) \right\} \phi(\omega)$$

$$= -\frac{\alpha_s^2 N_c}{\pi} \frac{1}{\phi_0} \chi\left(1 - \frac{\gamma_{\mathrm{crit}}}{\omega_{\mathrm{crit}}} \omega\right) \int \phi(\omega - \omega') \phi(\omega') \frac{\mathrm{d}\omega'}{2\pi i}. \tag{2.26}$$

As a result, $\phi(y, r)$ becomes

$$\phi(y, Q^2) = \frac{1}{2\pi i} \int \mathrm{d}\omega \, \phi(\omega) \exp\left[\omega(y - y_0) - \frac{\bar{\gamma}_{\mathrm{crit}}}{\omega_{\mathrm{crit}}} \omega(r - r_0) \right]. \tag{2.27}$$

Introducing the new variable

$$z = y - y_0 - \frac{\bar{\gamma}_{\text{crit}}}{\omega_{\text{crit}}}(r - r_0) \tag{2.28}$$

we notice that ϕ depends only upon z *, and eq. (2.26) takes the simple form

$$\left\{\frac{\mathrm{d}}{\mathrm{d}z} - \frac{\alpha_s N_c}{\pi}\chi\left(1 - \frac{\bar{\gamma}_{\text{crit}}}{\omega_{\text{crit}}}\frac{\mathrm{d}}{\mathrm{d}z}\right)\right\}\phi(z)$$

$$= -\frac{\alpha_s^2 N_c}{\pi}\frac{1}{\phi_0}\chi\left(1 - \frac{\bar{\gamma}_{\text{crit}}}{\omega_{\text{crit}}}\frac{\mathrm{d}}{\mathrm{d}z}\right)\phi^2(z). \tag{2.29}$$

It can even further be simplified by the following rescaling:

$$\bar{z} = \frac{\omega_{\text{crit}}}{\bar{\gamma}_{\text{crit}}}z, \qquad \bar{\phi} = \frac{\alpha_s}{\phi_0}\cdot\phi, \tag{2.30}$$

where use has been made of eq. (2.18). The equation then becomes

$$\left\{\frac{\chi(1 - \bar{\gamma}_{\text{crit}})}{\bar{\gamma}_{\text{crit}}}\frac{\mathrm{d}}{\mathrm{d}\bar{z}} - \chi\left(1 - \frac{\mathrm{d}}{\mathrm{d}\bar{z}}\right)\right\}\bar{\phi}(\bar{z})$$

$$= -\chi\left(1 - \frac{\mathrm{d}}{\mathrm{d}\bar{z}}\right)\bar{\phi}^2(\bar{z}). \tag{2.31}$$

It is an ordinary nonlinear differential equation which can be studied by standard methods [14].

However, in this form it is still too complicated and we begin our study with a somewhat simpler version. Namely, instead of the evolution equation (1.1) we switch to the double-log approximation (DLA) which have been defined in (2.10) and (2.11). It should be noted that in this approximation eq. (1.1) is replaced by the simpler nonlinear evolution equation for $F = x_B G$:

$$\frac{\partial^2 F(y, r)}{\partial y\, \partial r} = \frac{N_c \alpha_s}{\pi}F(y, r) - \frac{N_c^2 \alpha_s^2}{\phi_0 \pi}\mathrm{e}^{-r}\, F^2(y, r). \tag{2.32}$$

In this approximation, eq. (2.31) becomes

$$\frac{1}{(1 - \bar{\gamma}(\omega_{\text{crit}}))\bar{\gamma}(\omega_{\text{crit}})}\left(1 - \frac{\mathrm{d}}{\mathrm{d}\bar{z}}\right)\frac{\mathrm{d}}{\mathrm{d}\bar{z}}\bar{\phi} - \bar{\phi} = -\bar{\phi}^2. \tag{2.33}$$

* This property of GLR equation was firstly noted in ref. [5].

Using eq. (2.10) for χ we can calculate ω_{crit} and $\bar{\gamma}_{\text{crit}}$ from eqs. (2.17) and (2.18). The result is

$$\bar{\gamma}_{\text{crit}} = \tfrac{1}{2}, \qquad \omega_{\text{crit}} = \frac{2N_c\alpha_s}{\pi}. \tag{2.34}$$

It means that

$$\bar{z} = \frac{4N_c\alpha_s}{\pi}(y - y_0) - (r - r_0). \tag{2.35}$$

With a few changes of notation eq. (2.33) takes the simple form

$$-4\frac{d^2\bar{\phi}}{d\bar{z}^2} + 4\frac{d\bar{\phi}}{d\bar{z}} - \bar{\phi} = -\bar{\phi}^2. \tag{2.36}$$

The analysis of this equation will be the subject of sect. 3.

