Symmetry scattering on the hyperboloid SO(2,1)/SO(2) in different coordinate systems

R. F. Wehrhahn
Theoretical Nuclear Physics, University of Hamburg, Luruper Chaussee 149, 2000 Hamburg 50, Germany
Yu. F. Smirnov and A. M. Shirokov
Institute of Nuclear Physics, Moscow State University, SU-117234 Moscow, Russia

(Received 10 March 1992; accepted for publication 17 March 1992)

The symmetry scattering theory based on the Harish-Chandra–Helgason theory of spherical functions on noncompact Riemannian symmetric spaces is extended to treat all spherical harmonics on the hyperboloid SO(2,1)/SO(2). The required conditions for an extension of the symmetry scattering theory to treat all spherical harmonics on arbitrary noncompact Riemannian symmetric spaces are elaborated.

I. INTRODUCTION

The symmetry scattering considers the symmetry of a system as a constraint on its evolution and calculates its scattering properties due to this effect.[1] Symmetry scattering thus studies the asymptotic behavior of the eigenfunctions of an algebra of differential operators invariant under some group of transformations, then compares that behavior with one of the eigenfunctions of differential operators invariant under a more trivial symmetry, i.e., corresponding to a flatter group. By this comparison, a scattering operator related to the interaction due to the different symmetries is defined.

This scattering theory is based on the theory of spherical functions on noncompact Riemannian symmetric spaces developed by Harish-Chandra and Helgason[2] since the required eigenfunctions of the invariant differential operators can be gained from expansions using spherical functions.

The attractive feature of this approach lies in the possibility to find explicit expressions for the quantities involved in scattering by group theoretical methods, i.e., it allows us to find the wave functions and the corresponding S-matrices without direct solution of the differential equations governing the system.

The Harish-Chandra–Helgason theory has also been applied to scattering problems by other authors: Perelomov[3] in connection with coherent states, Semenov-Tyan-Shanskii[4] in connection with automorphic functions, and Freund[5] when dealing with scattering on p-adic spaces. In spite of this vast range of applications, symmetry scattering can only treat special scattering problems due to the restriction of the Harish-Chandra–Helgason theory. In particular, this theory is not valid for pseudo-Riemannian spaces. For example, it allows the construction of the spherical functions on the hyperboloid SO(2,1)/SO(2), but it is not applicable to the one-sheeted hyperboloid SO(2,1)/SO(1,1) (see, for example, Ref. 6). Further, it allows the construction of spherical functions only in the coordinate system adapted to the decomposition $G=KAK$, where G is the studied symmetry group, K is its maximal compact subgroup, and A is an Abelian subgroup.[2] On the hyperboloid this coordinate system is the spherical one.

Now, in a series of papers devoted to the group theoretical approach to scattering[7] it was shown that the algebraic methods allow us to solve the Laplace equation on both types of hyperboloids, as well as in different coordinate systems (spherical and hyperbolic). Hence, it is natural to search for extensions of the symmetry scattering theory to deal with (i) pseudo-Riemannian spaces and (ii) different coordinate systems.

Concerning the first point, there is considerable work done[8] so that an extension of the symmetry scattering along this direction should be possible but is not discussed here. In this paper a first attempt to extend the symmetry scattering along the second direction is made. For this purpose we treat the simple example of the symmetric space of rank one SO(2,1)/SO(2) but in a manner amenable to be generalized to other spaces.

The paper is organized in six sections. After the Introduction, in Sec. II the Harish-Chandra–Helgason theory is used to construct the spherical functions on the hyperboloid SO(2,1)/SO(2), i.e., the spherical harmonics in spherical coordinates. In Sec. III a similar procedure is applied to the construction of the spherical harmonics on the same hyperboloid but using hyperbolic coordinates. The connection of the spherical harmonics with quantum scattering problems is indicated in Sec. IV. In Sec. V the spherical functions for arbitrary noncompact Riemannian symmetric spaces are given in the standard language of harmonic analysis to facilitate the understanding of the proposed extension of the symmetry scattering theory. The results of our considerations are summarized in Sec. VI.

