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# Quasi quantum group covariant $q$-oscillators *

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If $q$ is a $p$th root of unity there exists a quasi-co-associative truncated quantum group algebra $U_q^T(sl_2)$ whose indecomposable representations are the physical representations of $U_q(sl_2)$, whose co-product yields the truncated tensor product of physical representations of $U_q(sl_2)$, and whose $R$-matrix satisfies quasi Yang–Baxter equations. These truncated quantum group algebras are examples of weak quasi quantum group algebras [2]. For primitive $p$th roots $q$, $q = e^{2\pi i/p}$, we consider a two-dimensional $q$-oscillator which admits $U_q^T(sl_2)$ as a symmetry algebra. Its wave function lie in a space $\mathscr{F}_q^T$ of "functions on the truncated quantum plane", i.e. of polynomials in noncommuting complex coordinate functions $z_a$, on which multiplication operators $Z_a$ and the elements of $U_q^T(sl_2)$ can act. This illustrates the concept of quasi quantum planes [1]. Due to the truncation, the Hilbert space of states is finite dimensional. The subspaces $\mathscr{F}^{T(n)}$ of monomials in $z_a$ of $n$th degree vanish for $n \geqslant p-1$, and $\mathscr{F}^{T(n)}$ carries the $(2J+1)$-dimensional irreducible representation of $U_q^T(sl_2)$ if $n = 2J$, $J = 0, \frac{1}{2}, \ldots, \frac{1}{2}(p-2)$. Partial derivatives $\partial_a$ are introduced. We find a $*$-operation on the algebra of multiplication operators $Z_i$ and derivatives $\partial_b$ such that the adjoints $Z_a^*$ act as differentiation on the truncated quantum plane. Multiplication operators $Z_a$ ("creation operators") and their adjoints ("annihilation operators") obey $q^{-1/2}$-commutation relations. The $*$-operation is used to determine a positive definite scalar product on the truncated quantum plane $\mathscr{F}_q^T$. Some natural candidates of hamiltonians for the $q$-oscillators are determined.

## 1. Introduction: Bargmann–Fock representation of harmonic oscillators

For pedagogical reasons let us first recall some well-known facts concerning standard harmonic oscillators in quantum theory. We begin with the one-dimensional harmonic oscillator. There are two operators, position $X$ and momentum $P$, which obey canonical commutation relations. The hamiltonian is $H = \frac{1}{2}[P^2 + X^2]$. In the position space Schrödinger representation, wave functions are functions of $x$, and $X$ acts as multiplication with $x$, while $P$ is a differential operator. However, one may also use the Bargmann–Fock or coherent state representation [4]. In this

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case, wave functions are holomorphic functions of a complex variable $z$, and the nonhermitian ("creation") operator

$$Z = \frac{1}{\sqrt{2}}[P + iX]$$

acts by multiplication with $z$, whereas the adjoint operator ("annihilation operator")

$$Z^* = \frac{1}{\sqrt{2}}[P - iX]$$

acts as a differential operator $\partial/\partial z$. The scalar product of two physical states with wave functions $\psi_1$ and $\psi_2$ may be written in two equivalent ways,

$$\langle\psi_1, \psi_2\rangle = \int d^2z\ \bar{\psi}_1\psi_2 \exp(-z\bar{z}/2) = \bar{\psi}_1(\partial/\partial z)\psi_2(z)\big|_{z=0}.$$

$d^2z$ is integration over real and imaginary part of $z$. The hamiltonian is $H = ZZ^* + \frac{1}{2}$. From the operators $Z$ and $Z^*$ one can recover $X$ and $P$ as linear combinations.

Let us now turn to the two-dimensional harmonic oscillator. There will be two operators $Z_1$ and $Z_2$ and their adjoints, and similarly for $X$ and $P$.

$$Z_1 = \frac{1}{\sqrt{2}}(P_1 + iX_1), \tag{1.1}$$

$$Z_2 = \frac{1}{\sqrt{2}}(P_2 + iX_2), \tag{1.2}$$

$$H = \sum_a Z_a Z_a^* + 1. \tag{1.3}$$

The canonical commutation relations read

$$[Z_a, Z_b] = 0 = [Z_a^*, Z_b^*], \tag{1.4}$$

$$[Z_a, Z_b^*] = \delta_{ab} \tag{1.5}$$

The two-dimensional harmonic oscillator admits the group U(2) of all unitary $2 \times 2$ matrices as a symmetry group. The Hilbert space of physical states carries a unitary representation $\mathscr{U}$ of this symmetry group.

$$\mathscr{U}(\xi)Z_a = Z'_a\mathscr{U}(\xi) \quad \text{with} \quad Z'_a = Z_b\xi_{ba}, \tag{1.6}$$

$$\mathscr{U}(\xi)H = H\mathscr{U}(\xi), \tag{1.7}$$

$$\mathscr{U}(\xi)|0\rangle = |0\rangle \tag{1.8}$$

for all $\xi = (\xi_{ab}) \in U(2)$. The SO(2) subgroup which consists of real matrices acts by rotation of the two-dimensional coordinate plane. The other symmetry transformations are canonical transformations which are not coordinate transformations. The ground state $|0\rangle$ of $H$ is invariant under the symmetry group.

It is convenient for generalization to rewrite the covariance properties (1.6) in another way. To this end let $\tau^{1/2}$ denote the fundamental two-dimensional representation of U(2) so that elements $\xi_{ab}$ are given by $\xi_{ab} = \tau^{1/2}_{ab}(\xi)$. The right-hand side of (1.6) becomes $Z_b \tau^{1/2}_{ba}(\xi)\mathscr{U}(\xi)$. Remember that the tensor product $\tau \boxtimes \tau'$ of representations $\tau, \tau'$ of groups is defined by $(\tau \boxtimes \tau')(\xi) = \tau(\xi) \otimes \tau'(\xi)$. With this notion of tensor products of representations, the covariance law assumes the final form

$$\mathscr{U}(\xi) Z_a = Z_b \big(\tau^{1/2}_{ba} \boxtimes \mathscr{U}\big)(\xi) \quad \text{for all } \xi \in U(2). \tag{1.9}$$

The invariance of the ground state can be re-expressed with help of the trivial one-dimensional representation $\epsilon(\xi) = 1 \in \mathbb{C}$ of U(2).

$$\mathscr{U}(\xi)\,|0\rangle = |0\rangle \epsilon(\xi) \quad \text{for all } \xi \in U(2). \tag{1.10}$$

In the Bargmann–Fock representation, $Z_a$ act as multiplication operators, whereas $Z_a^* = \partial_a$ act as differential operators. A basis in the Hilbert space $\mathscr{H}$ is spanned by wave functions of the form

$$Z_2^n Z_1^m\,|0\rangle.$$

They are eigenstates of $H$ to eigenvalue $n + m + 1$.

In this paper we will consider a two-dimensional $q$-oscillator. It differs from the standard oscillator as follows. The multiplication operators $Z_a$ and their adjoints will obey braid relations in place of canonical commutation relations, and the U(2) symmetry group will be replaced by the truncated quantum group algebra $U_q^T(sl_2)$ which is canonically associated with the quantum group algebra $U_q(sl_2)$ when $q$ is a primitive $p$th root of unity.

The main technical problem will be to determine the action of the annihilation operators ( = adjoints of creation operators) on the Hilbert space of physical states. As for the standard oscillator, the annihilation operators can be expressed in terms of certain partial derivatives $\partial_a$.

As hamiltonian we may choose

$$H_1 = \Big| \sum_{ab} (Z \times \bar{Z})_{ab} g^{ab} \Big| + 1 \tag{1.11}$$

or

$$H_2 = \tfrac{1}{2} \sum_{ab} \Big[ (Z \times \bar{Z})_{ab} g^{ab} + \big( (Z \times \bar{Z})_{ab} g^{ab} \big)^* \Big] + 1. \tag{1.12}$$

Here the $\times$ denotes a "covariant product" that substitutes for the ordinary product and the tensor $g^{ab}$ projects to the invariant part. This will be explained in sect. 3. The modulus $|.|$ in $H_1$ and the * in $H_2$ refer to a *-operation which is defined in sect. 4. $\bar{Z}$ denotes some kind of "Dirac adjoint" of the multiplication operators $Z$, i.e. a certain linear combination of operators $Z^*$ (with operator valued coefficients). $\bar{Z}_a$ will act as $\partial_a$. We are not prepared to argue for (1.11) or (1.12) on the basis of physical principles aside from invariance property (1.7) and hermiticity.

## 2. Truncated quantum group algebras

The quantum group algebras $U_q(sl_2)$ form a family of deformations of the universal enveloping algebra $U(sl_2)$. They are indexed by a complex number $q$. In the present context we are interested in the cases, where $q$ is a root of unity, $q^p = 1$. For these values of the deformation parameter $q$, $U_q(sl_2)$ is not semisimple. In this section we will shortly review how a semisimple truncated quantum group algebra $U_q^T(sl_2)$ is canonically associated with the quantum group algebra $U_q(sl_2)$, $q^p = 1$. $U_q^T(sl_2)$ is not a Hopf algebra but a weak quasi Hopf algebra. For details cf. ref. [2].