3. The nonlinear differential equation

First it may be instructive to view this equation as an equation of motion of a point particle in a classical potential. If we put $\bar{\phi} = x$ and $z = t$, the equation becomes

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - \tfrac{1}{4}(x - x^2)$$

or

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - \frac{\partial}{\partial x}\left(\tfrac{1}{8}x^2 - \tfrac{1}{12}x^3\right). \tag{3.1}$$

i.e. the potential has the form

$$U(x) = \tfrac{1}{4}\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right). \tag{3.2}$$

(fig. 1). One further notes the friction term which has the sign opposite to the usual case. Starting from the origin $x = 0$ and moving in the positive direction we expect three classes of solutions: (i) Solutions with small initial energy will come to a stop, turn back and pass the point $x = 0$ in the negative direction. When it reaches the potential wall on the negative-x axis it is reflected again and starts to move in the positive direction. Due to the "antifriction" term, it accelerates all the time, and when it reaches the barrier on the r.h.s., it will have enough energy to pass and to move to infinity; (ii) Solutions with sufficiently large initial energy will

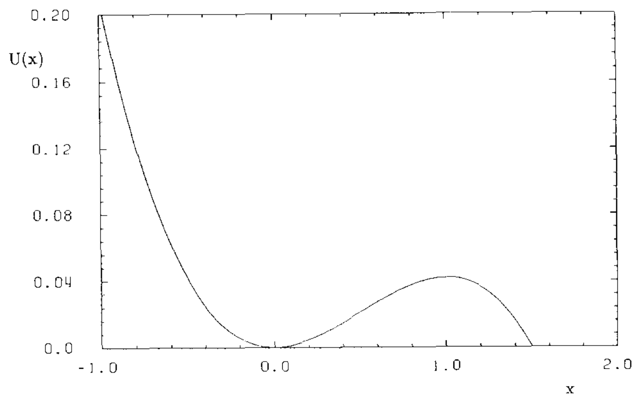


Fig. 1. The potential function eq. (3.2).

directly climb over the potential barrier and escape to infinity; (iii) In between these two classes there is one distinguished solution which comes to a stop just on the top of the barrier and remains there forever.

For a closer investigation we turn to a fixed point analysis of (3.1). Let us introduce $y = \dot{x}$ and rewrite the equation

$$\dot{y} = y - \tfrac{1}{4}x(1-x), \tag{3.3}$$

$$\dot{x} = y. \tag{3.4}$$

There are two fixed points, at $(x, y) = (0, 0)$ and $(x, y) = (1, 0)$. We linearize the equations of motion in the vicinity of these points:

$$\dot{y} = y - \tfrac{1}{4}x, \tag{3.5}$$

$$\dot{x} = y \tag{3.6}$$

near $x = 0$ and

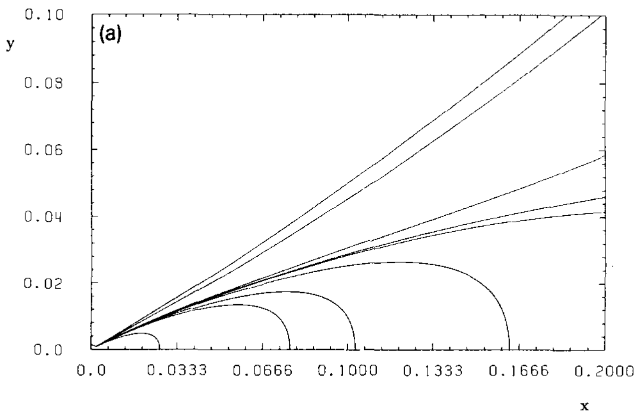
$$\dot{y} = y - \tfrac{1}{4}\bar{x}, \tag{3.7}$$

$$\dot{\bar{x}} = -y \tag{3.8}$$

near $x = 1$, defining $\bar{x} = 1 - x$. In the first case we find degeneracy (both eigenvalues are equal to $\frac{1}{2}$), and the solutions are

$$x(t) = A\ \mathrm{e}^{1/2t} + Bt\ \mathrm{e}^{1/2t},$$

$$y(t) = \tfrac{1}{2}[A\ \mathrm{e}^{1/2t} + Bt\ \mathrm{e}^{1/2t}] + B\ \mathrm{e}^{1/2t}. \tag{3.9}$$



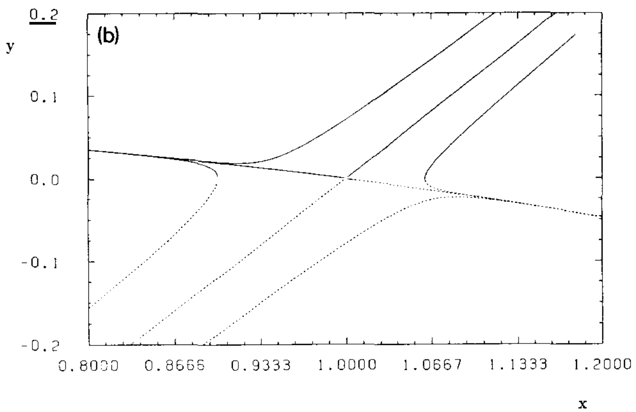


Fig. 2. (a) Solutions near the fixed point at the origin; (b) Solutions near the second fixed point (1, 0).