 0022-2488/92/072384-06$03.00

II. SPHERICAL HARMONICS ON THE HYPERBOLOID SO(2,1)/SO(2) AND THE HARISH-CHANDRA METHOD

Omitting all algebraic details of the Harish-Chandra methods explained, for instance, in Refs. 2 and 6, we treat here the essence of this approach on the example of the Riemannian symmetric space $X=\mathrm{SO}(2,1)/\mathrm{SO}(2)$, i.e., on the hyperboloid parametrized by $x_3^2-x_1^2-x_2^2=r^2>0$. The construction of the spherical functions on this hyperboloid is equivalent to diagonalize the Laplace operator

$$\Delta=\frac{\partial^2}{\partial x_3^2}-\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2},$$

or, more precisely, to obtain the angular dependent parts of its eigenfunctions. We call an arbitrary solution of the Laplace equation on the hyperboloid a spherical harmonic on X irrespectively of its quantum numbers and of the chosen coordinate system.

On the hyperboloid there are three main coordinate systems:[9]

(a) spherical:

$$x_1=r\sinh(\theta)\cos(\phi),$$
$$x_2=r\sinh(\theta)\sin(\phi), \tag{1a}$$
$$x_3=r\cosh(\theta);$$

(b) hyperbolic:

$$x_1=r\sinh(\omega),$$
$$x_2=r\cosh(\omega)\sinh(\eta), \tag{1b}$$
$$x_3=r\cosh(\omega)\cosh(\eta);$$

and

(c) parabolic or horispherical:

$$x_1=r[\sinh(\alpha)-\tfrac{1}{2}\beta^2e^{\alpha}],$$
$$x_2=r\beta e^{\alpha}, \tag{1c}$$
$$x_3=r[\cosh(\alpha)+\tfrac{1}{2}\beta^2e^{\alpha}].$$

In our case the coordinate system (1a) will in fact be a cylindrical coordinate system, but for more general spaces, $\mathrm{SO}(n,1)/\mathrm{SO}(n)$, similar coordinate system are called spherical.

After separating the radial variable r, the angular-dependent parts of the Laplace operator in these three coordinate systems are

$$\Delta_{\theta,\phi}=\frac{1}{\sinh(\theta)}\frac{\partial}{\partial\theta}\left(\sinh(\theta)\frac{\partial}{\partial\theta}\right)+\frac{1}{\sinh^2(\theta)}\frac{\partial^2}{\partial\phi^2}, \tag{2a}$$

$$\Delta_{\omega,\eta}=\frac{1}{\cosh(\omega)}\frac{\partial}{\partial\omega}\left(\cosh(\omega)\frac{\partial}{\partial\omega}\right)+\frac{1}{\cosh^2(\omega)}\frac{\partial^2}{\partial\eta^2}, \tag{2b}$$

and

$$\Delta_{\alpha,\beta}=\frac{\partial^2}{\partial\alpha^2}+\frac{\partial}{\partial\alpha}+e^{-2\alpha}\frac{\partial^2}{\partial\beta^2}, \tag{2c}$$

respectively.

As it is well known,[10] only irreducible representations D^j with $j=-\frac{1}{2}+ik$, $0<k<\infty$ belonging to the principal continuous series of the unitary representations of the $\mathrm{SO}(2,1)$ group can be realized on the two-sheeted hyperboloid. For them, the eigenvalue of the Laplace–Beltrami operator is equal to $j(j+1)=-\frac{1}{4}-k^2$. Thus, to construct the spherical functions on the hyperboloid X it is necessary to find the eigenfunctions of the operator (2) corresponding to this eigenvalue.

The remarkable achievement of Harish-Chandra was the discovery of the simple solution of the Laplace equation for arbitrary symmetric spaces which is a generalization of the plane waves.[2] The Harish-Chandra plane waves for the hyperboloid X are the eigenfunctions of the Laplace operator in horispherical coordinates that *do not* depend on the variable β. It is clear from expression (2) that these waves are given by

$$\begin{aligned} f_j(\alpha)&=e^{\alpha j}\\ &=[\cosh(\theta)+\sinh(\theta)\cos(\phi)]^{-1/2+ik}\\ &=[\sinh(\omega)+\cosh(\omega)\cosh(\eta)]^{-1/2+ik}. \end{aligned} \tag{3}$$

Above, the explicit expressions for the Harish-Chandra plane waves in the three coordinate systems are given.