Before we describe the construction of $U_q^T(sl_2)$ let us fix some notations concerning $U_q(sl_2)$. The quantum group algebra $U_q(sl_2)$ is generated by unit $e$ and $q^{\pm H/2}$, $S_\pm$ subject to the relations

$$q^{H/2}q^{-H/2} = q^{-H/2}q^{H/2} = e, \qquad q^{H/2}S_\pm = q^{\pm 1/2}S_\pm q^{H/2},$$

$$[S_+, S_-] = \frac{q^H - q^{-H}}{q^{1/2} - q^{-1/2}}. \tag{2.1}$$

A co-product $\Delta_q$, co-unit $\epsilon$ and antipode $\mathscr{S}$ are defined, turning $U_q(sl_2)$ into a Hopf algebra. Explicitly they read

$$\Delta_q(S_\pm) = S_\pm \otimes q^{H/2} + q^{-H/2} \otimes S_\pm, \qquad \Delta_q(q^{\pm H/2}) = q^{\pm H/2} \otimes q^{\pm H/2}, \tag{2.2}$$

$$\epsilon(S_\pm) = 0, \qquad \epsilon(q^{\pm H/2}) = 1, \tag{2.3}$$

$$\mathscr{S}(S_\pm) = -q^{\pm 1/2}S_\pm, \qquad \mathscr{S}(q^{\pm H/2}) = q^{\mp H/2}. \tag{2.4}$$

Co-associativity of the co-product means that $(\Delta_q \otimes \mathrm{id})\Delta_q(\xi) = (\mathrm{id} \otimes \Delta_q)\Delta_q(\xi)$ for all $\xi \in U_q^T(sl_2)$. If $\sigma : U_q(sl_2) \otimes U_q(sl_2) \to U_q(sl_2) \otimes U_q(sl_2)$ denotes the permutation automorphism $\sigma(\eta \otimes \xi) = \xi \otimes \eta$, we can define a second co-product by $\Delta'_q = \sigma \circ \Delta_q$. The universal $R$-matrix $R_q \in U_q(sl_2) \otimes U_q(sl_2)$ intertwines between

$\Delta_q$ and $\Delta'_q$ in the sense that $R_q \Delta_q(\eta) = \Delta'_q(\eta) R_q$ for all $\eta \in \mathrm{U}_q(\mathrm{sl}_2)$. Explicitly $R_q$ is given by

$$\mathscr{R}_q = q^{H\otimes H} \sum_{n\geqslant 0} \frac{\left(1-q^{-1}\right)^n}{[n]_q!} q^{-\frac{1}{4}n(n-1)} q^{nH/2} S_+^n \otimes q^{-nH/2} S_-^n . \tag{2.5}$$

Here $[n]_q = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$ and $[n]_q!$ is recursively defined by $[0]_q! = [1]_q! = 1$ and $[n]_q! = [n]_q [n-1]_q!$. It can easily be checked that the following definition of $S_\pm^*$, $(q^{\pm H/2})^*$ extends to a $*$-operation on $\mathrm{U}_q(\mathrm{sl}_2)$ (it is consistent with the defining relations (2.1) of the quantum group algebra $\mathrm{U}_q(\mathrm{sl}_2)$):

$$S_\pm^* = S_\mp, \qquad \left(q^{\pm H/2}\right)^* = q^{\mp H/2}. \tag{2.6}$$

This turns $\Delta_q$, $\epsilon$, $\mathscr{S}$ into $*$-(anti-)homomorphisms provided that we define $*$ on $\mathrm{U}_q(\mathrm{sl}_2) \otimes \mathrm{U}_q(\mathrm{sl}_2)$ by

$$(\eta \otimes \xi)^* = \xi^* \otimes \eta^* \quad \text{for all } \xi, \eta \in \mathrm{U}_q(\mathrm{sl}_2). \tag{2.7}$$

Moreover it results from an explicit calculation that $R_q$ is unitary in the following sense:

$$R_q^* = R_q^{-1} = (\mathscr{S} \otimes \mathrm{id})(R_q).$$

The "quantum dimension" $d_\tau$ of a representation $\tau$ of $\mathrm{U}_q(\mathrm{sl}_2)$ is defined by

$$d_\tau = \mathrm{tr}\ \tau(q^H). \tag{2.8}$$

As we mentioned earlier, if $q$ is a (primitive $p$th) root of unity, then $\mathrm{U}_q(\mathrm{sl}_2)$ is not semisimple, and tensor products of its irreducible representations are in general not fully reducible. Its irreducible representations $\tau^J$ with nonzero quantum dimension are called "physical" representations. They are labelled by $J = 0, \frac{1}{2}, \ldots, \frac{1}{2}(p-2)$. They have dimension $2J+1$.

We denote the tensor product of $\mathrm{U}_q(\mathrm{sl}_2)$-representations by $\boxtimes_q$. Given two representations $\tau$, $\tau'$ it is defined with help of the co-product $\Delta_q$,

$$\left(\tau \underset{q}{\boxtimes} \tau'\right)(\xi) = (\tau \otimes \tau') \Delta_q(\xi) \quad \text{for all } \xi \in \mathscr{U}_q(\mathrm{sl}_2).$$

The tensor product of $\tau^I \boxtimes_q \tau^J$ of two physical representations decomposes in general into physical representations, plus unphysical representations with quantum dimension 0. If we multiply $(\tau^I \boxtimes_q \tau^J)(\xi)$ with a projection operator $P_{IJ}$ which cuts away the contribution with zero quantum dimension, one obtains what is known as the truncated tensor product of physical representations of $\mathrm{U}_q(\mathrm{sl}_2)$.

One introduces a new algebra $U_q^T(sl_2)$ by $U_q^T(sl_2) = U_q(sl_2)/\mathcal{J}$, where $\mathcal{J}$ is the ideal which is annihilated by all the physical representations $\tau^I$, $2I = 0, \dots, p-2$, of $U_q(sl_2)$. $U_q^T(sl_2)$ is semisimple, its representations are fully reducible, and the irreducible ones are precisely the physical representations of $U_q(sl_2)$. Let

$$u(I, J) = \min\{|I+J|, p-2-I-J\} \tag{2.9}$$

and let $P_{IJ}$ be the projector on the physical subrepresentations $K$, $|I-J| \leqslant K \leqslant u(I, J)$ of the tensor product $\tau^I \boxtimes_q \tau^J$ of $U_q(sl_2)$ representations. There exists $P \in U_q^T(sl_2) \otimes U_q^T(sl_2)$ such that $P_{IJ} = (\tau^I \otimes \tau^J)(P)$. The co-product in $U_q^T(sl_2)$ is determined in terms of the co-product $\Delta_q$ in $U_q(sl_2)$ as

$$\Delta(\xi) = P\Delta_q(\xi), \tag{2.10}$$

hence $\Delta(e) = P \neq e \otimes e$. This co-product specifies a tensor product $\boxtimes$ according to

$$(\tau \boxtimes \tau')(\xi) = (\tau \otimes \tau')(\Delta(\xi)). \tag{2.11}$$

It is equal to the truncated tensor product of physical $U_q(sl_2)$ representations. Thus

$$\tau^I \boxtimes \tau^J = \bigoplus_{|I-J| \leqslant K \leqslant u(IJ)} \tau^K. \tag{2.12}$$

However, this co-product is not co-associative but there exists an element $\phi \in U_q^T(sl_2) \otimes U_q^T(sl_2) \otimes U_q^T(sl_2)$ such that $\phi$ has the following intertwining property:

$$\phi(\Delta \otimes \mathrm{id})\Delta(\eta) = (\mathrm{id} \otimes \Delta)\Delta(\eta)\phi \quad \text{for all } \eta \in U_q^T(sl_2). \tag{2.13}$$

$\phi_{IJK} = (\tau^I \otimes \tau^J \otimes \tau^K)(\phi)$ implements the well-known unitary equivalence of the truncated tensor products $\tau^I \boxtimes (\tau^J \boxtimes \tau^K)$ and $(\tau^I \boxtimes \tau^J) \boxtimes \tau^K$. A truncated tensor product $\boxtimes$ is defined also for basis vectors $\hat{e}_i^I$ in the dual representation spaces $\hat{V}^I$ on which $U_q^T(sl_2)$ acts from the right, viz. $\hat{e}_i^I \boxtimes \hat{e}_j^J = \hat{e}_i^I \otimes \hat{e}_j^J P_{IJ}$. The map $\phi_{IJK}$ can be specified by its action on triple truncated products of basis vectors, together with the condition $\phi = (\mathrm{id} \otimes \Delta)\Delta(e)\phi$, viz.

$$\sum_{ijkp} \begin{bmatrix} I & P & L \\ i & p & l \end{bmatrix}_q \begin{bmatrix} J & K & P \\ j & k & p \end{bmatrix}_q \hat{e}_i^I \otimes \hat{e}_j^J \otimes \hat{e}_k^K \phi = \sum_{Q,ijkq} F_{PQ} \begin{bmatrix} J & I \\ K & L \end{bmatrix} \begin{bmatrix} I & J & Q \\ i & j & q \end{bmatrix}_q \times \begin{bmatrix} Q & K & L \\ q & k & l \end{bmatrix}_q \hat{e}_i^I \otimes \hat{e}_j^J \otimes \hat{e}_k^K \tag{2.14}$$

with fusion matrices given by $6J$-symbols,

$$F_{PQ}\begin{bmatrix} J & I \\ K & L \end{bmatrix} = \begin{Bmatrix} K & J & P \\ I & L & Q \end{Bmatrix}_q.$$

$[\because]_q$ are Clebsch–Gordan coefficients for $U_q(sl_2)$ (and at the same time for $U_q^T(sl_2)$). $\phi$ has a quasi-inverse, denoted by $\phi^{-1}$, such that

$$\phi\phi^{-1} = (\mathrm{id} \otimes \Delta)\Delta(e), \qquad \phi^{-1}\phi = (\Delta \otimes \mathrm{id})\Delta(e). \tag{2.15}$$

The universal $R$-element $R \in U_q^T(sl_2) \otimes U_q^T(sl_2)$ of $U_q^T(sl_2)$ has the intertwining property

$$R\Delta(\eta) = \Delta'(\eta)R, \qquad \Delta' = \sigma \circ \Delta. \tag{2.16}$$