Eliminating the time variable we obtain for $y(x)$ at $x \to 0$ or $t \to -\infty$:

$$y = \tfrac{1}{2}x + \frac{x}{2 \ln x} + O\left(\frac{\ln \ln x}{\ln^2 x} \right). \tag{3.10}$$

Close to the fixed point (0, 0) all solutions approach the direction $y = \frac{1}{2}x$ (fig. 2a). The dependence upon A and B appears in the expansion (3.10) only in nonleading terms of the order of $1/\ln^2 x$.

For the case of the second fixed point we have two different eigenvalues with opposite signs:

$$\alpha_1 = \frac{1 + \sqrt{2}}{2}, \qquad \alpha_2 = \frac{1 - \sqrt{2}}{2}. \tag{3.11}$$

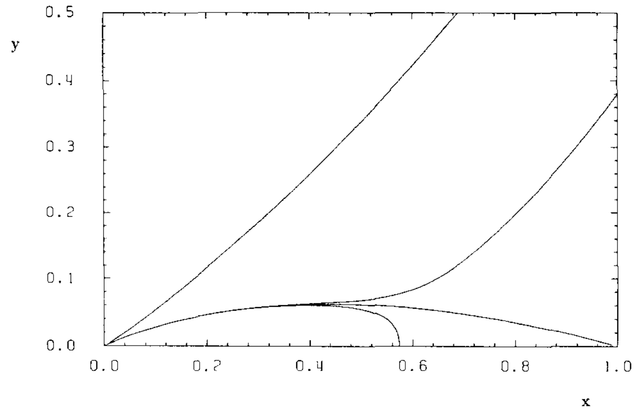


Fig. 3. Connecting the two fixed points.

Hence the fixed point is hyperbolic, and the direction of incoming and outgoing solutions are

$$y = -\alpha_{1,2}\bar{x} \tag{3.12}$$

(fig. 2b). Combining the two figures in one common map (fig. 3), it becomes clear that there is one unique solution, $x_{\text{sat}}(t)$, that leaves the origin and ends at the second fixed point (the subscript "sat" stands for "saturation" and will be explained below). All the other solutions that emanate from the origin will miss the second fixed point and either will stay below this distinguished solution and turn backwards (until x becomes negative and, hence, unphysical), or it lies above and moves upwards towards infinity.

We will now argue that the unique solution x_{sat} which ends at the second fixed point is the only acceptable one. Namely consider the ratio y/x. For all solutions other than x_{sat} it can be shown that it tends to infinity for sufficiently large time t. To see this we first note that all solutions except for x_{sat} will sooner or later become large. We therefore study eqs. (3.3) and (3.4) in the region of large x. Combine the two equations into a relation for $\mathrm{d}y/\mathrm{d}x$:

$$y\frac{\mathrm{d}y}{\mathrm{d}x} = y - \tfrac{1}{4}x(1-x) \tag{3.13}$$

and introduce the new variable $v = 1/x$. For $v \to 0$ we find

$$yv^2\frac{\mathrm{d}y}{\mathrm{d}v} = -y - \frac{1}{4v^2}. \tag{3.14}$$

We can have one of the two cases: either $v^2(\mathrm{d}y/\mathrm{d}v) \gg 1$, and eq. (3.13) becomes

$$y v^2 \frac{\mathrm{d}y}{\mathrm{d}v} = -\frac{1}{4v^2} \tag{3.15}$$

with the solution

$$y = \frac{1}{\sqrt{6}} v^{-3/2}. \tag{3.16}$$

Consequently,

$$\frac{y}{x} = yv = \frac{1}{\sqrt{6}} \frac{1}{\sqrt{v}} \to \infty. \tag{3.17}$$

Alternatively, assume $v^2(\mathrm{d}y/\mathrm{d}v) \ll 1$. Then (3.13) gives

$$y = -\frac{1}{4v^2}, \tag{3.18}$$

and again

$$\frac{y}{x} \to \infty. \tag{3.19}$$

Therefore all solutions which come close to infinity have the property that y/x gets large. On the other hand, this ratio has a simple meaning. If we write, for $x(t)$, the integral representation

$$x(t) = \int \frac{\mathrm{d}\omega}{2\pi i} \tilde{x}(\omega)\, \mathrm{e}^{\omega t}, \tag{3.20}$$

then

$$\frac{y}{x} = \frac{\int \omega \tilde{x}(\omega)\, \mathrm{e}^{\omega t}\, \mathrm{d}\omega}{\int \tilde{x}(\omega)\, \mathrm{e}^{\omega t}\, \mathrm{d}\omega}, \tag{3.21}$$

i.e. y/x is the mean value of ω. So, if $y/x \to \infty$, the mean value (i.e. the dominant value in the integral representation (3.20)) of ω becomes large. But as we have discussed before, for $\omega > \omega_{\mathrm{crit}}$ we should return to the perturbative solution. In other words, a solution to the nonlinear equation in the nonperturbative region which receives its main contribution from large ω-values has to be rejected since it does not match the perturbative solution.