From the parametrization of X it is seen that the Harish-Chandra plane wave describes a wave whose front lies on the parabola cut on the hyperboloid by the plane $x_1+x_3=e^{\alpha}=\mathrm{const}$. The movement of the wave corresponds to the parallel translation of this cutting plane.

From the Harish-Chandra plane waves it is then possible to construct generalized spherical functions, $Y_{km}(\theta,\phi)=\Upsilon_{km}(\theta)e^{im\phi}$. The idea is to expand a Harish-Chandra plane wave in terms of the spherical functions $Y_{km}(\theta,\phi)$ corresponding to the same eigenvalue of the Laplace operator, i.e.,

$$f_j(\alpha)=\sum_{n=0\pm 1,\pm 2,\ldots} c_nY_{kn}(\theta,\phi), \tag{4}$$

where $j=-\frac{1}{2}+ik$, and then to project this expansion into one of its sum terms.

For this purpose we introduce a projector $\wp_m$ in the following manner:

$$\wp_m: f(x) \mapsto \int_{SO(2)} \psi_m^*(\zeta) \check{R}(\zeta) f(x) d\mu(\zeta), \qquad (5)$$

where $\psi_m(\zeta)$ is a matrix element of the irreducible representation of SO(2) characterized by the number m, i.e., $\psi_m^*(\zeta)=e^{-im\zeta}$, $\check{R}(\zeta)$ is a rotation through the angle ζ, i.e., an element of SO(2), $d\mu(\zeta)$ is the normalized invariant measure on SO(2), and $f(x)$ is some eigenfunction of the Laplace operator (2).

By applying $\wp_m$ to $f_j(\alpha)$ we obtain

$$\wp_m f_j(\alpha) = \frac{1}{2\pi} \sum_{n=0\pm1,\pm2,...} c_n \Upsilon_{kn}(\theta) e^{in\phi} \int_0^{2\pi} e^{i(n-m)\zeta} d\zeta$$

$$= c_m Y_{km}(\theta,\phi). \qquad (6)$$

Note that for the hyperboloid the above projection amounts to a Fourier expansion since the unknown function $\Upsilon_{km}(\theta)$ follows from

$$\Upsilon_{km}(\theta) \sim \int_0^{2\pi} e^{-im\phi} f_j(\alpha) d\phi. \qquad (7)$$

Using Eq. (6) yields

$$\Upsilon_{km}(\theta) = \frac{2^m \Gamma(m+1) \Gamma(\frac{1}{2}+ik-m)}{\sqrt{k}\Gamma(\frac{1}{2}+ik+m)} P_{-1/2+ik}^{-m}(\cosh(\theta)), \qquad (8)$$

with

$$P_{-1/2+ik}^{-m}(\cosh(\theta)) = \frac{\tanh^m(\theta)\cosh^{-1/2+ik}(\theta)}{2^m\Gamma(m+1)} \times {}_2F_1(\tfrac{1}{2}(m-ik+\tfrac{1}{2}),\tfrac{1}{2}(m-ik+\tfrac{3}{2});m+1;\tanh^2(\theta)). \qquad (9)$$

Above, the relation[11]

$$P_\nu^\mu(\cosh(\theta)) = \frac{\Gamma(\nu+\mu+1)}{2\pi\Gamma(\nu+1)} \int_0^{2\pi} e^{i\mu\phi} [\cosh(\theta) + \sinh(\theta)\cos(\phi)]^\nu d\phi \qquad (10)$$

has been taken into account.

The normalizing factor is chosen to satisfy the condition

$$\int_0^\infty \Upsilon_{k'm}^*(\theta) \Upsilon_{k'm}^*(\theta) \sinh(\theta) d\theta = \delta(k-k'). \qquad (11)$$

This solution of the Laplace equation describes a wave whose front lies on a circle cut on the hyperboloid by the plane $x_3=$const. The movement of this wave corresponds to the translation of this cutting plane along the x_3 axis from infinity to the vertex of the hyperboloid and then after reflection back to infinity.

