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Scale dependence of the average potential around the maximum in ϕ^4 theories

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The average potential describes the physics at a length scale k^{-1} by averaging out the degrees of freedom with characteristic momenta larger than k. The dependence on k can be described by differential evolution equations. We solve these equations for the non-convex part of the potential around the origin in ϕ^4 theories, in the phase with spontaneous symmetry breaking. The average potential is real and approaches the convex effective potential in the limit $k \to 0$. Our calculation is relevant for processes for which the shape of the potential at a given scale is important, such as tunneling phenomena or inflation.

1. Introduction

Many properties of the quantum field theory for scalar fields can be investigated with the help of the effective potential [1]. In particular, the amount of spontaneous symmetry breaking is given by the location of the minimum of this potential. For practical applications the effective potential is usually calculated by the standard loop expansion (to which we shall refer as "naive" perturbation theory) [2]. For a four-dimensional theory with real scalar fields ϕ^a the one-loop contribution to the effective potential at the origin ($\phi^a = 0$) takes the form:

$$U^{(1)} \sim \left(V'(\rho)\right)^2 \ln\left(V'(\rho)/M^2\right), \tag{1.1}$$

where $V(\rho)$ is the classical potential, M an appropriate mass scale and

$$V'(\rho) = \frac{\partial V}{\partial \rho}, \qquad \rho = \tfrac{1}{2}\phi^a\phi_a. \tag{1.2}$$

Obviously, this contribution is real only for $V' \geqslant 0$, whereas $U^{(1)}$ develops an imaginary part if the curvature at the origin becomes negative. There have been attempts to give an interpretation to this imaginary part [3]. In any case, it is clear that eq. (1.1) cannot be a valid approximation to the effective potential which is, by its definition, real and convex. Indeed, the saddle-point approximation for the functional integral, which is the basic ingredient of perturbation theory, is no longer valid if unstable modes are present. For an expansion around the constant configuration $\rho = 0$ the mass term of the fluctuations is given by V'. "Naive" perturbation theory cannot be applied for negative V'. The same problem appears for non-vanishing ρ if the mass term of some of the fluctuations becomes negative. In spontaneously broken theories this occurs generically for a certain region in field space around the origin. One concludes that the loop expansion breaks down if ρ approaches the region where some of the fluctuations become unstable.

As long as all dimensionless couplings are small one may try to remedy this situation by an appropriate partial resummation of the loop contributions. Let us introduce in the functional integral some infrared cutoff characterized by a mass scale k. Then only quantum fluctuations with squared momenta larger than k^2 are included for the computation of the correspondingly modified effective potential U_k. One can treat U_k perturbatively, even at the origin, as long as $k^2 \gtrsim -V'$. We will study what happens with U_k as the infrared cutoff k decreases. The evolution of U_k with k can be described by differential evolution equations which are analogous to the renormalization group equations. The solution of these equations corresponds to a partial resummation of the "naive" perturbative contributions. There are various possible ways to implement an infrared cutoff. For example, one could introduce an additional mass term such that $V' \to V' + k^2$ *. We use here a different infrared cutoff which is particularly well suited for our purpose. We study the average action as formulated in ref. [4]. For a detailed presentation and discussion we refer the reader to refs. [4–6].

The average action is the effective action for averages of fields over a volume $\sim k^{-d}$. It describes the physics for processes which have a characteristic length scale $\sim k^{-1}$ acting as an effective infrared cutoff. All degrees of freedom with momenta larger than k are integrated out by the averaging. The average potential U_k is real and does not have to be convex (as opposed to the effective potential). It is the appropriate quantity for the study of processes, such as tunneling or inflation, which depend on the full range of the potential and especially on its non-convex part. It can be proven [4] that the average potential U_k approaches the effective potential U as $k \to 0$. It must therefore become convex in this limit. Nevertheless, for processes with a characteristic infrared cutoff the quantum fluctuations with $q^2 < k^2$ should not be included, as they are in the effective potential. It is therefore U_k and not U which is relevant for such processes.

We concentrate in this paper on the N-component ϕ^4 theory in arbitrary dimensions. We neglect the wave function renormalization effects. The corresponding anomalous dimensions are small [5] and do not change the qualitative

* This procedure gives similar qualitative results as the method employed in this paper.

behaviour of U_k in the region of interest. The one-loop contribution to the average potential reads [4,6]

$$U_k^{(1)}(\rho) = \tfrac{1}{2}(2\pi)^{-d}\int \mathrm{d}^d q\{\ln(P(q) + V'(\rho) + 2V''(\rho)\rho) + (N-1)\ln(P(q) + V'(\rho))\}. \tag{1.3}$$

This is the same formula as in the "naive" perturbation theory (1.1), except that the inverse propagator q^2 is now replaced by $P(q)$

$$P(q) = \frac{q^2}{1 - f_k^2(q)}, \tag{1.4}$$

$$f_k(q) = \exp\left\{-a(q^2/k^2)^\beta\right\}, \tag{1.5}$$

where a and β are constants of order 1. This form of P provides for an effective infrared cutoff for all modes with $q^2 \ll k^2$. In contrast, the contributions from the modes with $q^2 \gg k^2$ are not modified. We also note that for $k \to 0$ one recovers directly the standard one-loop contribution to the effective potential (1.1).

We now take the logarithmic derivative with respect to k and substitute U_k for V in the integral. This "renormalization group improvement" [4] gives the evolution equation ($t = \ln(k/\Lambda)$, $x = q^2$)

$$\frac{\partial}{\partial t}U_k = v_d \int_0^\infty \mathrm{d}x\, x^{d/2-1}\frac{\partial P}{\partial t}\left(\frac{1}{P + U_k' + 2U_k''\rho} + \frac{N-1}{P + U_k'}\right), \tag{1.6}$$

with

$$v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2). \tag{1.7}$$

Eq. (1.6) is the master equation for our investigation. It is valid as long as

$$\begin{aligned} &\bar{k}^2 + U_k'(\rho) > 0, \\ &\bar{k}^2 + U_k'(\rho) + 2U_k''(\rho)\rho > 0, \end{aligned} \tag{1.8}$$

where

$$\bar{k}^2 = \min P(q) \sim k^2. \tag{1.9}$$

If (1.8) is not fulfilled, the saddle-point approximation around a constant scalar configuration breaks down. One has to expand around a spin-wave solution [4,6] for $N > 2$, or around a vortex ($N = 2$), or a kink ($N = 1$).