It is not difficult to translate these results into the behavior of the structure function (cf. fig. 4). Our time variable was defined before (2.28) and in (2.30): one

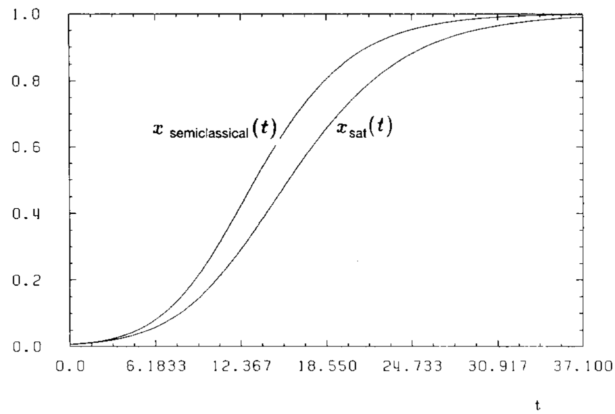


Fig. 4. Comparison of $x_{\rm sat}$ (lower curve) and the semiclassical solution (see eq. (3.32); upper curve). As a common starting point we have chosen $x_{\rm sat}(0) = x_{\rm semiclassical}(0) = 6\times10^{-3}$.

particular value belongs to the critical line, and we choose it to be $\bar{z} = t = 0$. The value of $x(0)$ can be specified only after the matching procedure which will be described below. At fixed q^2, $t\to\infty$ means $x_{\rm B}\to 0$. Solutions in fig. 3 which are above $x_{\rm sat}$, therefore, belong to structure functions which grow very strongly. Those which are below turn back and eventually become negative (and hence unphysical). Both classes are rejected. The only acceptable one is $x_{\rm sat}$ which ends at the second fixed point: $x_{\rm sat}$ approaches a constant value, i.e. the structure function saturates [15] (this also explains the subscript "sat").

Let us discuss a few properties of the solution $x_{\rm sat}$. Collecting the notations (eqs. (2.2), (2.3), (2.19) and (2.30)) (or, alternatively, using directly (2.32)), we find for the structure function

$$F\left(x_{\rm B}, Q^2\right) = {\rm e}^{r-r_0}\,\phi_0\frac{Q_0^2}{\alpha_{\rm s}}\bar{\phi} = {\rm e}^{r-r_0}\,\phi_0\frac{Q_0^2}{\alpha_{\rm s}}x(t). \tag{3.22}$$

We define the two slopes with respect to r and y:

$$\gamma\,|_{x_{\rm B}\,\rm fixed} = \frac{1}{F}\frac{\partial F}{\partial r}\bigg|_{x_{\rm B}\,\rm fixed} = 1 - \frac{1}{x}\frac{{\rm d}x}{{\rm d}t} = 1 - \frac{y}{x}, \tag{3.23}$$

where $r - r_0 = \ln(Q^2/Q_0^2)$ and

$$\gamma\,|_{Q^2\,\rm fixed} = \frac{1}{F}\frac{\partial F}{\partial y}\bigg|_{q^2\,\rm fixed} = \frac{4N_c\alpha_{\rm s}}{\pi}\frac{1}{x}\frac{{\rm d}x}{{\rm d}t} = \frac{4N_c\alpha_{\rm s}}{\pi}\frac{y}{x}. \tag{3.24}$$

Near the two fixed points solutions move very slowly. Therefore we expect for the distinguished solution $x_{\rm sat}$, provided the value $x(0)$ is sufficiently small, that y/x

remains close to $\frac{1}{2}$ for a long time, hence both γ's are approximately constant. Consequently, we can approximate the structure function in the vicinity of the critical line. At fixed x_B:

$$F(x_B, Q^2) \propto \sqrt{\frac{Q^2}{Q_0^2(x_B)}}, \tag{3.25}$$

where $Q_0^2(x_B)$ is the solution of the equation for the critical line, namely $\bar{z}(x_B, Q_0^2) = t(x_B, Q_0^2) = 0$, and at fixed Q^2:

$$F(x_B, Q^2) \propto \left(\frac{x_0(q^2)}{x_B}\right)^{2N_c\alpha_s/\pi}, \tag{3.26}$$

where x_0 again satisfies $t(x_0, Q^2) = 0$.