III. SPHERICAL FUNCTIONS IN HYPERBOLIC COORDINATES ON THE HYPERBOLOID

To find the spherical harmonics in hyperbolic coordinates,

$$Y_{k\mu}(\omega,\eta) = (1/\sqrt{2\pi}) \Upsilon_{k\mu}(\omega) e^{i\mu\eta}$$

using the Harish-Chandra approach, we use similar steps as in Sec. II. We introduce the projector

$$\wp_\mu: f(x) \mapsto \int_{SO(1,1)} \psi_\mu^*(\zeta) \check{T}(\zeta) f(x) d\mu(\zeta), \qquad (12)$$

where $\psi_\mu(\zeta)$ is a matrix element of the irreducible representation of SO(1,1) characterized by the number μ, i.e., $\psi_\mu^*(\zeta)=e^{-i\mu\zeta}$, $\check{T}(\zeta)$ is an element of SO(1,1), i.e., a "rotation" through the "angle" ζ, $d\mu(\zeta)$ is the normalized invariant measure on SO(1,1), and $f(x)$ is some eigenfunction of the Laplace operator (2).

Now applying this projector to a Harish-Chandra plane wave yields

$$\Upsilon_{k\mu}(\omega) = 2\frac{\Gamma(\frac{1}{2}-ik-i\mu)}{\Gamma(\frac{1}{2}-ik)} Q_{-(1/2)-ik}^{i\mu}(\sinh(\omega)) e^{\mu\pi}$$

$$= \sqrt{\frac{2\pi}{\cosh(\omega)}} \frac{\Gamma(\frac{1}{2}-ik-i\mu)\Gamma(\frac{1}{2}-ik+i\mu)}{\Gamma(\frac{1}{2}-ik)} \times P_{-(1/2)-i\mu}^{ik}(\tanh(\omega))$$

$$= \sqrt{\frac{2\pi}{\cosh(\omega)}} \frac{\Gamma(\frac{1}{2}-ik-i\mu)\Gamma(\frac{1}{2}-ik+i\mu)}{\Gamma(\frac{1}{2}-ik)} \times \left(\frac{\xi+1}{\xi-1}\right)^{ik/2} {}_2F_1\left(\frac{1}{2}+i\mu,\frac{1}{2}-i\mu;1-ik;\frac{1-\xi}{2}\right), \qquad (13)$$

with $\xi=\tanh(\omega)$. Here the relation[11]

$$Q_\lambda^\sigma(z) = e^{i\sigma\pi} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\sigma+1)} \int_0^\infty [z+\cosh(t)\sqrt{z^2-1}]^{-\lambda-1} \times \cosh(\sigma t) dt, \qquad (14)$$

with $\mathrm{Re}(\lambda+\sigma)>1$, has been used.

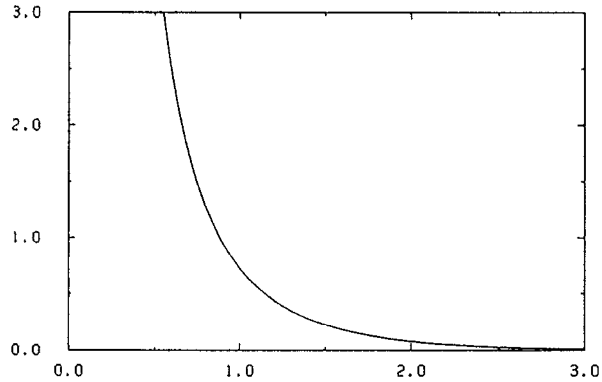

FIG. 1. The Pöschl–Teller potential $V(\theta)=(m^2-\frac{1}{4})/\sinh^2(\theta)$. The meaning of the quantities appearing here is indicated in the text. The axes are in arbitrary units.