In the following, we solve eq. (1.6) for $\rho \sim 0$, where the "naive" perturbative calculation of the effective potential breaks down. We derive analytic expressions for the average potential U_k at a given scale k. We obtain a simple picture for the behaviour of the potential as a function of k and we describe how convexity is approached in the limit $k \to 0$.

In sect. 2 we parametrize U_k in terms of its ρ-derivatives at $\rho = 0$. We solve the resulting infinite system of differential equations approximately, by truncating at some finite number of equations, retaining only a finite number of derivatives of U_k. In sect. 3, we solve the full evolution equation for the first derivative of U_k. This solution is the most important result of our investigation. It corresponds to the solution of the system of sect. 2 with an infinite number of equations. A summary and our conclusions are presented in sect. 4. Several technical points of the investigation are discussed in the appendices.

2. Solution of the evolution equation by truncation

We are interested in solving eq. (1.6) in the region around the origin ($\rho \sim 0$). As a first attempt we parametrize U_k in terms of its successive derivatives at $\rho = 0$ and truncate the resulting infinite system of differential equations at some finite number of equations. This is achieved by setting $\partial^n U_k(0)/\partial\rho^n = 0$ for all n bigger than some maximal number. For simplicity of the presentation we keep here only the first and second derivative of U_k. The effect of the third and fourth derivative will be discussed later. We define

$$-\bar{\mu}^2 = U_k'(0), \tag{2.1}$$

$$\bar{\lambda} = U_k''(0), \tag{2.2}$$

and obtain:

$$\frac{\mathrm{d}\bar{\mu}^2}{\mathrm{d}t} = -(N+2)v_d k^{d-2}\bar{\lambda} L_1^d(-\bar{\mu}^2), \tag{2.3}$$

$$\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}t} = -(N+8)v_d k^{d-4}\bar{\lambda}^2 L_2^d(-\bar{\mu}^2), \tag{2.4}$$

where

$$L_n^d(-\bar{\mu}^2) = -nk^{2n-d}\int_0^\infty \mathrm{d}x\, x^{d/2-1}\frac{\partial P}{\partial t}(P-\bar{\mu}^2)^{-(n+1)}. \tag{2.5}$$

The integrals L_n^d are always negative and diverge for $\bar{\mu}^2 \to \bar{k}^2$. Their properties for $|\bar{\mu}^2| \ll k^2$ can be found in ref. [5] and their behaviour near the pole at $\bar{\mu}^2 = \bar{k}^2$ is discussed in appendix A. As a result, $\bar{\mu}^2$ decreases for decreasing k. In the symmetric phase $\bar{\mu}^2$ vanishes for some finite scale k_s and becomes negative for $k < k_s$. The pole at $\bar{\mu}^2 = \bar{k}^2$ is not relevant. In the spontaneously broken phase, however, the mass term decreases slower than k^2 and becomes of the order of k^2 at some scale. The evolution equations are then dominated by the pole of L_n^d. We will see that the behaviour for k^2 near $\bar{\mu}^2$ is quite different from the usual one-loop renormalization group equations which correspond to a vanishing argument of L_n^d. On the other hand, the "naive" perturbative results remain valid in four dimensions for $k^2 \gg \bar{\mu}^2$. (One should include, however, the logarithmic k dependence of the quartic coupling and the "quadratic renormalization" [4] of $\bar{\mu}^2$.)

The leading pole in the functions L_n^d behaves like $\sim (\bar{k}^2 - \bar{\mu}^2)^{-(n+1/2)}$. In the vicinity of the pole we find

$$L_1^d = -K\epsilon^{-3/2},$$

$$L_2^d = -\tfrac{3}{2}K\epsilon^{-5/2}, \tag{2.6}$$

with

$$\epsilon = \frac{\bar{k}^2 - \bar{\mu}^2}{k^2} = \hat{k}^2 - \frac{\bar{\mu}^2}{k^2}. \tag{2.7}$$

Here K is a dimensionless constant defined in appendix A and $\hat{k}^2$ reads in our parametrization (A.6)

$$\hat{k}^2 = \frac{1}{1 - \exp(-2a)}. \tag{2.8}$$

For $a \gtrsim 1$, $\hat{k}^2$ is effectively independent of a and $\hat{k}^2 \sim 1$. Since we are interested in the behaviour of the evolution equations close to the pole it is convenient to rewrite eqs. (2.3), (2.4) in terms of ϵ. We can also absorb the explicit dependence on k and on various constants by defining the dimensionless quantity

$$\tilde{\lambda} = v_d K \lambda = v_d K k^{d-4} \bar{\lambda}. \tag{2.9}$$

Thus we obtain

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 2(\hat{k}^2 - \epsilon) - A_1 \tilde{\lambda} \epsilon^{-3/2}, \tag{2.10}$$

$$\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}t} = -(4-d)\tilde{\lambda} + B_1 \tilde{\lambda}^2 \epsilon^{-5/2}, \tag{2.11}$$

$$A_1 = (N+2), \qquad B_1 = \tfrac{3}{2}(N+8). \tag{2.12}$$

For any non-zero value of $\tilde{\lambda}$ the right-hand side of (2.10) and (2.11) blows up as $\epsilon \to 0$.