It is given in terms of the $R$-element $R_q$ for $U_q(sl_2)$ by

$$R = R_q\Delta(e) = \Delta'(e)R_q. \tag{2.17}$$

$R$ is not invertible, but has quasi-inverse $R^{-1}$ such that

$$RR^{-1} = \Delta'(e), \qquad R^{-1}R = \Delta(e). \tag{2.18}$$

Since $\Delta(e)^* = \Delta(e^*) = \Delta(e)$, the unitarity of $R_q$ implies that

$$R^* = R^{-1}. \tag{2.19}$$

Antipode, co-unit and $*$-operation are the same as in $U_q(sl_2)$. It is shown in ref. [2] that the defining properties of a weak quasi-triangular quasi Hopf algebra are satisfied. Here we state only some of the defining features. Co-unit is related to the co-product by

$$(\epsilon \otimes \mathrm{id})\Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon)\Delta. \tag{2.20}$$

Moreover, $\phi$ and $R$ satisfy Drinfeld's relations [6],

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\phi)(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = (\mathrm{e} \otimes \phi)(\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\phi)(\phi \otimes \mathrm{e}), \tag{2.21}$$

$$(\mathrm{id} \otimes \Delta)(R) = \phi_{231}^{-1}R_{13}\phi_{213}R_{12}\phi^{-1}, \tag{2.22}$$

$$(\Delta \otimes \mathrm{id})(R) = \phi_{312}R_{13}\phi_{132}^{-1}R_{23}\phi. \tag{2.23}$$

We used the standard notation. If $R = \sum r_a^1 \otimes r_a^2$ then

$$R_{13} = \sum r_a^1 \otimes e \otimes r_a^2, \qquad R_{12} = \sum r_a^1 \otimes r_a^2 \otimes e, \tag{2.24}$$

etc.

If $s$ is any permutation of 123 and $\phi = \Sigma \phi_\sigma^1 \otimes \phi_\sigma^2 \otimes \phi_\sigma^3$ then

$$\phi_{s(1)s(2)s(3)} = \sum_\sigma \phi_\sigma^{s^{-1}(1)} \otimes \phi_\sigma^{s^{-1}(2)} \otimes \phi_\sigma^{s^{-1}(3)}. \tag{2.25}$$

Eqs. (2.22) and (2.23) imply validity of quasi Yang–Baxter equations,

$$R_{12}\phi_{312}R_{13}\phi_{132}^{-1}R_{23}\phi = \phi_{321}R_{23}\phi_{231}^{-1}R_{13}\phi_{213}R_{12}, \tag{2.26}$$

and this guarantees that $R$ together with $\phi$ determines a representation of the braid group [7]. Let us recall that the braid group $B_n$ on $n$ threads is generated by elements $\sigma_i$ and $\sigma_i^{-1}$ $(i = 1, \ldots, n-1)$ which obey the Artin relations

$$\sigma_i\sigma_k = \sigma_k\sigma_i \quad \text{if } |k-i| \geq 2, \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

$$\sigma_i\sigma_i^{-1} = \mathbf{1} = \sigma_i^{-1}\sigma_i. \tag{2.27}$$

The unit element of $B_n$ is written as $\mathbf{1}$. We introduce some notations. Write

$$e^n = e \otimes \ldots \otimes e \quad (n \text{ factors}) \tag{2.28}$$

and similarly for $\mathrm{id}^n$. In addition we abbreviate $U_q^T(\mathrm{sl}_2)^{\otimes n} = U_q^T(\mathrm{sl}_2) \otimes \ldots \otimes U_q^T(\mathrm{sl}_2)$ ($n$ factors), and

$$\Delta^n = (\mathrm{id}^{n-1} \otimes \Delta)\ldots(\mathrm{id} \otimes \Delta)\Delta \quad \text{for } n \geqslant 2, \tag{2.29}$$

$$\Delta^1 = \Delta, \qquad \Delta^0 = \mathrm{id}, \qquad \Delta^{-1} = \epsilon. \tag{2.30}$$

The permutation of factors in $U_q^T(\mathrm{sl}_2) \otimes U_q^T(\mathrm{sl}_2)$ can be implemented by the multiplication operator $\mathscr{P} \in U_q^T(\mathrm{sl}_2) \otimes U_q^T(\mathrm{sl}_2)$ with the properties

(i) $\mathscr{P}\xi \otimes \eta = \eta \otimes \xi\mathscr{P}$ for all $\xi, \eta \in U_q^T(\mathrm{sl}_2)$,

(ii) $\tau \otimes \tau'(\mathscr{P}) = 0$ if $\tau, \tau'$ are inequivalent irreducible representations of $U_q^T(\mathrm{sl}_2)$.

*Theorem 2.1.* (Artin relations) [1] Let $\hat{R}^+ = \mathscr{P}R$ and $\hat{R}^- = R^{-1}\mathscr{P}$, and $n = r + k + 1$, $r \geqslant 0$. Define $\sigma_k^{n\pm} \in U_q^T(\mathrm{sl}_2)^{\otimes n}$ by

$$\sigma_k^{n\pm} = \Delta^{n-1}(e)(\mathrm{id}^{n-k+1} \otimes \Delta^{k-2})(e^{n-k-1} \otimes \phi(\hat{R}^\pm \otimes e)\phi^{-1}). \tag{2.31}$$

Then the $\sigma_k^{n\pm}$ obey Artin relations (2.27) with $\mathbf{1} = \Delta^{n-1}(e)$.

## 3. Truncated quantum planes

Quantum planes $\mathscr{F}_q$ are generalizations of the algebra $\mathscr{F}$ of polynomials in two complex variables [8]. In analogy to the action of $U(\mathrm{sl}_2)$ on $\mathscr{F}$, the quantum group algebra $U_q(sl_2)$ acts on the quantum plane $\mathscr{F}_q$ by generalized derivations [9]. Similar results exist for the truncated quantum groups $U_q^T(sl_2)$, but they require the concept of truncated quantum planes $\mathscr{F}_q^T$. Their construction will be reviewed in this section. In addition we introduce the analog of partial derivatives $\partial_a$ for the truncated quantum plane. For details of the general construction of quasi quantum planes the reader is refered to ref. [1].

*Definition 3.1.* Let $\mathscr{A}$ be an associative algebra that contains the truncated quantum group $U_q^T(sl_2)$ as a subalgebra and let $\tau$ denote some $n$-dimensional representation of $U_q^T(sl_2)$. An $n$-tupel $(F_\alpha)_{\alpha=1,\dots,n}$ of elements $F_\alpha \in \mathscr{A}$ is said to transform covariantly according to $\tau$ if

$$\xi F_\alpha = \sum_\beta F_\beta(\tau_{\beta\alpha} \otimes \mathrm{id})(\Delta(\xi)) = \sum_\beta F_\beta \tau_{\beta\alpha}(\xi_\sigma^1)\xi_\sigma^2. \tag{3.1}$$

Here $\Delta(\xi) = \Sigma\, \xi_\sigma^1 \otimes \xi_\sigma^2$ and a sum over $\sigma$ is understood throughout.

The tensor product for representations of the truncated quantum group was defined in (2.11) with the help of the co-product $\Delta$. It follows by comparison with (1.9) that the notion of covariance introduced in definition 3.1 is an obvious generalization of the notion we had for the group algebra $U(sl_2)$.

*Definition 3.2.* Let $(F_\alpha)$, $(G_\beta)$ be $n$, $m$-tupels that transform covariantly under representations $\tau$, $\tau'$. Then the covariant product of $(F_\alpha)$ and $(G_b)$ is an $nm$-tupel defined by

$$(F \times G)_{\alpha\beta} = \sum F_\gamma G_\delta(\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta} \otimes \mathrm{id})(\phi). \tag{3.2}$$

The fundamental importance of this definition derives from the following proposition.

*Proposition 3.3.* [1] In the notations of definition 3.2 the $nm$-tupel $((F \times G)_{\alpha\beta})$ transforms covariantly according to the tensor product representation $\tau \boxtimes \tau'$.

It can be shown that the covariant product is not associative but only quasi-associative [1] in the sense that covariant products with different positions of brackets are linear combinations of each other. Provided that $e \in U_q^T(sl_2)$ acts as the identity of $\mathscr{A}$, covariant products can be converted into ordinary products of $n$,$m$-tupels by

$$F_\alpha G_\beta = (F \times G)_{\gamma\delta}(\tau_{\gamma\alpha} \otimes \tau'_{\delta\beta} \otimes \mathrm{id})(\phi^{-1}). \tag{3.3}$$

The ordinary product of $n, m$-tupels is not covariant in general, since $\Delta$ is not co-associative.

*Definition 3.4.* *(Algebra $\mathscr{B}$) Let* $\tau^{1/2}$ denote the two-dimensional representation of $U_q^T(sl_2)$ and $R$ the $R$-matrix given by (2.17). The associative algebra $\mathscr{B}$ is generated by elements $Z_a$ $(a = 1, 2)$, and the elements of $U_q^T(sl_2)$. The unit element $e$ of $U_q^T(sl_2)$ acts as a unit element of $\mathscr{B}$ so that
(1) $Z_a e = Z_a = e Z_a$,
and the following further relations are imposed:
(2) ($U_q^T(sl_2)$ covariance) $\xi Z_a = Z_b(\tau_{ba}^{1/2} \otimes \mathrm{id})(\Delta(\xi))$ for $\xi \in U_q^T(sl_2)$;
(3) (braid relations) $(Z \times Z)_{ab} = q^{-1/4}(Z \times Z)_{dc}(\tau_{ca}^{1/2} \otimes \tau_{db}^{1/2})(R)$.