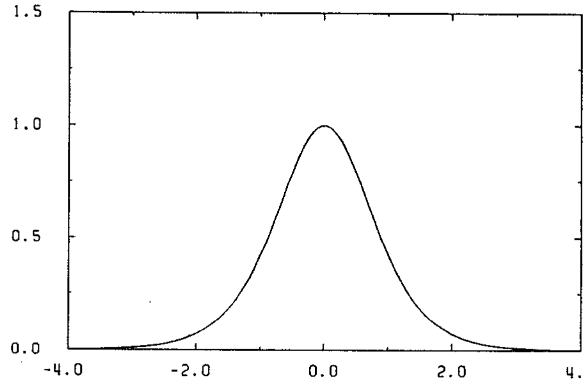

FIG. 2. The Pöschl–Teller barrier $V(\omega)=(\mu^2+\frac{1}{4})/\cosh^2(\omega)$. The meaning of the quantities appearing here is indicated in the text. The axes are in arbitrary units.

This solution of the Laplace equation describes a wave whose front lies on the hyperbola cut on the hyperboloid by the plane $x_1=$const. The evolution of this wave corresponds to the motion of its front along the x_1 axis going from $+\infty$ to $-\infty$ and suffering a partial reflection around the vertex of the hyperboloid.

IV. CONNECTION OF THE SPHERICAL FUNCTIONS ON THE HYPERBOLOID WITH THE SCATTERING THEORY

The substitution of the spherical function $\Upsilon_{km}(\theta)$ by the new function $\Phi_{km}(\theta)=\sqrt{\sinh(\theta)}\Upsilon_{\rm km}(\theta)$ transforms the Laplace equation

$$\Delta_{\theta,\phi}\Upsilon_{km}(\theta)=-(k^2+\tfrac{1}{4})\Upsilon_{km}(\theta)$$

into the Schrödinger equation

$$-\frac{\partial^2\Phi_{km}(\theta)}{\partial^2\theta}+\frac{m^2-\frac{1}{4}}{\sinh^2(\theta)}\Phi_{km}(\theta)=k^2\Phi_{km}(\theta), \qquad (15)$$

where the real non-negative number k fixing the irreducible representation of the SO(2,1) group coincides with the wave number of the particle in the potential scattering theory.

Using Eq. (8) it is easy to find the explicit expression of the S-matrix for the scattering on the repulsive (if $|m|\neq 0$) Pöschl–Teller potential,

$$S=-\frac{\Gamma(ik)\Gamma\{[(1+|m|)/2]-ik\}}{\Gamma(-ik)\Gamma\{[(1+|m|)/2]+ik\}}. \qquad (16)$$

Thus, the knowledge of the spherical function on the hyperboloid in spherical coordinates solves automatically the scattering problem for the Pöschl–Teller potential represented in Fig. 1.

Now, the Laplace equation in hyperbolic coordinates

$$\Delta_{\omega,\eta}Y_{k\mu}(\omega,\eta)=-(k^2+\tfrac{1}{4})Y_{k\mu}(\omega,\eta),$$

after the substitution of the function $\Upsilon_{k\mu}$ appearing in the

$$Y_{k\mu}(\omega,\eta)=(1/\sqrt{2\pi})\Upsilon_{k\mu}(\omega)e^{i\mu\eta}$$

by the new function

$$\Xi_{km}(\omega)=\sqrt{\cosh(\omega)}\Upsilon_{k\mu}(\omega),$$

is transformed into the Schrödinger equation

$$-\frac{\partial^2\Xi_{k\mu}(\omega)}{\partial^2\omega}+\frac{\mu^2+\frac{1}{4}}{\cosh^2(\omega)}\Xi_{k\mu}(\omega)=k^2\Xi_{k\mu}(\omega). \qquad (17)$$

Thus, knowing the function $\Upsilon_{k\mu}(\omega)$ on the upper sheet of the two-sheeted hyperboloid solves automatically the scattering problem for the Pöschl–Teller barrier shown in Fig. 2. It should be noted that in this case the scattering problem is on the infinite straight line instead of being on the infinite ray as it was in the previous case. Therefore, instead of one scattering quantity, i.e., the scattering amplitude, two quantities, the reflection and the transmission coefficients, must be calculated. Again, from the analysis of the asymptotic properties for $\omega\to\pm\infty$ of the wave given in Eq. (13) the transmission and reflection coefficients, D and R, can be found. For instance, we have[12]