A new set of variables

$$\chi = \tilde{\lambda}\epsilon^{-3/2}, \tag{2.13}$$

$$\tau = \tau(t), \quad \text{with} \quad \mathrm{d}\tau = \frac{\mathrm{d}t}{\epsilon}, \tag{2.14}$$

makes the behaviour of the system eqs. (2.10), (2.11) more transparent

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} = \left(2\hat{k}^2 - 2\epsilon - A_1\chi\right)\epsilon, \tag{2.15}$$

$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = \left(B_1 + \tfrac{3}{2}A_1\right)\chi^2 - \left(3\hat{k}^2 + (1-d)\epsilon\right)\chi. \tag{2.16}$$

Depending on the number of dimensions, the last system of differential equations has two or three fixed points. Only one is infrared stable, namely

$$(\epsilon_{\mathrm{f}}, \chi_{\mathrm{f}}) = \left(0, \frac{3\hat{k}^2}{\frac{3}{2}A_1 + B_1}\right). \tag{2.17}$$

The other fixed points are discussed in appendix B and turn out to be neither infrared nor ultraviolet stable. For ϵ, χ sufficiently close to the fixed point (2.17) we can linearize eq. (2.15) in ϵ and neglect the terms $\sim \epsilon\chi$ in eq. (2.16). We also write

$$\chi = \chi_{\mathrm{f}} + (\delta\chi), \tag{2.18}$$

and neglect terms $\sim (\delta\chi)^2$. The equations simplify considerably

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} = C\epsilon, \tag{2.19}$$

$$\frac{\mathrm{d}(\delta\chi)}{\mathrm{d}\tau} = 3\hat{k}^2(\delta\chi), \tag{2.20}$$

with

$$C = 2\hat{k}^2 - A_1\chi_{\mathrm{f}} = \frac{2B_1\hat{k}^2}{B_1 + \frac{3}{2}A_1}. \tag{2.21}$$

With the boundary conditions

$$\tau(t_0) = 0, \qquad \epsilon(\tau = 0) = \epsilon_0, \qquad (\delta\chi)(\tau = 0) = (\delta\chi)_0, \tag{2.22}$$

one obtains from eqs. (2.19), (2.14)

$$t = t_0 + \frac{\epsilon_0}{C}(\exp(C\tau) - 1). \tag{2.23}$$

This indicates that t approaches a finite value t_f, for $\tau \to -\infty$

$$t_f \equiv \lim_{\tau \to -\infty} t(\tau) = t_0 - \frac{\epsilon_0}{C}. \tag{2.24}$$

The behaviour of the system becomes now apparent: Both ϵ and $\delta\chi$ vanish for $t \to t_f$

$$\epsilon = \epsilon_0 + C(t - t_0) = C(t - t_f), \tag{2.25}$$

$$(\delta\chi) = (\delta\chi)_0 \left\{ \frac{C}{\epsilon_0}(t - t_f) \right\}^{3\hat{k}^2/C}. \tag{2.26}$$

In terms of the original variables k, $\bar{\mu}^2$, $\bar{\lambda}$ our solution reads

$$\bar{\mu}^2 = \bar{k}^2 - Ck^2 \ln\left(\frac{k}{k_f}\right), \tag{2.27}$$

$$\bar{\lambda} = \frac{3C^{5/2}}{2v_d KB_1} k^{4-d} \left[\ln\left(\frac{k}{k_f}\right)\right]^{3/2} \left\{1 + c_0 \left[\ln\left(\frac{k}{k_f}\right)\right]^{3\hat{k}^2/C}\right\}, \tag{2.28}$$

where, in an obvious notation

$$k_f = k_0 \exp\left(-\frac{\epsilon_0}{C}\right). \tag{2.29}$$

The integration constant c_0 is determined by the initial condition for $\bar{\lambda}$. The behaviour indicated by our solution can be summarized as follows: As soon as $\bar{\mu}^2$ approaches the vicinity of the pole, the system becomes strongly attracted towards the fixed point. The ratio $\bar{\mu}^2/\bar{k}^2$ increases with decreasing k and reaches the boundary value one for a finite value k_f. The quartic coupling $\bar{\lambda}$ evolves $\sim \epsilon^{3/2}$ (plus small corrections) and reaches zero at the same value k_f. For $k = k_f$ the average potential is purely quadratic, namely

$$U_{k_f}(\rho) = -\bar{k}_f^2 \rho \tag{2.30}$$

We cannot continue our solution for $k < k_f$, since for $\bar{\mu}^2 > \bar{k}^2$ the condition (1.8) is violated. In this case the constant solution is not a minimum of the constrained action [6], and the expansion should be performed around the true minimum (spin waves for $N > 2$, vortices for $N = 2$, or kinks for $N = 1$). For $N > 2$ the average potential obtained from an expansion around the spin-wave solution, has, in the tree approximation, the form [6]

$$U_k(\rho) = -\bar{k}^2\rho. \tag{2.31}$$

This matches exactly (2.30) for $k = k_f$. It is remarkable how the evolution of the average potential, with an expansion around the constant solution, leads naturally to the form implied by the expansion around the spin-wave configuration.

We next enlarge the truncated system of differential equations and include also the third and fourth derivative of U_k at the origin. The technical discussion of the infrared fixed point behaviour is similar as above and is presented in appendix C. The main conclusion is that the fixed point value for χ_f increases (see table C.1) as the number of differential equations is enlarged. As a result the decrease of ϵ becomes slower and the ratio k_f/k_0 decreases. The series for χ_f converges only slowly as the number of derivatives is increased. There are two points which are not yet settled and require further investigation:

(i) The fixed point at $\epsilon = 0$ is infrared attractive as long as

$$2\hat{k}^2 - A_1\chi_f > 0. \tag{2.32}$$

Since χ_f seems to increase with the number of equations of the truncated system, one may wonder whether the asymptotic value of χ_f, without a truncation (i.e. an infinite number of equations) still satisfies the above relation.

(ii) In the truncated system $\epsilon = 0$ is reached at a finite k_f, which decreases with the number of equations. It is an interesting possibility that k_f reaches zero without a truncation. In sect. 3 we address both the above questions. We show that $\epsilon = 0$ is a stable infrared fixed point of the full evolution equation for the first derivative of the potential. Moreover, $\epsilon = 0$ is reached asymptotically only for $k \to 0$.