Relation (2) states that $(Z_a)$ is a 2-tupel transforming covariantly according to the representation $\tau^{1/2}$. Covariance of $(Z \times Z)$ and the intertwining property (2.16) of $R$ ensures that the relation (3) is consistent with the transformation law (2) in the sense that left- and right-hand side of (3) transform in the same way. If covariant products are converted into ordinary products, (3) reads

$$Z_a Z_b = Z_d Z_c \left(\tau_{ca}^{1/2} \otimes \tau_{db}^{1/2} \otimes \mathrm{id}\right)\left(\phi_{213} R_{12} \phi^{-1}\right). \tag{3.4}$$

Note that the right-hand side involves the generator $\sigma_2$ of the braid group. This explains why relations (3) are called "braid relations". The algebra $\mathscr{B}$ generates the truncated quantum plane (space of "noncommutative polynomials") out of a unique invariant ground state ("constant polynomial").

*Definition 3.5.* (Truncated quantum plane). A truncated quantum plane $\mathscr{F}_q^T$ is the unique $\mathscr{B}$-left-module with cyclic and $U_q^T(sl_2)$ invariant element $|0\rangle$. Here $U_q^T(sl_2)$-invariance of $|0\rangle$ means that $\xi\,|0\rangle = |0\rangle\ \epsilon(\xi)$ for all $\xi \in U_q^T(sl_2)$.

It is instructive to describe the structure of $\mathscr{F}_q^T$ in some detail. If $\Delta = \sum \xi_\sigma^1 \otimes \xi_\sigma^2$ the relation (2) in definition 3.4 reads explicitly

$$\xi Z_a = \sum Z_b \tau_{ba}^{1/2}(\xi_\sigma^1)\xi_\sigma^2.$$

This tells us how to shift elements $\xi \in U_q^T(sl_2)$ through factors $Z_a$ from left to right. Together with the invariance of the ground state $|0\rangle$ it follows that the truncated quantum plane is spanned by states of the form

$$Z_\alpha^n \equiv Z_{a_1} \dots Z_{a_n} |0\rangle \tag{3.5}$$

with multi-index $\alpha = (a_1, \dots, a_n)$. We abbreviate

$$\tau^{(n)} = \left(\tau^{1/2} \boxtimes \left(\tau^{1/2} \boxtimes \dots \left(\tau^{1/2} \boxtimes \tau^{1/2}\right) \dots\right)\right), \quad n \text{ factors}.$$

The transformation law of states (3.5) can be calculated using covariance of $Z_a$ and invariance of the ground state $|0\rangle$.

$$\xi Z_\alpha^n |0\rangle = Z_\beta^n |0\rangle\ \tau_{\beta\alpha}^{(n)}(\xi). \tag{3.6}$$

Let $\mathscr{F}_q^{\mathrm{T}(n)}$ denote the subspace of $\mathscr{F}_q^{\mathrm{T}}$ spanned by elements $Z_\alpha^n|0\rangle$ for fixed $n$. Then $\mathscr{F}_q^{\mathrm{T}(n)}$ carries a subrepresentation of $\tau^{(n)}$. One can prove the following theorem [1].

*Theorem 3.6.* (Structure of $\mathscr{F}_q^{\mathrm{T}}$). The truncated quantum plane $\mathscr{F}_q^{\mathrm{T}}$ decomposes into a direct sum of subspaces $\mathscr{F}_q^{\mathrm{T}(n)}$,

$$\mathscr{F}_q^{\mathrm{T}} = \bigoplus_{n \geqslant 0} \mathscr{F}_q^{\mathrm{T}(n)}.$$

For $q^p = 1$, $p \geqslant 4$ we obtain
(i) $\mathscr{F}_q^{\mathrm{T}(n)} = 0$ for all $n \geqslant p-1$;
(ii) $\mathscr{F}_q^{\mathrm{T}(n)}$ carries the $(n+1)$-dimensional irreducible representation of $\mathrm{U}_q^{\mathrm{T}}(\mathrm{sl}_2)$ if $n \leqslant p-2$. Moreover a basis of $\mathscr{F}_q^{\mathrm{T}(n)}$ is provided by states

$$|m, m'\rangle = Z_2^m Z_1^{m'}|0\rangle, \quad m + m' = n, \quad n \leqslant p-2.$$

Some details of the proof can be found in appendix C.

In ref. [1] the non-associative covariant product $\times$ is used to introduce a product for elements of the truncated quantum plane. Due to the truncation (i) in theorem 3.6 this product turns out to be associative in the special example under consideration. Thus truncated quantum planes $\mathscr{F}_q^{\mathrm{T}}$ fit into the framework of noncommutative geometry. We do not want to discuss this further, since in the present context we are only concerned with the linear structure of the truncated quantum plane.

Let us explain how to introduce differentiation operators in a consistent manner. As discussed in sect. 1, they should transform according to the contragredient representation $\tilde{\tau}^{1/2}$. $\tilde{\tau}^{1/2}$ is defined by

$$\tilde{\tau}^{1/2}(\xi) = {}^{\mathrm{t}}\,\tau^{1/2}(\mathscr{S}^{-1}(\xi)). \tag{3.7}$$

Here ${}^{\mathrm{t}}$ denotes the transpose of the matrix and $\mathscr{S}^{-1}$ the inverse of the antipode. Explicit formulas can be found in appendix A. The contragredient representation $\tilde{\tau}^{1/2}$ is equivalent to the fundamental representation $\tau^{1/2}$. From the formula (2.12) for the decomposition of tensor products we deduce that $\tilde{\tau}^{1/2} \otimes \tau^{1/2}$ contains the one-dimensional representation $\tau^0 = \epsilon$ as a subrepresentation. This implies that there exists a metric tensor $g_{ab}$ such that

$$g_{cd}\big(\tilde{\tau}_{ca}^{1/2} \boxtimes \tau_{db}^{1/2}\big)(\xi) = g_{ab}\epsilon(\xi). \tag{3.8}$$

It is given by

$$g = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}. \tag{3.9}$$

The inverse tensor $g^{ab} = \mathrm{diag}(q^{1/2}, q^{-1/2})$ obeys $(\tilde{\tau}_{ca}^{1/2} \boxtimes \tau_{db}^{1/2})(\xi) g^{ab} = g^{ab}\epsilon(\xi)$.

*Definition 3.7.* (Algebra $\mathscr{C}$) The associative algebra $\mathscr{C}$ is generated by elements $\xi \in U_q^T(sl_2)$, $Z_a$ and $\partial_a$ subject to the following relations:
(i) The unit element $e \in U_q^T(sl_2)$ is also unit element of $\mathscr{C}$.
(ii) Covariance: For $\xi \in U_q^T(sl_2)$

$$\xi Z_a = Z_b(\tau_{ba} \otimes \text{id})(\Delta(\xi)), \tag{3.10}$$

$$\xi \partial_a = \partial_b(\tilde{\tau}_{ba}^{1/2} \otimes \text{id})(\Delta(\xi)). \tag{3.11}$$

(iii) Braid relations

$$(Z \times Z)_{ab} = (Z \times Z)_{dc} q^{-1/4}(\tau_{ca} \otimes \tau_{db})(R), \tag{3.12}$$

$$(\partial \times \partial)_{ab} = (\partial \times \partial)_{dc} q^{-1/4}(\tilde{\tau}_{ca}^{1/2} \otimes \tilde{\tau}_{db}^{1/2})(R), \tag{3.13}$$

$$(\partial \times Z)_{ab} = g_{ab} e + (Z \times \partial)_{dc} q^{-3/4}(\tilde{\tau}_{ca}^{1/2} \otimes \tau_{db})(R). \tag{3.14}$$

The phase factors we wrote in the braid relations can be calculated from the eigenvalues of the symmetric $R$-matrix. The general formulas are given in ref. [1]. A proper choice of the phase factors guarantees that $\mathscr{B} \subset \mathscr{C}$. This is a nontrivial result which is proven in ref. [1]. It is possible to develop a full differential calculus including differential forms of order $n$ and an exterior derivative $d$, which enjoys the usual properties. This differential calculus is not associative but only quasi-associative [1].

Again, we may write all the braid relations (3.12)–(3.14) in terms of ordinary products in place of covariant products. Thus (3.14) shows how to shift operators $\partial_a$ through multiplication operators $Z_a$ from left to right. As a consequence, a general element in $\mathscr{C}$ is a linear combination of elements

$$Z_{a_1} \cdots Z_{a_n} \partial_{b_1} \cdots \partial_{b_m} \xi, \quad \xi \in U_q^T(sl_2).$$

The truncated quantum plane $\mathscr{F}_q^T$ was introduced as a $\mathscr{B}$-left-module. Let us denote the left action of $X \in \mathscr{B}$ by $\pi(X)$. This action can be extended to $\mathscr{C}$, if we set

$$\pi(\partial_b) \,|0\rangle = 0.$$

With the extended left action $\pi$, the truncated quantum plane becomes a $\mathscr{C}$-module. In the following we will also write $\hat{Z}_a$, $\hat{\partial}_a$, $\xi$ instead of $\pi(Z_a)$, $\pi(\partial_a)$, $\pi(\xi)$.

*Definition 3.9.* (Algebra $\mathscr{P}_q^T$) The associative algebra $\mathscr{P}_q^T$ is defined by

$$\mathscr{P}_q^T = \mathscr{C}/\ker(\pi).$$

In other words, $\mathscr{P}_q^{\mathrm{T}}$ is the algebra generated by $\hat{Z}_a$, $\hat{\partial}_b$ and $\xi \in \mathrm{U}_q^{\mathrm{T}}(\mathrm{sl}_2)$. It is a subalgebra of the algebra $\mathscr{B}(\mathscr{F}_q^{\mathrm{T}})$ of linear transformations on $\mathscr{F}_q^{\mathrm{T}}$.