As a second result, we establish the behaviour of the deep inelastic structure function at $x_B \to 0$:

$$F(x_B, Q^2) = Q^2 \frac{\phi_0}{\alpha_s}\left\{1 - C \exp\left(-\frac{(\sqrt{2}-1)}{2}\left(\frac{4N_c\alpha_s}{\pi}\ln\frac{x_0}{x_B} - \ln\frac{Q^2}{Q_0^2}\right)\right)\right\}. \tag{3.27}$$

It is interesting to note that we could use this result for the profile function of quark–quark scattering at high energies. In ref. [16] a picture of hadron–hadron sattering at high energies has been developed in which the size of the constituent quarks is small enough to allow for the use of perturbative QCD for quark–quark scattering. If this picture is accepted, we could use (3.27) and parametrize the profile function in the following way:

$$a(s, b_t) = \left(1 - C\left(\frac{s}{s_0}\right)^{\omega_0}\right) \Theta(R(s) - b_t), \tag{3.28}$$

where $\omega_0 = (\sqrt{2} - 1)2N_c\alpha_s/\pi$ and $R(s) \propto \ln s$.

Finally, it should be stressed that our solution x_{sat} is quite different from the semiclassical one that has been used in refs. [6,10] for the solution of GLR equation to the right of the critical line. We demonstrate this by applying the semiclassical approximation to (3.10). We make the ansatz

$$x(t) = e^{\psi(t)} \tag{3.29}$$

and assume ψ to be a smooth function of t. If furthermore $\ddot{\psi} \ll \dot{\psi}^2$ we are allowed to put

$$\frac{d^2x}{dt^2} = \dot{\psi}^2 e^{\psi} \tag{3.30}$$

and obtain a simple equation for ψ:

$$\dot{\psi}^2 - \dot{\psi} + \tfrac{1}{4} = \tfrac{1}{4}e^{\psi}. \tag{3.31}$$

It can be solved easily:

$$x = \frac{1}{(1 + C\,e^{-t/4})^2}. \tag{3.32}$$

(the factor C has to be specified by imposing suitable the initial condition). From this solution one can see that it does not reproduce the correct behaviour of the deep inelastic structure function at $t \to +\infty$ (cf. eq. (3.27)): in fig. 4 we compare the behavoiur of $x_{sat}(t)$ with the semiclassical solution of eq. (3.32), matching them at $t = 0$ in such a way that $x_{sat}(0) = x_{semiclassical}(0) = 6 \times 10^{-3}$.

4. Higher-order corrections and matching procedure

So far we have restricted ourselves to leading approximations. In this section we consider various corrections. First we adress the question of stability, i.e. we look at solutions in the vicinity of those considered in sect. 3 and show that deviations remain small. In sect. 3 we have restricted ourselves to a special class of solutions, namely those which depend upon z (eq. (3.10)) but not x and q^2 separately. In this section we shall investigate more general solutions, but stay in the vicinity of the solutions of sect. 3. We return to the DLA equation (2.32) and search for solutions of the form

$$F = \frac{\phi_0}{\alpha_s} e^r\, x_{sat}(t) + \Delta(t, r), \tag{4.1}$$

where

$$t = \frac{4\alpha_s N_c}{\pi}(y - y_0) - (r - r_0). \tag{4.2}$$

Our goal is to show that Δ is small. Calculating derivatives of F and retaining, in eq. (2.32), only terms up to the order Δ, we obtain the following equation for Δ:

$$\frac{4\alpha_s N_c}{\pi}\frac{\partial\dot{\Delta}}{\partial r} = \frac{\alpha_s N_c}{\pi}\Delta - 2\frac{\alpha_s^2 N_c}{\phi_0 \pi}\frac{\phi_0}{\alpha_s} x_{sat}\Delta \tag{4.3}$$

or

$$\frac{\partial\dot{\Delta}}{\partial r} = \tfrac{1}{4}[1 - 2x(t)]\Delta, \tag{4.4}$$

where $\dot{\Delta} = \partial\Delta/\mathrm{d}t$. Solutions are found easily:

$$\Delta = \int \frac{\mathrm{d}\nu}{2\pi i} f(\nu) \exp\left(\nu r + \frac{1}{4\nu} \int_0^t [1 - 2x(t')] \, \mathrm{d}t' \right) \tag{4.5}$$

or

$$\Delta = \int \frac{\mathrm{d}\nu}{2\pi i} f(\nu) \exp\left(\nu r + \frac{1}{4\nu} \int_{x(0)}^{x(t)} [1 - 2x'] \frac{\mathrm{d}x'}{y(x')} \right). \tag{4.6}$$

If we now chose the initial condition $\Delta(t = 0) = 0$, we can specify $f(\nu)$ in eq. (4.6), namely $f(\nu) = 1$, and (4.6) becomes the integral representation of the modified Bessel function:

$$\Delta(t, r) = \sqrt{\frac{1}{r} \int_{x(0)}^{x(t)} [1 - 2x'] \frac{\mathrm{d}x'}{y(x')}} \cdot I_1\left(\sqrt{r \int_{x(0)}^{x(t)} [1 - 2x'] \frac{\mathrm{d}x'}{y(x')}} \right). \tag{4.7}$$