3. The full evolution equation for the potential at the origin

In this section we treat eq. (1.6) as a partial differential equation depending on two variables t and ρ. This corresponds to an infinite number of ordinary differential equations for an infinite number of higher ρ-derivatives of U_k, considered as functions of t only. In particular, we are interested in the derivative with respect to ρ of eq. (1.6)

$$\frac{\partial}{\partial t}U_k'(\rho) = (N-1)v_d k^{d-2}U_k''(\rho)L_1^d(U_k'(\rho))$$
$$+ v_d k^{d-2}(3U_k''(\rho) + 2\rho U_k'''(\rho))L_1^d(U_k'(\rho) + 2\rho U_k''(\rho)). \tag{3.1}$$

In analogy to (2.7) we introduce the function

$$\epsilon(k,\rho) = k^{-2}U_k'(\rho) + \hat{k}^2, \tag{3.2}$$

and we use the dimensionless field variable

$$\tilde{\rho} = k^{2-d}\rho. \tag{3.3}$$

In terms of these variables eq. (3.1) reads

$$\frac{\partial \epsilon}{\partial t} = (d-2)\tilde{\rho}\frac{\partial \epsilon}{\partial \tilde{\rho}} + 2(\hat{k}^2 - \epsilon) + (N-1)v_d\frac{\partial \epsilon}{\partial \tilde{\rho}}L_1^d(\epsilon)$$
$$+ v_d\left(3\frac{\partial \epsilon}{\partial \tilde{\rho}} + 2\tilde{\rho}\frac{\partial^2 \epsilon}{\partial \tilde{\rho}^2}\right)L_1^d\left(\epsilon + 2\tilde{\rho}\frac{\partial \epsilon}{\partial \tilde{\rho}}\right). \tag{3.4}$$

Here the partial derivative $\partial/\partial t$ is now taken at fixed $\tilde{\rho}$ (not fixed ρ!). For small ϵ we can use the approximation (2.6) and write:

$$\frac{\partial \epsilon}{\partial t} = 2\hat{k}^2 - Q\frac{\partial \epsilon}{\partial \tilde{\rho}}\epsilon^{-3/2} + \Delta, \tag{3.5}$$

with

$$Q = (N+2)v_d K. \tag{3.6}$$

The contribution

$$\Delta = (d-2)\tilde{\rho}\frac{\partial \epsilon}{\partial \tilde{\rho}} - 2\epsilon + 2v_d\tilde{\rho}\frac{\partial^2 \epsilon}{\partial \tilde{\rho}^2}L_1^d\left(\epsilon + 2\tilde{\rho}\frac{\partial \epsilon}{\partial \tilde{\rho}}\right)$$
$$+ 3v_d\frac{\partial \epsilon}{\partial \tilde{\rho}}\left\{L_1^d\left(\epsilon + 2\tilde{\rho}\frac{\partial \epsilon}{\partial \tilde{\rho}}\right) + K\epsilon^{-3/2}\right\} + (N-1)v_d\frac{\partial \epsilon}{\partial \tilde{\rho}}\left\{L_1^d(\epsilon) + K\epsilon^{-3/2}\right\} \tag{3.7}$$

does not influence the leading behaviour for small ϵ and $\tilde{\rho}$, and will be neglected in the following.

In terms of the variables

$$\gamma = \left(\frac{\epsilon}{\hat{k}^2}\right)^{-1/2}, \qquad y = \frac{\hat{k}^3}{Q}\tilde{\rho}, \tag{3.8}$$

eq. (3.5) reads

$$\frac{\partial \gamma}{\partial t} = -\gamma^3\left(1 + \frac{\partial \gamma}{\partial y}\right). \tag{3.9}$$

The most general solution of the above partial differential equation satisfies the relation

$$\gamma + y = F\left(\frac{1}{2\gamma^2} - t\right). \tag{3.10}$$

The arbitrary function F is undetermined until boundary conditions for $\gamma(t, y)$ are imposed.

In order to implement the boundary conditions it is more convenient to return to the variables ρ, ϵ. Then at some scale k_0 we impose

$$\epsilon(0, \rho) = \epsilon_0(\rho). \tag{3.11}$$

The function F is now completely determined through the equation:

$$F\left(\frac{\epsilon_0(\rho)}{2\hat{k}^2} - t_0\right) = T\rho + \left(\frac{\epsilon_0(\rho)}{\hat{k}^2}\right)^{-1/2}, \tag{3.12}$$

where

$$T = \frac{\hat{k}^3 k_0^{2-d}}{Q}. \tag{3.13}$$

The most notable property of the above equation is that it specifies the essential features of F for the most general class of boundary conditions which are physically relevant. We will only assume that, for a ϕ^4 theory, $U'_{k_0}(\rho)$ is a monotonically increasing function of ρ which diverges for $\rho \to \infty$. Then eqs. (3.2) and (3.12) imply that, for a sufficiently large argument, F is a monotonically increasing function, which approaches infinity for an infinite argument. This property is independent of the precise form of the boundary conditions. The asymptotic behaviour of ϵ or U'_k for $k \to 0$ can now be presented in a concise form. It is given by the relation

$$U'_k(\rho) = -\bar{k}^2 + k^2\epsilon(k, \rho) = -\bar{k}^2 + k^2 \frac{\epsilon(k, 0)}{\left\{1 - \epsilon(k, 0)^{1/2}(\hat{k}^2/Q)k^{2-d}\rho\right\}^2}, \tag{3.14}$$

with

$$\epsilon(k, 0) = \frac{\hat{k}^2}{\{F(\ln(\Lambda/k))\}^2}. \tag{3.15}$$

We conclude that $\epsilon(k, 0)$ vanishes for $k \to 0$. Its precise functional dependence on k is determined by the initial conditions for $U_{k_0}(\rho)$. Eq. (3.14) is the most essential outcome of our investigation. The leading term for $U_k'(\rho)$ is in exact agreement with the result (2.31) which was obtained in ref. [6] from an expansion around the spin-wave configuration. It is remarkable that eq. (3.14) has been obtained through an expansion around the configuration of a constant scalar field, which remains now valid for all values of k. The essential physics has been reproduced by the "renormalization group improvement" without ever having to resort to the use of the spin-wave configuration. Eq. (3.14) contains also the deviations from the leading behaviour (2.31) for non-vanishing values of k. We believe that very close to the origin these corrections are actually more reliably estimated by the present method than by the loop expansion around the spin-wave configuration whose shortcomings have been discussed in ref. [4].