In the last part of this section we deduce explicit formulas for the action of $Z_a$, $\partial_a$ and elements $\xi \in \mathrm{U}_q^{\mathrm{T}}(\mathrm{sl}_2)$ on $\mathscr{F}_q^{\mathrm{T}}$. As explained earlier, the action of $\xi \in \mathrm{U}_q^{\mathrm{T}}(\mathrm{sl}_2)$ involves the covariant transformation law of $Z_a$. When evaluated on states this simplifies due to the following calculation:

$$\begin{aligned}
\xi \hat{Z}_a Z_\alpha^n |0\rangle &= \hat{Z}_b\big(\tau_{ba}^{1/2} \otimes \mathrm{id}\big)\big(\Delta(\xi)\big) Z_\alpha^n |0\rangle \\
&= \hat{Z} Z_\beta^n |0\rangle \big(\tau_{ba}^{1/2} \otimes \tau_{\beta\alpha}^{(n)}\big)\big(\Delta(\xi)\big) \\
&= \hat{Z} Z_\beta^n |0\rangle \big(\tau_{ba}^{1/2} \otimes \tau_{\beta\alpha}^{(n)}\big)\big(\Delta(e)\Delta_q(\xi)\big) \\
&= e\hat{Z} Z_\beta^n |0\rangle \big(\tau_{ba}^{1/2} \otimes \tau_{\beta\alpha}^{(n)}\big)\big(\Delta_q(\xi)\big) \\
&= \hat{Z}_b\big(\tau_{ba}^{1/2} \otimes \mathrm{id}\big)\big(\Delta_q(\xi)\big) Z_\alpha^n |0\rangle .
\end{aligned}$$

So we found that $\xi \hat{Z}_a = \hat{Z}_b(\tau_{ba}^{1/2} \otimes \mathrm{id})(\Delta_q(\xi))$ and $\Delta_q$ is formally defined by (2.2). Explicitly we get

$$\begin{aligned}
&q^{\pm H/2}\hat{Z}_1 = \hat{Z}_1 q^{\pm 1/4} q^{\pm H/2}, &&q^{\pm H/2}\hat{Z}_2 = \hat{Z}_1 q^{\mp 1/4} q^{\pm H/2}, \\
&S_-\hat{Z}_1 = \hat{Z}_1 q^{-1/4} S_- + \hat{Z}_2 q^{H/2}, &&S_-\hat{Z}_2 = \hat{Z}_2 q^{1/4} S_-, \\
&S_+\hat{Z}_1 = \hat{Z}_1 q^{-1/4} S_-, &&S_+\hat{Z}_2 = \hat{Z}_1 q^{1/4} S_+ + \hat{Z}_1 q^{H/2}.
\end{aligned} \tag{3.15}$$

To derive formulas for the action of $\hat{Z}_a$, $\hat{\partial}_a$ on the truncated quantum plane one makes use of the braid relations. By definition, braid relations for $\hat{Z}_a$, $\hat{\partial}_b$ are obtained from (3.12)–(3.14) if we substitute $\hat{Z}_a$ for $Z_a$ and $\hat{\partial}_b$ for $\partial_b$. When expressions of appendix A are inserted they read *

$$(\hat{Z} \times \hat{Z})_{12} = (\hat{Z} \times \hat{Z})_{21} q^{-1/2}, \tag{3.16}$$

$$(\hat{\partial} \times \hat{\partial})_{12} = (\hat{\partial} \times \hat{\partial})_{21} q^{1/2}, \tag{3.17}$$

$$(\hat{\partial} \times \hat{Z})_{11} = q^{-1/2} + q^{-1}(\hat{Z} \times \hat{\partial})_{11} + q^{-1}(\hat{Z} \times \hat{\partial})_{22}(q^{-1} - 1), \tag{3.18}$$

$$(\hat{\partial} \times \hat{Z})_{21} = q^{-1/2}(\hat{Z} \times \hat{\partial})_{21}, \tag{3.19}$$

* To compare with the formulas in ref. [8], $\hat{\partial}_1$, $\hat{\partial}_2$ have to be rescaled by factors $q^{1/2}$, $q^{-1/2}$, respectively.

$$(\hat{\partial} \times \hat{Z})_{21} = q^{-1/2}(\hat{Z} \times \hat{\partial})_{12}, \tag{3.20}$$

$$(\hat{\partial} \times \hat{Z})_{22} = q^{1/2} + q^{-1}(\hat{Z} \times \hat{\partial})_{22}. \tag{3.21}$$

The relations involve covariant products $\times$. By lemma B.1 in appendix B, these covariant products are equal to ordinary products when acting on states $\mathscr{F}_q^{\mathrm{T}} - \mathscr{F}_q^{\mathrm{T}(p-2)}$ ($-$ is to be understood in the set theoretic sense). Only if covariant products act on states in $\mathscr{F}_q^{\mathrm{T}(p-2)}$ they cannot be trivially converted into ordinary products. Evaluating the action of $Z_a$, $\partial_b$ on $\mathscr{F}_q^{\mathrm{T}}$, we apply at most one generator $(Z_a, \partial_b)$ on states in $\mathscr{F}_q^{\mathrm{T}} \supset \mathscr{F}_q^{\mathrm{T}(p-2)}$. In conclusion, the calculations leading to theorem 3.10 below are not effected by the distinction between ordinary and covariant products.

*Theorem 3.10.* The action of $Z_a$, $\partial_a$, $S_\pm$ and $q^{\pm H/2}$ on the truncated quantum plane is given by

$$S_- \,|\, n, m\rangle = q^{(n+m-1)/4} \frac{1-q^{-m}}{1-q^{-1}} \,|\, n+1, m-1\rangle, \tag{3.22}$$

$$S_+ \,|\, n, m\rangle = q^{(n+m-1)/4} \frac{1-q^{-m}}{1-q^{-1}} \,|\, n-1, m+1\rangle, \tag{3.23}$$

$$q^{\pm H/2} \,|\, n, m\rangle = q^{\pm(m-n)/4} \,|\, n, m\rangle, \tag{3.24}$$

$$\hat{Z}_1 \,|\, n, m\rangle = q^{-n/2} \,|\, n, m+1\rangle \quad \text{for } n+m < p-2, \tag{3.25}$$

$$\hat{Z}_2 \,|\, n, m\rangle = |\, n+1, m\rangle \quad \text{for } n+m < p-2, \tag{3.26}$$

$$\hat{Z}_a \,|\, n, m\rangle = 0 \quad \text{for } n+m = p-2, \tag{3.27}$$

$$\hat{\partial}_1 \,|\, n, m\rangle = q^{-(n+1)/2} \frac{1-q^{-m}}{1-q^{-1}} \,|\, n, m-1\rangle, \tag{3.28}$$

$$\hat{\partial}_2 \,|\, n, m\rangle = q^{1/2} \frac{1-q^{-n}}{1-q^{-1}} \,|\, n-1, m\rangle. \tag{3.29}$$

*Proof.* One should once see lemma B.1 in action to feel confident about its implications. To this aim we study $\hat{Z}_1 \,|\, n, m\rangle$. In the following $P^J \in \mathrm{U}_q^{\mathrm{T}}(\mathrm{sl}_2)$ should denote the minimal central projection belonging to the $(2J+1)$-dimensional representation of $U_q^T(sl_2)$. It acts as identity on the subspace $\mathscr{F}_q^{T(2J)}$.

$$\begin{aligned}\hat{Z}_1 | n, m\rangle &= \hat{Z}_1 \hat{Z}_2 | n-1, m\rangle = P^{(m+n+1)/2} \hat{Z}_1 \hat{Z}_2 | n-1, m\rangle \\ &= \hat{Z}_a \hat{Z}_b | n-1, m\rangle \left(\tau_{a1}^{1/2} \otimes \tau_{a2}^{1/2} \otimes \tau^{(n+m-1)/2}\right)(\mathrm{id} \otimes \Delta)\Delta(P^{(n+m+1)/2}) \\ &= \hat{Z}_a \hat{Z}_b | n-1, m\rangle \left(\tau_{a1}^{1/2} \otimes \tau_{a2}^{1/2} \otimes \tau^{(n+m-1)/2}\right)\left((\mathrm{id} \otimes \Delta)\Delta(P^{(n+m+1)/2})\phi\right) \\ &= \hat{Z}_a \hat{Z}_b \left(\tau_{a1}^{1/2} \otimes \tau_{b2}^{1/2} \otimes \mathrm{id}\right)(\phi) | n-1, m\rangle = (\hat{Z} \times \hat{Z})_{12} | n-1, m\rangle.\end{aligned}$$

Now we can apply the braid relation and write the resulting expression in terms of ordinary products. Thus we obtain $\hat{Z}_1 \hat{Z}_2 | n-1, m\rangle = \hat{Z}_2 \hat{Z}_1 | n-1, m\rangle q^{-1/2}$. An $n$-fold iteration gives the above result. Proofs of the other formulas are similar. □

## 4. The $*$-operation on $\mathscr{P}_q^T$ and scalar product on $\mathscr{F}_q^T$

In sect. 3 we defined an algebra $\mathscr{P}_q^T$ of operators acting on the truncated quantum plane $\mathscr{F}_q^T$. $\mathscr{P}_q^T$ contains $U_q^T(sl_2)$ as a subalgebra. From sect. 2 we know that a $*$-operation (an involutive antiautomorphism) on $U_q^T(sl_2)$ is given by (2.6). In this section we show how this $*$-operation can be extended to $\mathscr{P}_q^T$. Adjoints of multiplication operators $\hat{Z}_i$ act as differentiation on the truncated quantum plane. "Creators" $\hat{Z}_a$ and "annihilators" $\hat{Z}_a^* \equiv \hat{Z}_a^+$ satisfy $q^{-1/2}$-commutation relations. After having defined a suitable positive definite scalar product, we end up with a Hilbert space interpretation of the truncated quantum plane. This last step works only for the primitive $p$th roots $q$, $q = e^{2\pi i/p}$. In the limit $p \to \infty$ the harmonic oscillator is recovered.