As to the magnitude of Δ, there are two potentially dangerous points in the integral of (4.7), namely $x \to 0$ and $x \to 1$. For $x \to 1$ the integral diverges logarithmically, but

$$\int_{x(0)}^{x} \frac{(1 - 2x')}{y(x')} \mathrm{d}x' < 0, \tag{4.8}$$

and (4.7) is oscillating and thus never grows (such oscillations have been seen first in the computer study of ref. [5]). For $x \to x_0$

$$\Delta \propto \mathrm{e}^{\sqrt{2r \ln(x/x_0)}}, \tag{4.9}$$

and this can be large of the order e^r only if $x/x(0) \approx \mathrm{e}^{r/2}$. But since $x/x(0) \leq 1/\alpha_s$ and does not depend on r, Δ will be small for all values $r \gg \ln(1/\alpha_s)$. We thus conclude that Δ never gets large, i.e. the solution x_{sat} of the previous section is stable against small perturbations.

As a second class of corrections, we leave the DLA approximation and return to eq. (2.31). We briefly investigate how much its solutions differ from those of the DLA equation (3.10). Let us rewrite this equation in the following form:

$$\left\{ \frac{\chi(1 - \bar{\gamma}_{crit})}{\bar{\gamma}_{crit} \chi\left(1 - \frac{\mathrm{d}}{\mathrm{d}\bar{z}}\right)} \frac{\mathrm{d}}{\mathrm{d}\bar{z}} - 1 \right\} \bar{\phi} = -\bar{\phi}^2. \tag{4.10}$$

Here the function $\chi(1 - \omega)$ has a rather complicated form, but we remember that our matching has been done at $\omega = \omega_{crit}$, and we therefore can expand around this point. We return to (2.26) and expand in ω: on the l.h.s. the first two terms cancel, and on the r.h.s. we retain the leading term. Substituting for ω d/d$\bar{z}$, we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}\bar{z}} - \gamma_{\mathrm{crit}}\right)^2 \bar{\phi} = -\gamma_{\mathrm{crit}}^2 \bar{\phi}^2. \tag{4.11}$$

This equation could be reduced to the DLA equation (3.1) changing $\bar{z}$ to

$$\bar{z} = 2\gamma_{\mathrm{crit}} z'.$$

Thus eq. (4.11) has two fixed points at $\bar{\phi} = 0$ and $\bar{\phi} = 1$, and the only difference lies in the eigenvalues near these fixed points.

Our general equation (4.10) still depends only upon $\bar{\phi}$ and $\dot{\bar{\phi}}$ and not on $\bar{z}$, but we cannot apply the same standard methods which we have used in the DLA case. Indeed it turns out that eq. (4.10) cannot be reduced to the differential equation of the definite order due to the very complicated form of the kernel $\chi(1 - \mathrm{d}/\mathrm{d}\bar{z})$.

Nevertheless we are able to investigate the behaviour of the solution in the vicinity of $\bar{\phi} \to 1$ making the following ansatz:

$$\bar{\phi} = 1 - \Delta\bar{\phi},$$

$$\Delta\bar{\phi} = \mathrm{e}^{-\alpha_1 \bar{z}},$$

$$\dot{\bar{\phi}} = \alpha_1 \Delta\bar{\phi}. \tag{4.12}$$

For α_1 we have, from (4.10),

$$\frac{\chi(1 - \bar{\gamma}_{\mathrm{crit}})}{\bar{\gamma}_{\mathrm{crit}} \chi(1 + \bar{\alpha}_1)} \cdot \bar{\alpha}_1 - 1 = 0. \tag{4.13}$$

Eq. (4.13) can be solved numerically, and it turns out that the minimal value of $\bar{\alpha}_1 \sim 0.85$. It means that the solution approaches the saturation limit ($\bar{\phi} \to 1$) much faster than in the DLA approximation, but at the moment we cannot connect the two fixpoints and hence cannot prove that the saturating solution exists.

The final point of this section is the determination of $f(\omega)$ in (2.23) or $\phi(\omega)$ in (2.27). It is obtained from the correct specification of boundary condition, which we will name "matching procedure". In short, we propose to proceed as follows. In the nonperturbative region we can accept only the distinguished solution x_{sat} of the previous section. The task is then to find solutions of the linear equation which on the critical line match this solution. Since the location of the critical line is known only up to its intercept on the y-axis, there is still some freedom: we still need to impose some initial condition in the perturbative region.