At this point some remarks are due on the region of validity of the solution (3.14). The contribution Δ, given by (3.7), which was neglected in our discussion of the differential equation (3.5), remains small as long as $\epsilon \ll 1$ and

$$\rho \ll \frac{Q}{\hat{k}^2} k^{d-2} \epsilon(k, 0)^{-1/2}. \tag{3.16}$$

In particular, one could infer from eq. (3.14) that U_k has a minimum at

$$\rho = \frac{Q}{\hat{k}^3} k^{d-2} \left\{ \hat{k} \epsilon(k, 0)^{-1/2} - 1 \right\}. \tag{3.17}$$

This point occurs, however, outside the range of validity (3.16) and should not be trusted. We note that for $d > 2$ the region of validity for ρ shrinks to zero with a power of k. In contrast, this region increases for $k \to 0$ in two dimensions. We also note that the determination (3.12) of F for very large arguments involves large values of ρ where neglecting Δ is is not justified. This may be relevant in two dimensions where no spontaneous symmetry breaking is expected for $N \geq 2$ (compare ref. [5]).

The higher derivatives of the potential near the origin can be computed from eq. (3.14). One finds

$$U_k''(\rho) = \frac{2\hat{k}^2}{Q} \epsilon(k, \rho)^{3/2} k^{4-d},$$

$$U_k'''(\rho) = \frac{6\hat{k}^4}{Q^2} \epsilon(k, \rho)^4 k^{6-2d},$$

$$U_k''''(\rho) = \frac{24\hat{k}^6}{Q^3} \epsilon(k, \rho)^{5/2} k^{8-3d}. \tag{3.18}$$

Inserting the definitions (C.1), (C.3) and (C.6) one obtains for the fixed-point values

$$\chi_{\mathrm{f}} = \frac{2\hat{k}^2}{N+2},$$

$$\psi_{\mathrm{f}} = \frac{6\hat{k}^4}{(N+2)^2},$$

$$\omega_{\mathrm{f}} = \frac{24\hat{k}^6}{(N+2)^3}. \qquad (3.19)$$

These values correspond to the fixed points without a truncation of the master equation (1.6). In the language of sect. 2 we now retain an infinite number of differential equations for the derivatives at the origin. We display the numerical values (for $\beta = 3$, $\hat{k}^2 = 1.175$) in table C.1, where they can be compared with the corresponding values for the truncated system discussed in sect. 2 and appendix C.

Before we conclude this section, it is instructive to discuss a specific class of boundary conditions, in order to explicitly obtain the function F for that class. We consider boundary conditions of the form:

$$U'_{k_0}(\rho) = -\bar{\mu}^2(k_0) + \bar{\lambda}(k_0)\rho. \qquad (3.20)$$

The function F can now be determined with the help of eqs. (3.2) and (3.12). One finds

$$F(x) = \frac{1}{\sqrt{2(x+t_0)}} + \frac{2}{S}(x+t_0) - \frac{R}{S}, \qquad (3.21)$$

where

$$R = 1 - \frac{\bar{\mu}^2(k_0)}{k_0^2\hat{k}^2},$$

$$S = \frac{Qk_0^{d-4}}{\hat{k}^5}\bar{\lambda}(k_0). \qquad (3.22)$$

For large x the first term on the right-hand side of the last expression becomes negligible, and $F(x)$ is a monotonically increasing function which diverges linearly for $x \to \infty$. For $k \to 0$ the average potential is given by (3.14) with

$$\epsilon(k, 0) = \frac{\hat{k}^2S^2}{4}\left\{\ln\left(\frac{k_0}{k}\right)\right\}^{-2}. \qquad (3.23)$$

4. Summary and conclusions

The average potential U_k describes the physics at a scale k by averaging out the degrees of freedom with momenta larger than k. It is a real quantity which does not have to be convex. It is the appropriate tool for the study of physical processes such as inflation or tunneling, where the non-convex part of the potential is needed. Typically, such processes have an inherent length scale l which acts as an infrared cutoff. By choosing $k \sim l^{-1}$ one accounts for all quantum fluctuations with squared momentum q^2 larger than l^{-2}. On the other hand, the contributions from modes with $q^2 < l^{-2}$ are not included, as appropriate for a physical infrared cutoff given by l. (Of course the relevant length scale l has to be determined by the physics of the process under investigation.)

In this paper we have solved the evolution equations for the average potential around the origin in spontaneously broken N-component ϕ^4 theories. This is the region where the potential is non-convex. We have used a saddle point approximation around a constant-field configuration. We have found that, for $k \to 0$, the potential displays a very simple behaviour

$$U_k \sim -\tfrac{1}{2}k^2\phi^a\phi_a. \tag{4.1}$$

The exact outcome of our investigation, which contains the first corrections to the above expression, is given by eq. (3.14). Our results are in agreement with a previous one-loop calculation of the non-convex part of the average potential for $N > 2$, which has been based on an expansion around a spin-wave configuration [6]. The remarkable aspect of our calculation is that we have never made use of the spin-wave configuration. The essential physics has been reproduced through the expansion around the constant configuration and the renormalization group evolution of the potential. In the limit $k = 0$ we recover the convex effective potential.

The non-convex part of the average potential is important for the calculation of the tunneling rate between different minima, for example between the two minima of the model with discrete symmetry ($N = 1$) if a small symmetry breaking term is added. Usually calculations of the tunneling rate use the perturbative potential (often even the classical potential) which has near the origin the form

$$U_\mathrm{P} = -\tfrac{1}{2}\mu_\mathrm{P}^2\phi^2. \tag{4.2}$$

The validity of these calculations depends on the question whether the physical infrared cutoff l^{-2} relevant for the tunneling process is larger than μ_P^2. If not, one has to replace (4.2) very near to the origin by

$$U \sim l^{-2}\phi^2, \tag{4.3}$$

with an appropriate modification for ϕ away from the origin. The true tunneling rate will disagree with the predictions of "naive" perturbation theory in this case. Similar problems may arise in inflationary cosmology when the non-convex part of the "naive" perturbative potential is used. It should be noted that our result (3.14) is valid only for a certain region around the origin. The potential around the minima has been calculated in ref. [5]. One has to perform a straightforward (although technically involved) calculation in order to obtain the form of the average potential for the remaining regions of the field space. We hope to report on this soon.

We would like to thank M. Drees for his help with the numerical checks of the results.