Let us start with some general considerations. Given an algebra $\mathscr{A}$ containing $U_q^T(sl_2)$ as a subalgebra, the transformation law (3.1) for a covariant tupel $(F_\alpha)$ tells us how to shift elements $\xi \in U_q^T(sl_2)$ through the element $F_\alpha$ from left to right. We may ask how to do the reverse, that is how to shift $\xi$ from right to left of $F_\alpha$. In the example of $\mathscr{P}_q^T$ the covariance laws are explicitly given in (3.15). It is easy to see that they fix completely how to move generators of $U_q^T(sl_2)$ from right to left of the elements $\hat{Z}_a$. The relations that can be obtained from (3.15) are only special examples of

$$F_\alpha \xi = \left(\tilde{\tau}_{\alpha\beta} \otimes \mathrm{id}\right)(\Delta(\xi)) F_b \quad \text{for all } \xi \in U_q^T(sl_2), \tag{4.1}$$

which should hold for arbitrary $n$-tupels $(F_\alpha)$ transforming covariantly according to the representation $\tau$. Consistency of (4.1) with the covariant transformation law (3.1) is discussed in ref. [2].

Assume next that the $*$-operation (2.6) on $U_q^T(sl_2)$ can be extended to $\mathscr{A}$. Then (4.1) furnishes a transformation law for the adjoint tupel $(F_\alpha^*)$,

$$\xi F_\alpha^* = F_\beta^* \left( \tilde{\tau}_{\beta\alpha} \otimes \mathrm{id} \right) \Delta'(\xi). \tag{4.2}$$

To derive (4.2) we used the definition (2.7) for the action of $*$ on $U_q^T(sl_2) \otimes U_q^T(sl_2)$ and unitarity of $\tau$, i.e. $\tau_{\alpha\beta}(\xi^*) = \overline{\tau_{\beta\alpha}(\xi)}$ for all $\xi \in U_q^T(sl_2)$. Observe that the adjoint tupel $(F_\alpha^*)$ is not covariant since in (4.2) $\Delta'$ appears in place of $\Delta$. We define the conjugate tupel $(\bar{F}_\alpha)$ by

$$\bar{F}_\alpha = F_\beta^* \left( \tilde{\tau}_{\beta\alpha} \otimes \mathrm{id} \right)(R). \tag{4.3}$$

Using the intertwining properties (2.16) of $R$ we deduce that $(\bar{F}_\alpha)$ transform covariantly according to representation $\tilde{\tau}$. We may invert (4.3) to express the adjoint tupel in terms of the conjugate,

$$F_\alpha^* = \bar{F}_\beta \left( \tilde{\tau}_{\beta\alpha} \otimes \mathrm{id} \right)(R^{-1}). \tag{4.4}$$

In $\mathscr{P}_q^T$, the pair $(\hat{\partial}_a)$ transforms covariantly according to the contragredient representation. Applying the linear transformation (4.4) we can construct a pair $(\hat{Z}_b^+)$, $\hat{Z}_b^+ \in \mathscr{P}_q^T$, which obeys the transformation law (4.2),

$$\hat{Z}_a^+ = \hat{\partial}_b ( \tilde{\tau}_{ba} \otimes \mathrm{id} )(R^{-1}) c \tag{4.5}$$

Here $c$ is an element in the center of $U_q^T(sl_2)$, which does not alter the transformations properties.

The study of transformation properties suggests to regard $\hat{Z}_a^+$ as a good candidate for the adjoint of $\hat{Z}_a$ under the $*$-operation to be found. Given some fixed central $c$, we would like to extend the map $\hat{Z}_a \to \hat{Z}_a^* = \hat{Z}_a^+$ to an involutive antihomomorphism of $\mathscr{P}_q^T$. Aside from the choice of $c$ there remains no freedom for this extension, since $\mathscr{P}_q^T$ is generated by $Z_a$, $\hat{Z}_b^+$ and elements $\xi \in U_q^T(sl_2)$. Consistency with the braid relations in $\mathscr{P}_q^T$ presents an important obstruction. However, it turns out that for properly chosen central $c$, consistency is achieved.

The center of the truncated quantum group $U_q^T(sl_2)$ is spanned by the minimal central projectors $P^J$, $J = 0, \frac{1}{2}, \dots, \frac{1}{2}(p-2)$, corresponding to the irreducible representations of $U_q^T(sl_2)$. The "number" operator $N$ is a special element. In terms of $P^J$ it is given by

$$N = \sum_{J=0}^{1/2(p-2)} (2J+1) P^J.$$

Elements $\hat{Z}_a$, $\hat{\partial}_a$ change the eigenvalues of $N$ by one,

$$N\hat{Z}_a = \hat{Z}_a(N+1), \qquad N\hat{\partial}_a = \hat{\partial}_a(N-1) \tag{4.6}$$

It will turn out that the pair $Z_a^+$ is the adjoint of $Z_a$ for some $*$-operation on $\mathscr{P}_q^{\mathrm{T}}$ provided that we choose

$$c = q^{N/4-1}. \tag{4.7}$$

*Theorem 4.1.* The operators $\hat{Z}_a^+ \equiv \hat{\partial}_b(\tilde{\tau}_{ba}^{1/2} \otimes \mathrm{id})(R^{-1})q^{N/4-1}$ are explicitly given by

$$\hat{Z}_1^+ = \left(\hat{\partial}_1 q^{H/2} + \hat{\partial}_2 q^{-1/4}\left(q^{1/2} - q^{-1/2}\right)S_-\right)q^{N/4-1}, \tag{4.8}$$

$$\hat{Z}_2^+ = \hat{\partial}_2 q^{-H/2} q^{N/4-1}. \tag{4.9}$$

Let $[A, B]_{\not{q}} \equiv AB - q^{-1/2}BA$ for all $A, B \in \mathscr{P}_q^{\mathrm{T}}$. Then the braid relations in $\mathscr{P}_q^{\mathrm{T}}$ can be rewritten in terms of the $q^{-1/2}$-commutator $[\,,]q^{-1/2}$ of $\hat{Z}_a$, $\hat{Z}_a^+$,

$$\left[\hat{Z}_1, \hat{Z}_2\right]_{q^{-1/2}} = 0, \tag{4.10}$$

$$\left[\hat{Z}_1^+, \hat{Z}_2^+\right]_{q^{-1/2}} = 0, \tag{4.11}$$

$$\left[\hat{Z}_a^+, \hat{Z}_b\right]_{q^{-1/2}} = \theta_{ab} \in \mathrm{U}_q^{\mathrm{T}}(\mathrm{sl}_2), \tag{4.12}$$

where

$$\theta_{ab} = g_{fg}\left(\tilde{\tau}_{fa}^{1/2} \otimes \tau_{gb}^{1/2} \otimes \mathrm{id}\right)\left(R_{12}^{-1}\phi_{213}^{-1}R_{13}^{-1}\right)q^{1/4}q^{N/4-1}.$$

Moreover, the following additional equations hold:

$$\hat{Z}_1\hat{Z}_2^+ = q^{-3/4}q^{-N/4+1}S_+, \qquad \hat{Z}_2^+\hat{Z}_1 = q^{-5/4}q^{-N/4+1}S_+P, \tag{4.13}$$

$$\hat{Z}_2\hat{Z}_1^+ = q^{3/4}q^{N/4-1}S_-, \qquad \hat{Z}_1^+\hat{Z}_2 = q^{+5/4}q^{N/4-1}S_-P, \tag{4.14}$$

$$\hat{Z}_a\hat{Z}_a^+ = [N_a]_q, \qquad \hat{Z}_a^+\hat{Z}_a = [N_a + 1]_q. \tag{4.15}$$

Here $P$ denotes the projector on $\mathscr{F}_q^{\mathrm{T}} - \mathscr{F}_q^{\mathrm{T}(p-2)}$ (set theoretic difference). The operator $N_a \in \mathscr{B}(\mathscr{F}_q^{\mathrm{T}})$ is defined by the action $N_a | n_1, n_2\rangle = n_a | n_1, n_2\rangle$ on the states $| n_1, n_2\rangle$ and we used the abbreviation $[x]_q = (q^{x/2} - q^{-x/2})/(q^{1/2} - q^{-1/2})$.

*Proof.* A procedure similar to the proof of (3.15) establishes the explicit formulas for $\hat{Z}_a^+$. One decomposes $R^{-1} = \Delta(e)R_q^{-1}$ and applies the transformation properties of states to verify $\hat{Z}_a^+ \equiv \hat{\partial}_b(\tilde{\tau}_{ba}^{1/2} \otimes \mathrm{id})(R^{-1})c = \hat{\partial}_b(\tilde{\tau}_{ba}^{1/2} \otimes \mathrm{id})(R_q^{-1})c$. $(\tilde{\tau}_{ba}^{1/2} \otimes \mathrm{id})(R_q^{-1})$ is evaluated in appendix A.