In somewhat more detail, we describe this "matching procedure" for finding the solution to the GLR equation in the full kinematical region of deep inelastic scattering in three steps:

(1) Start from some initial distribution of gluons at fixed $Q^2 = Q_0^2$. We need to know the structure function as well as its slope since our linear equation is of the second order. Thus we should use as initial conditions:

$$F(y = \ln(1/x_B),\ r = r_0) = F^{\text{input}}(y) \tag{4.14}$$

and

$$\gamma_r^{\text{input}} = \frac{1}{F^{\text{input}}(y)} \frac{\partial F(y,\ r)}{\partial r}\bigg|_{r=r_0} \tag{4.15}$$

or

$$\gamma_y^{\text{input}} = \frac{1}{F^{\text{input}}(y)} \frac{\partial F(y,\ r)}{\partial y}\bigg|_{r=r_0}. \tag{4.16}$$

(2) Find the value of y_0 and specify the scale of our variable t in a such way that

$$F^{\text{input}}(y_0) = \frac{\phi_0 Q_0^2}{\alpha_s} \cdot x_{\text{sat}}(0) \tag{4.17}$$

and

$$\gamma_r^{\text{input}}(y_0) = 1 - \frac{\dot{x}_{\text{sat}}(0)}{x_{\text{sat}}(0)} \tag{4.18}$$

or

$$\gamma_y^{\text{input}}(y_0) = \frac{4N_c\alpha_s}{\pi} \cdot \frac{\dot{x}_{\text{sat}}(0)}{x_{\text{sat}}(0)}. \tag{4.19}$$

(3) Solve the linear evolution equation

$$\frac{\partial^2 F}{\partial y\, \partial r} = \frac{\alpha_s N}{\pi} F \tag{4.20}$$

with the following initial condition:

$$\begin{gathered} F(y,\ r) = \frac{\phi_0 Q_0^2}{\alpha_s} x(0); \qquad \gamma_r = 1 - \frac{\dot{x}_{\text{sat}}(0)}{x_{\text{sat}}(0)}; \\ \text{at } t = 0 \quad \text{and} \quad y > y_0; \\ F(y,\ r = r_0) = F^{\text{input}}(y); \qquad \gamma_r(y,\ r = r_0) = \gamma_r^{\text{input}}(y); \\ \text{at } y < y_0 \quad \text{and} \quad r = r_0. \end{gathered} \tag{4.21}$$

The following example may help to clarify this matching procedure. Assume that

$$F^{\text{input}}(y)=\left(\frac{1}{x_{\text{B}}}\right)^{a}=\mathrm{e}^{ay},\qquad \gamma_y^{\text{input}}(y)=a.$$

One can find the value of y_0 from the equations

$$a=\frac{4N_c\alpha_s}{\pi}\cdot\frac{\dot{x}_{\text{sat}}(0)}{x_{\text{sat}}(0)},$$

$$\mathrm{e}^{ay_0}=\frac{\phi_0 Q_0^2}{\alpha_s}\cdot x(0).$$

(note that $\dot{x}_{\text{sat}}(0)$ should be considered as a function of x_{sat} and therefore is not an independent variable). As an example, we chose $a=\frac{1}{2}$, $\alpha_s=0.2$, and $Q_0^2=10\ \text{GeV}^2$. Then $x_{\text{sat}}(0)=6\times10^{-3}$ and $y_0=0.4$. Fig. 4 shows how x_{sat} and the semiclassical solution (3.32) differ from each other: with the same starting point at $x=6\times10^{-3}$, the semiclassical approximation reaches saturation earlier than the exact solution.

However one also sees that this matching procedure cannot work for arbitrary values of a. Indeed, for large z,

$$a>\frac{2N_c\alpha_s}{\pi},$$

we are not able to find y_0 since for all values of t the ratio $\dot{x}_{\text{sat}}(0)/x_{\text{sat}}(0)$ is smaller than $\frac{1}{2}$. The physical meaning of this observation is very simple. The nonlinear term in the evolution equation imposes some limitation on the increase of the structure function. It means that not all input distributions garantee that with our starting value $Q^2=Q_0^2$ we are below the critical line in the perturbative region. The steeper the input distribution is, the lower (and more to the right) lies the critical line.

Of course all this discussion is only qualitative. To deal with realistic situations we need to study the solution of the evolution equation with a more complicated kernel; in particular we should take into account the running coupling constant α_s.

5. Conclusions

In this paper we have tried to investigate solutions of the nonlinear evolution equation in the nonperturbative region. It is completely unclear whether these equations are correct beyond the perturbative region, but we just wanted to see what kind of constraint information from perturbative QCD can provide for the continuation into the nonperturbative regime. All previous insight had been based upon the semiclassical approximation. In this paper, we have avoided to make use of this approximation, but tried to use alternative analytic methods as much as possible. As a result of this intention, we had to ignore the dependence of the coupling constant upon the momentum scale, i.e. we have used a fixed α_s.