Appendix A

The integral

$$L_n^d(w) = -nk^{2n-d}\int_0^\infty \mathrm{d}x\, x^{d/2-1}\frac{\partial P}{\partial t}(P+w)^{-(n+1)}, \tag{A.1}$$

$$P(x) = \frac{x}{1-f_k^2(x)} = \frac{x}{1-\exp\left\{-2a\left(x/k^2\right)^\beta\right\}}, \tag{A.2}$$

has a pole at $w = -\bar{k}^2$, where

$$\bar{k}^2 = \min P(x). \tag{A.3}$$

We are interested here in the behaviour of L_n^d in the vicinity of the pole. (See ref. [5] for $w \geq 0$.) By defining the dimensionless quantities

$$y = \frac{x}{k^2},\quad \hat{P} = \frac{P}{k^2},\quad \hat{w} = \frac{w}{k^2},\quad \hat{k}^2 = \frac{\bar{k}^2}{k^2}, \tag{A.4}$$

we can rewrite the integral as

$$L_n^d(\hat{w}) = 2n\int_0^\infty \mathrm{d}y\, y^{d/2+1}\frac{\partial\left(\hat{P}/y\right)}{\partial y}\left(\hat{P}+\hat{w}\right)^{-(n+1)}. \tag{A.5}$$

We work with the particular family of parametrizations [5]

$$\beta = \frac{\exp(2a)-1}{2a}, \tag{A.6}$$

for which $\hat{P}$ has its minimum at $y=1$ with the value

$$\hat{P}(1)=\hat{k}^2=\frac{1}{1-\exp(-2a)}. \tag{A.7}$$

Since $\hat{P}$ is a monotonic function of y on either side of the minimum we can define a new variable z, such that

$$\hat{P}=\hat{k}^2+z^2. \tag{A.8}$$

The variable z is related to $\hat{P}(y)$ through the equations

$$z=+\left(\hat{P}-\hat{k}^2\right)^{1/2} \quad \text{for } y>1,$$

$$z=-\left(\hat{P}-\hat{k}^2\right)^{1/2} \quad \text{for } y<1. \tag{A.9}$$

We also define

$$\epsilon=\hat{w}+\hat{k}^2, \tag{A.10}$$

and substitute in (A.5). We find

$$L_n^d(\epsilon)=-2\int_{-\infty}^{\infty}\mathrm{d}z\, G_n^d(z)(z^2+\epsilon)^{-(n+1)}, \tag{A.11}$$

$$G_n^d(z)=2nzy^{d/2}\left(\frac{\hat{P}}{y}\left(\frac{\partial\hat{P}}{\partial y}\right)^{-1}-1\right). \tag{A.12}$$

Due to the relation

$$L_{n+1}^d=-\frac{1}{n}\frac{\partial}{\partial\epsilon}L_n^d \tag{A.13}$$

we only have to evaluate L_1^d. Force $\epsilon\to 0$ most of the contribution to the integral comes from the region $y\sim 1$. Therefore, it is a good approximation to expand

$$\hat{P}=\hat{k}^2+a_2(y-1)^2+a_3(y-1)^3+\dots \tag{A.14}$$

with

$$a_2=\frac{1}{2}\frac{\mathrm{d}^2\hat{P}}{\mathrm{d}y^2}\bigg|_{y=1}=\frac{\exp(2a)}{2}\left(\frac{1}{1-\exp(-2a)}-\frac{1}{2a}\right). \tag{A.15}$$

The inversion is now straightforward and we obtain

$$z = a_2^{1/2}(y-1)\left\{1 + \frac{a_3}{2a_2}(y-1)\right\}, \tag{A.16}$$

$$y = 1 + a_2^{-1/2}z - \tfrac{1}{2}a_2^{-2}a_3 z^2, \tag{A.17}$$

$$\mathrm{d}y = \left(a_2^{-1/2} - a_2^{-2}a_3 z\right)\mathrm{d}z. \tag{A.18}$$

The function $G(z)$ can now be expanded around $z=0$. Near the pole of L_1^d one finds

$$L_1^d(\epsilon) = -2G_1^d(0)\int_{-\infty}^{\infty}\mathrm{d}z\left(z^2+\epsilon\right)^{-2} - 2G_1^{d\prime}(0)\int_{-\infty}^{\infty}\mathrm{d}z\, z\left(z^2+\epsilon\right)^{-2}$$

$$= -K\epsilon^{-3/2} - R\epsilon^{-1}, \tag{A.19}$$

with

$$K = \pi a_2^{-1/2}\hat{k}^2, \tag{A.20}$$

$$R = \left(\frac{d-2}{a_2} - \frac{2a_3}{a_2^2}\right)\hat{k}^2 - 4. \tag{A.21}$$

The next correction is proportional to $\epsilon^{-1/2}$. In summary, the leading contributions to the first four integrals are given by the following expressions:

$$L_1^d(\epsilon) = -K\epsilon^{-3/2}, \qquad L_2^d(\epsilon) = -\tfrac{3}{2}K\epsilon^{-5/2},$$

$$L_3^d(\epsilon) = -\tfrac{15}{8}K\epsilon^{-7/2}, \qquad L_4^d(\epsilon) = -\tfrac{35}{16}K\epsilon^{-9/2}. \tag{A.22}$$

Appendix B

We want to investigate the stability of the possible fixed points of the system of differential equations

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} = \left(2\hat{k}^2 - 2\epsilon - A_1\chi\right)\epsilon,$$

$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = -(4-d)\chi\epsilon + B_1\chi^2 - \tfrac{3}{2}\chi\left(2\hat{k}^2 - 2\epsilon - A_1\chi\right). \tag{B.1}$$

We are looking for values (ϵ_f, χ_f) which annihilate the right-hand side of these equations. Two solutions are

$$(\epsilon_1, \chi_1) = \left(0, \frac{3\hat{k}^2}{\frac{3}{2}A_1 + B_1}\right), \tag{B.2}$$

$$(\epsilon_2, \chi_2) = (0, 0). \tag{B.3}$$

For $d < 4$, there is another solution

$$(\epsilon_3, \chi_3) = \left(\frac{B_1\hat{k}^2}{B_1 + (4-d)\frac{1}{2}A_1}, \frac{(4-d)\hat{k}^2}{B_1 + (4-d)\frac{1}{2}A_1}\right). \tag{B.4}$$