$$D=\frac{\sinh^2(\pi k)}{\sinh^2(\pi k)+\cosh^2(\pi\mu)}. \qquad (18)$$

V. THE SPHERICAL FUNCTIONS FOR ARBITRARY NONCOMPACT RIEMANNIAN SYMMETRIC SPACES

Let G/K be a noncompact Riemannian symmetric space. To find the spherical functions on it, we begin to construct its Harish-Chandra plane waves. These waves are constructed by introducing an appropriate set of coordinates on G/K, the horispherical coordinates. These coordinates $G \ni x = nhk$ are related to the Iwasawa decomposition of the group $G = NAK$. If $\mathscr{G}$ is the Lie algebra of G and $\mathscr{G} = \mathscr{P} + \mathscr{K}$ is a Cartan decomposition, then the Iwasawa decomposition of G is the exponentiation of the decomposition $\mathscr{G} = \mathscr{N} + \mathscr{A} + \mathscr{K}$, where $\mathscr{A}$ is any maximal Abelian subspace of $\mathscr{P}$, and $\mathscr{N}$ is the sum of the root spaces of $\mathscr{G}$ taken over the set of all positive simple roots with respect to some ordering.

The Harish-Chandra plane waves are the eigenfunctions of the invariant differential operators on G/K under G which *do not* depend on (n), so the only dependence left is on the coordinate (h). However, since $\mathscr{A}$ is Abelian, the invariant differential operators acting only on the coordinates (h) have constant coefficients. Hence, the Harish-Chandra plane waves are the exponential functions depending on the variable $h \in A$ or equivalently on the variable $\alpha \in \mathscr{A}$. With $\lambda \in \mathscr{A}^*$, the set of complex-valued linear functions on $\mathscr{A}$, they can be written in the following manner:

$$\Psi_\lambda(xK) = e^{\lambda \mathbf{A}(x)}, \tag{19}$$

where $x \in G$, $\mathbf{A}: G \mapsto \mathscr{A}$, $\mathbf{A}(x) = \mathbf{A}(ne^a k) = a$.

To retain the standard notation of Harish-Chandra,[2] we supplement λ with half the sum of the positive roots of the Cartan algebra,

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} \alpha;$$

the eigenfunctions become

$$\Psi_\lambda(xK) = e^{(i\lambda + \rho)\mathbf{A}(x)}. \tag{20}$$

The inclusion of the factor ρ is convenient when dealing with invariant measures on G adapted to its Iwasawa decomposition.

From these plane waves, the generalized spherical functions are gained. Harish-Chandra and Helgason proved that the folding of Ψ_λ with the characters of the representations of K provide all generalized spherical functions on G, i.e., all eigenfunctions of the invariant differential operators with $f(kxk') = f(kx) = \chi(k) f(x)$, where $\chi(k)$ is the character of a representation of K. We explicitly have the following theorem.

Theorem:[2] The generalized spherical functions on G are given by

$$f_{\lambda,\chi}(x) = \int_{K/M} e^{(i\lambda+\rho)(A(kMx))} \chi(k^{-1}M) dk_M. \tag{21}$$

Here are $kM \in K/M$, M the centralizer of A in K, and

$$\int_{K/M} dk_M = 1.$$

Now, this theorem is nothing but the projecting procedure explained in Secs. II and III.

Find in G/K a set of coordinates that separates the action of the invariant differential operators; choose a complete set of eigenfunctions depending on a subset of the coordinates whose generators do not lie in $\mathscr{A}$; then, the projection of Ψ_λ into this set of eigenfunctions provides the generalized spherical functions depending on the complementary subset of coordinates and behaving like the chosen eigenfunctions with respect to the initial subset of coordinates. In the theorem, the complete set of eigenfunctions is given by the characters of the representations of K. Note that G/K still contains the subgroup K.

In Sec. III we use the reduction $SO(2,1) \supset SO(1,1)$ and the complete set of eigenfunctions are gained from the representations of $SO(1,1)$, i.e., the functions $e^{i\mu\eta}$, μ, $\eta \supset \mathbf{R}$. Now, these functions form a complete set of eigenfunctions. Further, the integrals over $SO(1,1)$ defining the projector operators, (12), are convergent, so that for this space the Harish-Chandra–Helgason approach is valid.