As the main result we have found (but without making any claim for mathematical rigor) that there is only one solution which matches the known perturbative solution; it becomes flat for $x_B \to 0$, i.e. it supports the idea of saturation. As an immediate consequence of this uniqueness, one should expect that the x_B distribution of xG at fixed low Q^2 which is usually considered to be a rather free input to the evolution equation has a well-defined continuation towards small x_B. This is in perfect qualitative agreement with experience from computer studies of the GLR equation: the nonlinear evolution equations clearly distinguish between "natural" and "unnatural" initial distribution, in that the latter ones are more strongly corrected in course of the Q^2 evolution than the former ones. More strongly, our discussion in this paper shows that certain initial distributions are unacceptable within our matching procedure. As an example we can use our solution for the natural parametrization of the initial gluon distribution at $Q^2 = q_0^2$. It should be stressed that such an initial distribution looks quite different from the one used in ref. [8].

A detailed analysis, within the well-known operator expansion and renormalization group equation, of the transition from perturbative to nonperturbative QCD is still missing. We can, however, say in general words how our analysis could be interpreted in this language. Starting in the $(Q^2 - \ln(1/x_B))$-plane from the lower-right part, i.e. in the perturbative region, we first have the linear evolution equation, and the Q^2-evolution is determined by the anomalous dimensions of the leading-twist operators. Since we have restricted ourselves to gluons only, it is the two-gluon operator which matters. Moving upwards in the kinematic plane towards small x_B (which is equivalent to $\omega \to 0$ or $n \to 1$), the anomalous dimensions of the two-gluon, four-gluon,... operators all threaten to become large individually (they go as $1/\omega$), and the neglect of nonleading-twist operators is no longer justified. The GLR equation with its nonlinear term is nothing but an attempt to find the sum of them. As soon as this nonlinear term becomes important, the Q^2 dependence of the solutions is described by a new function, $\gamma_{\text{nonpert}}(\omega)$, which has no longer the meaning of a single anomalous dimension but represents the collective effect of all nonleading-twist terms in the operator expansion. The $1/\omega$ singular disappears and is replaced by the regular $C \cdot \omega$ behavior with $C \propto 1/\alpha_s$. Within our matching procedure, the continuation into the nonperturbative region requires (and determines!) nonperturbative terms. On the critical line, perturbative and nonperturbative description match; for large Q^2 it coincides with the result obtained from the semiclassical approximation.

It should be emphazised that most of this discussion would also remain valid if we add higher-order terms to the GLR equation: ϕ^3, ϕ^4,... The essential step in our matching procedure is expressed (2.9) and can easily be generalized to

$$\bar{\gamma}_{\text{nonpert}}(\omega) = n\bar{\gamma}_{\text{nonpert}}\left(\frac{\omega}{n}\right), \tag{5.1}$$

and it has always the solution (2.10). This indicates that our results may be of rather general validity.

One of us (E.L.) would like to thank the DESY Theory Division for its kind hospitality.

References

[1] L.V. Gribov, E.M. Levin and M.G. Ryskin, Phys. Rep. 100 (1983) 1
[2] V.A. Abramovski, V.N. Gribov and O.V. Kancheli, Sov. J. Nucl. Phys. 18 (1973) 308
[3] A.H. Mueller, Nucl. Phys. B (Proc. Suppl.) 18C (1991) 125
[4] E.M. Levin, DESY-91-110 (September 1991), Mini-rapporteur talk at EP–HEP 91 Conference, Geneva, Switzerland, July–August 1991
[5] J. Bartels, J. Bluemlein and G. Schuler, Z. Phys. C50 (1991) 91
[6] J.C. Collins and J. Kwiecinski, Nucl. Phys. B258 (1985) 601
[7] J.C. Collins and J. Kwiecinski, Nucl. Phys. B316 (1989) 307
[8] J. Kwiecinski, A.D. Martin, W.J. Stirling and R.G. Roberts, Phys. Rev. D42 (1990) 3645
[9] J. Kwiecinsky, A.D. Martin and P.J. Sutton, Phys. Let. B264 (1991) 199
[10] J.C. Collins and J. Kwiecinski, Nucl. Phys. B335 (1990) 89;
M.G. Ryskin and V.T. Kim, DESY-91-064 (September 1991)
[11] Ya.Ya. Balitskij and L.N. Lipatov, Sov. J. Nucl. Phys. 28 (1978) 822
[12] Yu.L. Dokshitzer, D.I. Dyakonov and S.I. Troyan, Phys. Rep. 58 (1980) 269
[13] E.M. Levin, G. Marchesini, M.G. Ryskin and B. Webber, Nucl. Phys. B357 (1991) 167
[14] A.A. Andronov, A.A. Witt and S.E. Khaikin, Theory of oscillators (Pergamon, Oxford, 1966)
[15] A.H. Mueller, Nucl. Phys. B307 (1988) 34; B317 (1989) 573; B335 (1991) 115
[16] E.M. Levin and M.G. Ryskin, Phys. Rep. 189 (1990) 267