For ϵ, χ sufficiently close to the fixed point we can write

$$\epsilon = \epsilon_f + (\delta\epsilon), \qquad \chi = \chi_f + (\delta\chi), \tag{B.5}$$

and, keeping terms only linear in $(\delta\epsilon)$, $(\delta\chi)$, obtain

$$\frac{d}{d\tau}\begin{pmatrix}(\delta\epsilon)\\(\delta\chi)\end{pmatrix} = \hat{M}_f\begin{pmatrix}(\delta\epsilon)\\(\delta\chi)\end{pmatrix}. \tag{B.6}$$

The last equation has the solution

$$\begin{pmatrix}(\delta\epsilon)\\(\delta\chi)\end{pmatrix} = \exp\left(\hat{M}_f\tau\right)\begin{pmatrix}(\delta\epsilon)_0\\(\delta\chi)_0\end{pmatrix} \tag{B.7}$$

For the 2×2 matrix $\hat{M}_f\tau$ we have the identity

$$\exp\left(\hat{M}_f\tau\right) = b_0(\tau) + b_1(\tau)\hat{M}_f. \tag{B.8}$$

Here the functions $b_i(\tau)$ are given by the relation

$$\exp(\lambda_i\tau) = b_0(\tau) + b_1(\tau)\lambda_i, \tag{B.9}$$

where λ_i are the two (non-equal) eigenvalues of $\hat{M}_f$. This gives:

$$b_0 = \frac{\lambda_1\exp(\lambda_2\tau) - \lambda_2\exp(\lambda_1\tau)}{\lambda_1 - \lambda_2},$$

$$b_1 = \frac{\exp(\lambda_1\tau) - \exp(\lambda_2\tau)}{\lambda_1 - \lambda_2}. \tag{B.10}$$

It is clear that a stable infrared (ultraviolet) fixed point requires both eigenvalues of $\hat{M}_f$ to be positive (negative). It is a straightforward exercise to calculate these eigenvalues for the points given in (B.2)–(B.4). For (ϵ_1, χ_1) they are

$$\begin{pmatrix} \lambda_1^1 \\ \lambda_2^1 \end{pmatrix} = \begin{pmatrix} \dfrac{2B_1\hat{k}^2}{B_1 + \frac{3}{2}A_1} \\ 3\hat{k}^2 \end{pmatrix}, \tag{B.11}$$

whereas for (ϵ_2, χ_2) one has

$$\begin{pmatrix} \lambda_1^2 \\ \lambda_2^2 \end{pmatrix} = \begin{pmatrix} 2\hat{k}^2 \\ -3\hat{k}^2 \end{pmatrix}. \tag{B.12}$$

Finally one finds for (ϵ_3, χ_3)

$$\begin{pmatrix} \lambda_1^3 \\ \lambda_2^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left\{\left[3\hat{k}^2 - (d+1)\epsilon_3\right] + \left\{\left[3\hat{k}^2 - (d+1)\epsilon_3\right]^2 + 8(4-d)\hat{k}^2\epsilon_3\right\}^{1/2}\right\} \\ \frac{1}{2}\left\{\left[3\hat{k}^2 - (d+1)\epsilon_3\right] - \left\{\left[3\hat{k}^2 - (d+1)\epsilon_3\right]^2 + 8(4-d)\hat{k}^2\epsilon_3\right\}^{1/2}\right\} \end{pmatrix}. \tag{B.13}$$

It is clear that (ϵ_1, χ_1) is the only infrared stable fixed point, while (ϵ_2, χ_2) and (ϵ_3, χ_3) are neither infrared nor ultraviolet stable. We identify (ϵ_3, χ_3) with the fixed point corresponding to the phase transition between the symmetric and the spontaneously broken phase. (It is infrared unstable for ϵ and infrared stable for $\tilde{\lambda}$.) Although we recover qualitatively the correct phase structure, our treatment of (ϵ_3, χ_3) is not very precise quantitatively, since ϵ is not small. For a better treatment of this last fixed point one has to evaluate the integrals L_n^d away from the pole [5].

Appendix C

In this appendix we enlarge the truncated system of evolution equations by taking into account the third and fourth derivative of the average potential at the origin. We define

$$-\bar{\mu}^2 = U_k'(0), \qquad \bar{\lambda} = U_k''(0),$$
$$\bar{\nu} = U_k'''(0), \qquad \bar{\sigma} = U_k''''(0), \tag{C.1}$$

and derive from (1.6) the following evolution equations:

$$\frac{\mathrm{d}\bar{\mu}^2}{\mathrm{d}t} = -(N+2)v_d k^{d-2}\bar{\lambda} L_1^d(-\bar{\mu}^2),$$

$$\frac{\mathrm{d}\bar{\lambda}}{\mathrm{d}t} = -(N+8)v_d k^{d-4}\bar{\lambda}^2 L_2^d(-\bar{\mu}^2) + (N+4)v_d k^{d-2}\bar{\nu} L_1^d(-\bar{\mu}^2),$$

$$\frac{\mathrm{d}\bar{\nu}}{\mathrm{d}t} = 2(N+26)v_d k^{d-6}\bar{\lambda}^3 L_3^d(-\bar{\mu}^2) - 3(N+14)v_d k^{d-4}\bar{\nu}\bar{\lambda} L_2^d(-\bar{\mu}^2)$$
$$+(N+6)v_d k^{d-2}\bar{\sigma} L_1^d(-\bar{\mu}^2),$$

$$\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}t} = -6(N+80)v_d k^{d-8}\bar{\lambda}^4 L_4^d(-\bar{\mu}^2) + 12(N+44)v_d k^{d-6}\bar{\nu}\bar{\lambda}^2 L_3^d(-\bar{\mu}^2)$$
$$-3(N+24)v_d k^{d-4}\bar{\nu}^2 L_2^d(-\bar{\mu}^2) - 4(N+20)v_d k^{d-4}\bar{\lambda}\bar{\sigma} L_2^d(-\bar{\mu}^2). \quad \text{(C.2)}$$