VI. CONCLUSION

Using the Harish-Chandra plane waves and the projection into irreducible representations of both SO(2) and SO(1,1), it was possible to extend the Harish-Chandra–Helgason formalism to treat all types of spherical harmonics on the hyperboloid SO(2,1)/SO(2).

The extension to arbitrary Riemannian symmetric spaces, G, depends on the existence in the different reductions $G \supset H$, of sufficiently many representations of H, and also of the convergence of the integrals of the type (5), (12), and (21) taken over these groups. Since these two issues are for noncompact groups by no means trivial, a general extension of the theory is still missing. However, the considerations of this paper provide, in particular cases, the way to extend the symmetry scattering theory to different coordinates.

ACKNOWLEDGMENTS

One of us, (YFS) would like to thank the Nuclear Theory Department of the University of Hamburg for its hospitality and the DAAD for the financial support. We all benefited from discussions with Professor N. Ya. Vilenkin, Professor K. A. Gridnev, Professor P. P. Kulish, Professor V. G. Neudatchin, Professor Yu. Kuperin, and Professor I. V. Komarov.

[1] R. F. Wehrhahn, Phys. Rev. Lett. **65**, 1294 (1990).
[2] Harish-Chandra, Am. J. Math. **80**, 241 (1958); **80**, 553 (1958); Acta Math. **116**, 1 (1966); S. Helgason, Adv. Math. **22**, 187 (1976); *Groups and Geometric Analysis, Integral Geometry, Invariant Differ-*

ential Operators, and Spherical Functions (Academic, New York, 1984); Ann. Math. **98**, 451 (1973).

[3] A. Perelomov, *Generalized Coherent States and Their Applications,* Texts and Monographs in Physics (Springer, Berlin, 1985).

[4] M. A. Semenov-Tyan-Shanskii, Cep. Matem. **40**, 562 (1976); P. Lax and R. S. Phillips, in *Scattering Theory and Automorphic Functions* (Mir, Moscow, 1979) (in Russian), Appendix 2, pp. 284–318.

[5] P. G. O. Freund, preprint Enrico Fermi Inst. 90-87; preprint Enrico Fermi Inst. 90-90.

[6] R. Wehrhahn, Scattering theory with symmetry as the interaction. Habilitationsschrift Universität Hamburg (1991); R. Wehrhahn, Yu. F. Smirnov, and A. M. Shirokov, J. Math. Phys. **32**, 3254 (1991).

[7] C. Wulfman and R. D. Levine, Chem. Phys. Lett. **60**, 372 (1979); Y. Alhassid, J. Engel, and J. Wu, Phys. Rev. Lett. **53**, 17 (1984); Y. Alhassid, F. Gürsey, and F. Iachello, Ann. Phys. (N.Y.) **167**, 181 (1986); N. Limic, J. Niederle, and R. Raczka, J. Math. Phys. **7**, 1861 (1966); **7**, 2026 (1966); **8**, 1079 (1967).

[8] I. M. Gel'fand, M. I. Graev, and N. Ya. Vilenkin, *Generalized Function 5* (Academic, New York, 1966); M. Mizony, Publ. Dept. Math. Lyon 2/P 1-27 (1982); Publ. Dept. Math. Lyon 3/A 1-50 (1987); SIAM J. Math. Anal. **14**(5), 987 (1983); M. Flensted-Jensen, *Analysis on Non-Riemannian Symmetric Spaces,* CBMS Regional Conf. Series 61 (American Mathematical Society, Providence, RI, 1986).

[9] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* (PWN–Polish Scientific Publishers, Warszawa, 1977).

[10] N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations,* Moscow, Nauka, 1965 (American Mathematical Society, Providence, RI, 1968).

[11] A. Erdély, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw–Hill, New York, 1953), Vol. I, p. 157 (16) and (17).

[12] L. D. Landau and E. M. Lifschitz, *Lehrbuch der theoretischen Physik* (Akad. Verlag, Berlin, 1974), Bd. III, p. 85.