All other equations in the theorem can be verified directly as an application of theorem 3.10. Alternatively, one can use the braid relations of $\hat{Z}_a$, $\hat{\partial}_b$ to argue for the $q^{-1/2}$-commutation relations in the theorem. In a first step we write (3.13) and (3.14) in terms of $\hat{Z}_a$ and $\hat{Z}_b^+$. The result is

$$\hat{Z}_a^+\hat{Z}_b^+ = \hat{Z}_d^+\hat{Z}_c^+\left(\tilde{\tau}_{ca}^{1/2}\otimes\tilde{\tau}_{db}^{1/2}\otimes \mathrm{id}\right)\left(\phi_{213}^{-1}R'_{12}\phi_{321}\right)q^{-1/4},$$

$$\hat{Z}_a^+\hat{Z}_b\left(\tilde{\tau}_{ac}^{1/2}\otimes\tau_{bd}^{1/2}\otimes \mathrm{id}\right)\left(\phi_{231}^{-1}\right) = g_{fg}\left(\tilde{\tau}_{fc}^{1/2}\otimes\tau_{gd}^{1/2}\otimes \mathrm{id}\right)\left(R_{12}^{-1}\phi_{213}^{-1}R_{13}^{-1}\right)cq^{1/4}$$
$$+\hat{Z}_d\hat{Z}_c^*q^{-1/2}.$$

Here the $'$ denotes permutation of the components of the $R$-matrix. The rest is again a standard application of lemma B.1 and appendix A. This is left as an exercise. □

*Theorem 4.2.* The map $*$ defined on generators $\hat{Z}_a$ by $\hat{Z}_a^* = \hat{Z}_a^+ = \hat{\partial}_b(\tilde{\tau}_{ba}^{1/2}\otimes \mathrm{id})(R_q^{-1})c$ and on $\xi\in \mathrm{U}_q^{\mathrm{T}}(\mathrm{sl}_2)$ by (2.6) can be extended to a $*$-operation on $\mathscr{P}_q^{\mathrm{T}}$.

*Proof.* In theorem 4.1, all the relations in $\mathscr{P}_q^{\mathrm{T}}$ are displayed in a form that is well suited for a proof of theorem 4.1. For example, by (4.11)

$$\left[\hat{Z}_1, \hat{Z}_2\right]_{q^{-1/2}}^* = \left[\hat{Z}_2^+, \hat{Z}_1^+\right]_{q^{1/2}} = q^{1/2}\left[\hat{Z}_1^+, \hat{Z}_2^+\right]_{q^{-1/2}} = 0.$$

At a first glance, one might be afraid of problems coming from (4.12). However, taking into account eqs. (4.13)–(4.15) relation (4.12) appears as an equation in $\mathrm{U}_q^{\mathrm{T}}(\mathrm{sl}_2)$, which is consistent with the $*$-operation. Consistency of (4.13)–(4.15) is obvious. □

*Corollary 4.3.* On generators $\hat{\partial}_a$ the $*$-operation acts as

$$\hat{\partial}_1^* = \left(Z_1q^{3/2}q^{H/2} + q^{5/4}\left(q^{1/2}-q^{-1/2}\right)Z_2S_+\right)q^{N/4-1}, \tag{4.16}$$

$$\hat{\partial}_2^* = Z_2q^{-1/2}q^{-H/2}q^{N/4}. \tag{4.17}$$

*Proof.* To derive the action of $*$ on $\hat{\partial}_a$ we solve (4.8) and (4.9) for $\hat{\partial}_b$, apply the $*$-operation and use the involution property $\hat{Z}_a^{+*} = (\hat{Z}_a^*)^* = \hat{Z}_a$. □

The $*$-operation on $\mathscr{P}_q^{\mathrm{T}}$ can be used to define a bilinear form for $\mathscr{F}_q^{\mathrm{T}}$. Let $\langle 0|$ denote the linear form on $\mathscr{F}_q^{\mathrm{T}}$ specified by

$$\langle 0|n, m\rangle = \delta_{n,0}\delta_{m,0}.$$

*Theorem 4.4.* (scalar product) The bilinear form

$$\langle\,|\,\rangle : \mathscr{F}_q^{\mathrm{T}}\times\mathscr{F}_q^{\mathrm{T}}\to\mathbb{C}, \tag{4.18}$$

$$(|n', m'\rangle\times|n, m\rangle)\to\langle n', m'|n, m\rangle \equiv \langle 0|\left(\hat{Z}_2^{n'}\hat{Z}_1^{m'}\right)^*\hat{Z}_2^n\hat{Z}_1^m|0\rangle \tag{4.19}$$

is symmetric and positive definite and thus defines a scalar product on $\mathscr{F}_q^{\mathrm{T}}$. Explicitly,

$$\langle n', m' \mid n, m\rangle = \delta_{n',n}\delta_{m',m}[n]_q![m]_q! \tag{4.20}$$

*Proof.* It is obvious that $\langle n', m' \mid n, m\rangle = 0$ if $n \neq n'$ or $m \neq m'$. It remains to calculate $\langle n, m \mid n, m\rangle$.

$$\langle n, m \mid n, m\rangle = \langle 0 \mid \hat{Z}_2^n * \hat{Z}_1^m * \hat{Z}_2^n \hat{Z}_1^m \mid 0\rangle$$
$$= [n]_q![m]_q! > 0.$$

We used (4.15) to get the last equality and $[n]_q! = [n]_q[n-1]_q \ldots [1]_q$. Thus we derived the formula $\langle n'm', n, m\rangle = \delta_{n,n'}\delta_{m,m'}[n]_q![m]_q!$ from which symmetry and positivity follow using that $[n]_q > 0$ if $n \leqslant p-1$, $q = e^{2\pi i/p}$. Thus theorem 4.4 is established. □

*Corollary 4.5.* An orthonormal basis of the truncated quantum plane is provided by

$$\mid n, m\rangle_0 \equiv \frac{1}{\sqrt{[n]_q![m]_q!}} \mid n, m\rangle. \tag{4.21}$$

It remains to discuss the properties of the hamiltonians proposed in sect.1. For the "Dirac adjoint" $\bar{Z}_a$ we substitute $\bar{Z}_a = \hat{Z}_b^*(\bar{\tau}_{ba}^{1/2} \otimes \mathrm{id})(R)q^{1-N/4} = \hat{\partial}_a$. Due to the properties of $g^{ab}$ (cf. text before definition 3.7), the expression $(Z \times \bar{Z})_{ab} g^{ab}$ is seen to be invariant. Invariance (1.7) of the hamiltonians $H_1$, $H_2$ is a direct consequence of this observation. The spectrum can be calculated with help of theorem 3.10.

*Proposition 4.6.* The hamiltonians $H_1$, $H_2$ are both diagonal on the states $\mid n, m\rangle_0$, i.e. $H_i \mid m, m\rangle_0 = \lambda^i_{n+m} \mid n, m\rangle_0$. For $r \leqslant p-2$, $\lambda^1_r = [r]_q + 1$ and $\lambda^2_r = \frac{1}{2}[2r]_q + 1$.

Let us conclude with a short remark on nonsingular values of the deformation parameter $q$ (i.e. $|q| = 1$, but $q$ not a root of unity). In these cases, no truncation is needed. With simple obvious changes, a $*$-operation on the quantum plane is still given by the formulas above. Also the explicit expression for the bilinear form $\langle\, ,\rangle$ in the proof of theorem 4.5 applies. But without truncation we have no upper limit for $n$, $m$ and the product $[n]_q![m]_q!$ will become negative for certain pairs $(n, m)$. Thus positivity is lost for generic values of $q$.

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## Appendix A

THE REPRESENTATION $\tau^{1/2}$

In this appendix we collect some standard formulas concerning the two-dimensional fundamental representation of the quantum group algebra $U_q(sl_2)$.

$\tau^{1/2}$ is defined by the following action on the generators of $U_q(sl_2)$:

$$\tau^{1/2}(S_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \tau^{1/2}(S_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\tau^{1/2}(q^{\pm H/2}) = \begin{pmatrix} q^{\pm 1/4} & 0 \\ 0 & q^{\mp 1/4} \end{pmatrix}. \tag{A.1}$$

Acting with $\tau^{1/2}$ on the first component of $R_q$ we obtain

$$(\tau^{1/2} \otimes \mathrm{id})(R_q) = \begin{pmatrix} q^{H/2} & q^{-1/4}(q^{1/2} - q^{-1/2})S_- \\ 0 & q^{-H/2} \end{pmatrix}. \tag{A.2}$$

The contragredient representation is given by

$$\tilde{\tau}^{1/2}(S_+) = \begin{pmatrix} 0 & 0 \\ -q^{-1/2} & 0 \end{pmatrix}, \qquad \tilde{\tau}^{1/2}(S_-) = \begin{pmatrix} 0 & -q^{1/2} \\ 0 & 0 \end{pmatrix},$$

$$\tilde{\tau}^{1/2}(q^{\pm H/2}) = \begin{pmatrix} q^{\mp 1/4} & 0 \\ 0 & q^{\pm 1/4} \end{pmatrix}. \tag{A.3}$$

The action of $\tilde{\tau}^{1/2}$ on the first component of $R_q$ gives

$$(\tilde{\tau}^{1/2} \otimes \mathrm{id})(R_q) = \begin{pmatrix} q^{-H/2} & 0 \\ q^{-3/4}(q^{-1/2} - q^{1/2})S_- & q^{H/2} \end{pmatrix}. \tag{A.4}$$

From (A.2), (A.4) and (A.1), (A.3) one can obtain the matrices $(\tau^{1/2} \otimes \tau^{1/2})(R_q)$, $(\tilde{\tau}^{1/2} \otimes \tau^{1/2})(R_q)$ and $(\tilde{\tau}^{1/2} \otimes \tilde{\tau}^{1/2})(R_q)$.

## Appendix B

LEMMA B.1

The lemma we prove in this appendix expresses that truncation does not effect reassociation of $(\tau^{1/2} \boxtimes \tau^{1/2}) \boxtimes \tau^K$ as long as $K$ is not maximal. If $K$ is maximal one has to be more careful.