The integrals L_n^d are discussed in the appendix A. In the vicinity of the pole at $\bar{\mu}^2 = \bar{k}^2$, it is convenient to define the dimensionless quantities

$$\epsilon = \frac{\bar{k}^2 - \bar{\mu}^2}{k^2} = \hat{k}^2 - \frac{\bar{\mu}^2}{k^2},$$

$$\tilde{\lambda} = v_d K k^{d-4}\bar{\lambda},$$

$$\tilde{\nu} = (v_d K)^2 k^{2d-6}\bar{\nu},$$

$$\tilde{\sigma} = (v_d K)^3 k^{3d-8}\bar{\sigma}. \quad \text{(C.3)}$$

(We work again with the parametrization (A.6), for which $\hat{k}^2$ and K are given by (A.7) and (A.20) respectively.) Substituting in eqs. (C.2) gives

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = 2(\hat{k}^2 - \epsilon) - A_1\tilde{\lambda}\epsilon^{-3/2}$$

$$\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}t} = -(4-d)\tilde{\lambda} + B_1\tilde{\lambda}^2\epsilon^{-5/2} - B_2\tilde{\nu}\epsilon^{-3/2},$$

$$\frac{\mathrm{d}\tilde{\nu}}{\mathrm{d}t} = -(6-2d)\tilde{\nu} - C_1\tilde{\lambda}^3\epsilon^{-7/2} + C_2\tilde{\lambda}\tilde{\nu}\epsilon^{-5/2} - C_3\tilde{\sigma}\epsilon^{-3/2},$$

$$\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}t} = -(8-3d)\tilde{\sigma} + D_1\tilde{\lambda}^4\epsilon^{-9/2} - D_2\tilde{\lambda}^2\tilde{\nu}\epsilon^{-7/2} + D_3\tilde{\nu}^2\epsilon^{-5/2} + D_4\tilde{\lambda}\tilde{\sigma}\epsilon^{-5/2}, \quad \text{(C.4)}$$

with

$$A_1 = N+2,$$
$$B_1 = \tfrac{3}{2}(N+8), \qquad B_2 = N+4,$$
$$C_1 = \tfrac{15}{4}(N+26), \qquad C_2 = \tfrac{9}{2}(N+14), \qquad C_3 = N+6,$$
$$D_1 = \tfrac{105}{8}(N+80), \quad D_2 = \tfrac{45}{2}(N+44), \quad D_3 = \tfrac{9}{2}(N+24), \quad D_4 = 6(N+20). \tag{C.5}$$

We rescale $\tilde{\lambda}$, $\tilde{\nu}$ and $\tilde{\sigma}$ by appropriate powers of ϵ

$$\chi = \tilde{\lambda}\epsilon^{-3/2}, \qquad \psi = \tilde{\nu}\epsilon^{-4/2}, \qquad \omega = \tilde{\sigma}\epsilon^{-5/2}, \tag{C.6}$$

and use the variable τ defined in eq. (2.14). In terms of the new variables eqs. (C.4) read

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} = \left(2\hat{k}^2 - 2\epsilon - A_1\chi\right)\epsilon,$$
$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau} = -(4-d)\chi\epsilon + B_1\chi^2 - B_2\psi - \tfrac{3}{2}\chi\left(2\hat{k}^2 - 2\epsilon - A_1\chi\right),$$
$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = -(6-2d)\psi\epsilon - C_1\chi^3 + C_2\chi\psi - C_3\omega - \tfrac{4}{2}\psi\left(2\hat{k}^2 - 2\epsilon - A_1\chi\right),$$
$$\frac{\mathrm{d}\omega}{\mathrm{d}\tau} = -(8-3d)\omega\epsilon + D_1\chi^4 - D_2\chi^2\psi + D_3\psi^2 + D_4\chi\omega - \tfrac{5}{2}\omega\left(2\hat{k}^2 - 2\epsilon - A_1\chi\right). \tag{C.7}$$

The last system of differential equations has a fixed point with $\epsilon_f = 0$ and χ, ψ, ω taking values that set the right-hand side of the equations to zero. This fixed point $(\epsilon_f, \chi_f, \psi_f, \omega_f)$ is infrared stable as long as $A_1\chi_f < 2\hat{k}^2$, and therefore is approached for $\tau \to -\infty$. In terms of the variable t it is then reached at a finite value t_f. We have solved numerically the system (C.4), for $\hat{k}^2 = 1.175$ which corresponds to $\beta = 3$, and for various initial conditions. We have verified that, as $\epsilon_f = 0$ is reached for a finite value t_f, (χ, ψ, ω) always approach $(\chi_f, \psi_f, \omega_f)$. The fixed point $(\epsilon_f, \chi_f, \psi_f, \omega_f)$ is independent of d. The quantitative role of the higher derivatives can be estimated by determining the fixed-point values for different truncations of the evolution equation. This is presented in table C.1 for various values of N. The first column lists the fixed points for the truncation which keeps only the second derivative. The third and fourth derivatives are added for the next

TABLE C.1
Fixed points for different truncations of the evolution equation. E is the number of the retained differential equations. We display the values for various N and use $\hat{k}^2 = 1.175$, which corresponds to $\beta = 3$

	$E=2$	$E=3$		$E=4$			$E=\infty$		
	χ_f	χ_f	ψ_f	χ_f	ψ_f	ω_f	χ_f	ψ_f	ω_f
$N=1$	0.196	0.365	0.223	0.475	0.476	0.507	0.783	0.921	1.442
$N=2$	0.168	0.312	0.157	0.403	0.331	0.281	0.588	0.518	0.608
$N=4$	0.131	0.239	0.088	0.303	0.176	0.101	0.392	0.230	0.180
$N=10$	0.078	0.137	0.026	0.165	0.046	0.011	0.196	0.058	0.023

two columns. Even though the values of the fixed points vary, it is clear that the qualitative behaviour of the solutions is unaffected by the higher derivatives. As ϵ runs to zero at a finite t_f, the couplings $\tilde{\lambda}$, $\tilde{\nu}$, $\tilde{\sigma}$ approach zero proportional to increasing powers of ϵ. In the last columns we list the asymptotic values of $(\chi_f, \psi_f, \omega_f)$ in the limit that keeps an infinite number of higher derivatives. These values were obtained in sect. 3 by solving the full evolution equation for the first derivative of the potential and are given by eqs. (3.19).

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