*Lemma B.1.* Let $K=0, \frac{1}{2}, 1, \ldots, (p-3)/2$ and $L=0, \frac{1}{2}, 1, \ldots, (p-2)/2$ or $K=(p-2)/2$ and $L=(p-4)/2$. Suppose $\tau$, $\tau'$ are two representations equivalent to $\tau^{1/2}$. Then

$$(\tau \otimes \tau' \otimes \tau^K)\big((e \otimes \Delta)\Delta(P^L)\phi\big) = (\tau \otimes \tau' \otimes \tau^K)\big((e \otimes \Delta)\Delta(P^L)\big). \quad \text{(B.1)}$$

Especially this applies to $\tau$, $\tau'$ being $\tau^{1/2}$ or $\tilde{\tau}^{1/2}$.

*Proof.* For the values of $K$, $L$ in lemma B.1 one can derive the following identity:

$$\begin{bmatrix} \frac{1}{2} & K & P \\ j & k & p \end{bmatrix}_q \begin{bmatrix} \frac{1}{2} & P & L \\ i & p & l \end{bmatrix}_q = \sum_{Q,q} \begin{Bmatrix} K & \frac{1}{2} & P \\ \frac{1}{2} & L & Q \end{Bmatrix}_q \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & Q \\ i & j & q \end{bmatrix}_q \begin{bmatrix} Q & K & L \\ q & k & l \end{bmatrix}_q . \quad \text{(B.2)}$$

This follows from the corresponding equation for Clebsch–Gordon coefficients and $6j$-symbols of $U_{q'}(sl_2)$ at generic values of $q'$ (i.e. $q$ not a root of unity) [11]. The restrictions on the allowed combinations of $K$, $L$ ensure that the limit $q' \to q$, $q^p = 1$ is regular. Inserting the identity into the definition (2.14) of $\phi$ we obtain

$$\sum_{ijkp} \begin{bmatrix} \frac{1}{2} & P & L \\ i & p & l \end{bmatrix}_q \begin{bmatrix} \frac{1}{2}K & P & \\ j & k & p \end{bmatrix}_q \hat{e}_i^{1/2} \otimes \hat{e}_j^{1/2} \otimes \hat{e}_k^K \phi$$

$$= \sum_{Q,ijkq} \begin{Bmatrix} K & \frac{1}{2} & P \\ \frac{1}{2} & L & Q \end{Bmatrix}_q \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & Q \\ i & j & q \end{bmatrix}_q \begin{bmatrix} Q & K & L \\ q & k & l \end{bmatrix}_q \hat{e}_i^{1/2} \otimes \hat{e}_j^{1/2} \otimes \hat{e}_k^K$$

$$= \sum_{ijkp} \begin{bmatrix} \frac{1}{2} & P & L \\ i & p & l \end{bmatrix}_q \begin{bmatrix} \frac{1}{2} & K & P \\ j & k & p \end{bmatrix}_q \hat{e}_i^{1/2} \otimes \hat{e}_j^{1/2} \otimes \hat{e}_k^K .$$

This proves the lemma for $\tau, \tau' = \tau^{1/2}$. To extend it to arbitrary representations $\tau$, $\tau'$ equivalent to $\tau^{1/2}$ is trivial. □

## Appendix C

### PROOF OF THEOREM 3.6

For convenience of the reader we give the details of the prof of theorem 3.6. The main ingredients of the proof are in the following two lemmas:

*Lemma C.1.* $\dim \mathcal{F}_q^{\mathrm{T}(n)} = n+1$ if $n \leqslant p-2$.

*Lemma C.2.* Suppose that $\mathcal{F}_q^{\mathrm{T}(r)}$ carries the irreducible representation $\tau$ and $\mathcal{F}_q^{\mathrm{T}(s)}$ carries the irreducible representation $\tau'$. Then $\mathcal{F}_q^{\mathrm{T}(r+s)}$ is either zero or it carries a subrepresentation of $\tau \boxtimes \tau'$.

*Proof of lemma C.1.* We recall that the tensor product of representations of $U_q^{\mathrm{T}}(\mathrm{sl}_2)$ is the truncated tensor product of physical representations of $\mathrm{U}_q(\mathrm{sl}_2)$, see sect. 2. No truncation appears in the tensor product $\tau^I \boxtimes \tau^J$ if $I+J \leqslant p-2-I-J$, i.e. if $I+J \leqslant \frac{1}{2}(p-2)$, so that the braid relations reduce to [8]

$$\hat{Z}_1 \hat{Z}_2 = q^{-1/2} \hat{Z}_2 \hat{Z}_1. \tag{C.1}$$

More rigorously, these relations can be obtained as a simple consequence of appendix B. This can be used to shift all factors $\hat{Z}_1$ to the right of all factors $\hat{Z}_2$. There can be $0,\dots,n$ factors $\hat{Z}_1$. It follows that the number of linearly independent vectors in $\mathcal{F}_q^{\mathrm{T}(n)}$ is $n+1$. □

*Proof of lemma C.2.* The lemma follows from a discussion similar to the one which precedes theorem 3.6 in sect. 3.

*Proof of theorem 3.6.* Validity of theorem 3.6 for $0 \leqslant n \leqslant (p-2)$ follows from the two lemmas and the tensor product decomposition (2.12) and (2.9) applied to $\tau^J \otimes \tau^{1/2}$.

To prove the first part of theorem 3.6 it suffices to show that $\mathcal{F}_q^{\mathrm{T}(p-1)} = 0$ since all higher-order polynomials contain factors of order $p-1$. In the following $P^J \in U_q^{\mathrm{T}}(\mathrm{sl}_2)$ should denote the minimal central element of $U_q^{\mathrm{T}}(\mathrm{sl}_2)$ that is associated with the irreducible $(2J+1)$-dimensional representation of $U_q^{\mathrm{T}}(\mathrm{sl}_2)$. To prove $\mathcal{F}_q^{\mathrm{T}(p-1)} = 0$ we use that

$$(Z \times Z)_{ab} \left(\tau_{ac}^{1/2} \otimes \tau_{bd}^{1/2}\right)\left(\Delta(P^0)\right) = 0. \tag{C.2}$$

This expresses the fact that there are no homogeneous polynomials of degree two which transform according to the trivial representation. Moreover we know that all polynomials of degree $p-2$ transform according to the $(p-1)$-dimensional representation. In mathematical terms this means that

$$(Z^{p-3} \times Z)_{\alpha a} = (Z^{p-3} \times Z)_{\beta b} \left(\tau_{\beta\alpha}^{(p-3)} \otimes \tau_{ba}^{1/2}\right)\left(\Delta(P^{\frac{1}{2}(p-2)})\right). \tag{C.2}$$

We wish to show that

$$\left(Z^{p-3} \times (Z \times Z)\right)_{\alpha ab} = 0.$$

In order to establish this we will show that all these polynomials $((Z^{p-3} \times Z) \times$

$Z)_{\beta cd}$ are linear combinations of the polynomials

$$\left(Z^{p-3}\times(Z\times Z)\right)_{\alpha ab}\left(\tau^{\frac{1}{2}(p-3)}_{\alpha\beta}\otimes\tau^{1/2}_{ac}\otimes\tau^{1/2}_{bd}\right)\left(e\otimes\Delta(P^{0})\right)$$

which vanish due to (C.2). More precisely we show that

$$\begin{aligned}&\left((Z^{p-3}\times Z)\times Z\right)_{\beta cd}a\\&\quad=\left(Z^{p-3}\times(Z\times X)\right)_{\alpha ab}\left(\tau^{\frac{1}{2}(p-3)}_{\alpha\beta}\otimes\tau^{1/2}_{ac}\otimes\tau^{1/2}_{bd}\right)\left(\left(e\otimes\Delta(P^{0})\right)A\right),\end{aligned}\qquad(C.4)$$

where $a\neq 0$ is a real number and $A\in U_q^{\mathrm{T}}(\mathrm{sl}_2)^{\otimes 3}$ is given by $A\equiv\phi(\Delta(P^{\frac{1}{2}(p-2)})\otimes e)$. It follows from eq. (C.3) by reassociation that the right-hand side of eq. (C.4) equals

$$\left((Z^{p-3}\times Z)\times Z\right)_{\alpha ab}\left(\tau^{\frac{1}{2}(p-3)}_{\alpha\beta}\otimes\tau^{1/2}_{ac}\otimes\tau^{1/2}_{bd}\right)\left(\left(\Delta(P^{\frac{1}{2}(p-2)})\otimes e\right)\phi^{-1}\left(e\otimes\Delta(P^{0})\right)A\right).\qquad(C.5)$$

Using the explicit expression (2.14) for $\phi$ it is easy to see that

$$\left(\Delta(P^{\frac{1}{2}(p-2)})\otimes e\right)\phi^{-1}\left(e\otimes\Delta(P^{0})\right)A=\left(\Delta(P^{\frac{1}{2}(p-2)})\otimes e\right)a,$$

where $a$ is some nonzero real number. So we get finally that the right-hand side of eq. (C.4) equals

$$\begin{aligned}&\left((Z^{p-3}\times Z)\times Z\right)_{\alpha ab}\left(\tau^{\frac{1}{2}(p-3)}_{\alpha\beta}\otimes\tau^{1/2}_{ac}\otimes\tau^{1/2}_{bd}\right)\left(\Delta(P^{\frac{1}{2}(p-2)})\otimes e\right)a\\&\quad=\left((Z^{p-3}\times Z)\times Z\right)_{\beta cd}a\end{aligned}\qquad(C.6)$$

by (C.3). This completes the proof that all products of the type $(Z^{p-3}\times Z)\times Z$ vanish. But since all other products of $p-1$ generators $Z$ can be obtained out of these by reassociation, we established that $\mathscr{F}_q^{\mathrm{T}(p-1)}=0$. This concludes the proof of theorem 3.6. □

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