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Computation of the running coupling in the SU(2) Yang–Mills theory

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A finite-size scaling technique is applied to the SU(2) gauge theory (without matter fields) to compute a non-perturbatively defined running coupling $\alpha(q)$ for a range of momenta q given in units of the string tension K. We find that already at rather low q, the evolution of $\alpha(q)$ is well described by the 2-loop approximation to the Callan–Symanzik β-function. At the highest momentum reached, $q = 20 \times \sqrt{K}$, we obtain $\alpha_{\overline{\mathrm{MS}}}(q) = 0.187 \pm 0.005 \pm 0.009$ for the running coupling in the $\overline{\mathrm{MS}}$ scheme of dimensional regularization.

1. Introduction

From the point of view of perturbation theory, the renormalized coupling $\alpha(q)$ in QCD is an input parameter, whose value at some reference momentum $q = q_0$ must be supplied by experiment. There is little doubt that QCD is a well-defined theory also at low energies, where the perturbation expansion does not apply. We may thus imagine that the parameters of the theory are fixed through the hadron spectrum, for example, or some other set of experimentally accessible quantities in the low-energy domain. The running coupling then becomes a computable function of momentum, in any renormalization scheme that one may choose.

A theoretical determination of $\alpha(q)$ at high energies is obviously desirable. In particular, one would be interested to know at which scale the perturbative evolution of the coupling sets in. Since one is concerned with the non-perturbative properties of QCD, the lattice formulation of the theory, combined with numerical

simulation techniques, is currently the most promising way to approach the problem [1–5]. The basic difficulty in any such calculation is that the momenta q of interest can be orders of magnitude greater than the masses of the light particles in the theory. Lattices sufficiently wide to avoid finite-volume effects and with a spacing a substantially smaller than $1/q$ thus tend to be much larger than what can be simulated on a computer.

Many years ago Wilson pointed out that this difficulty may be overcome, in principle, by introducing a renormalization group transformation which allows one to step up the energy scale in a recursive manner [1]. No simulation of an exceedingly large lattice would then be required, while all physical scales are kept at a safe distance from the ultra-violet cutoff $1/a$.

The finite-size scaling technique described in ref. [2] may be regarded as a particular realization of this basic idea, even though the details are quite different from Wilson's formulation. The method has already been shown to work well in the case of the two-dimensional non-linear σ-model. In the present paper it is applied to the pure SU(2) gauge theory, the simplest asymptotically free theory in four dimensions. As a result we shall be able to compute the running coupling (in a certain adapted scheme [3]) over a large range of momenta, q, reaching energies far above the scale set by the string tension.

Other strategies to compute the running coupling in non-abelian gauge theories have recently been put forward by El-Khadra et al. [4] and Michael [5]. In both cases one considers a single lattice which covers all relevant distance scales, and one is, therefore, limited to rather low momenta q. In ref. [5], for example, the coupling is determined from the static quark potential at short distances. The largest lattice currently available has 56×48^3 points and a spacing a roughly equal to 0.03 fm [21]. The distances at which the coupling can be calculated with some confidence are thus greater than 0.1 fm or so, a limitation which will be difficult to alleviate.

2. Finite-size technology

The finite-size scaling technique of ref. [2] is based on a renormalized coupling

$$\alpha(q) = \frac{\bar{g}^2(L)}{4\pi}, \qquad q = 1/L, \tag{2.1}$$

which runs with the linear extent L of the lattice. As proposed in ref. [3], we define $\bar{g}^2(L)$ through the response of the system to a constant colour-electric background field. For detailed explanations the reader is referred to refs. [2,3]. Here we only list the basic definitions and outline our strategy.

2.1. BOUNDARY CONDITIONS AND FUNCTIONAL INTEGRAL

We choose to set up the theory on a hyper-cubic euclidean lattice with spacing a and size $L \times L \times L \times L$. In particular, the possible values of the time coordinate x^0 of a lattice point x are $x^0 = 0, a, 2a, \ldots, L$ (L is taken to be an integer multiple of a). The spatial sublattices at fixed times are thought to be wrapped on a torus, i.e. we assume periodic boundary conditions in these directions.

A gauge field U on the lattice is an assignment of a matrix $U(x, \mu) \in \mathrm{SU}(2)$ to every pair $(x, x + a\hat{\mu})$ of nearest neighbor lattice points ($\hat{\mu}$ denotes the unit vector in the μ-direction and $\mu = 0, 1, 2, 3$). At the top and bottom of the lattice, the link variables are required to satisfy inhomogenous Dirichlet boundary conditions,

$$U(x, k)\,|_{x^0=0} = W(\boldsymbol{x}, k), \qquad U(x, k)\,|_{x^0=L} = W'(\boldsymbol{x}, k), \tag{2.2}$$

for all $k = 1, 2, 3$, where W and W' are prescribed spatial gauge fields. They will be set to some particular values below.

The action of a lattice gauge field U is taken to be

$$S[U] = \frac{1}{g_0^2} \sum_p w(p)\ \mathrm{tr}\{\mathbb{1} - U(p)\}, \tag{2.3}$$

with g_0 being the bare coupling. The sum in eq. (2.3) runs over all *oriented* plaquettes p on the lattice and $U(p)$ denotes the parallel transporter around p. The weight $w(p)$ is equal to 1 in all cases except for the spatial plaquettes at $x^0 = 0$ and $x^0 = L$ which are given the weight $\frac{1}{2}$.

The partition function of the system,

$$\mathcal{Z} = \int \mathrm{D}[U]\ \mathrm{e}^{-S[U]}, \qquad D[U] = \prod_{x,\mu} \mathrm{d}U(x, \mu), \tag{2.4}$$

involves an integration over all fields U with fixed boundary values W and W'. $\mathcal{Z}$ is also referred to as the Schrödinger functional, because it is just the (euclidean) propagation kernel for going from the initial field configuration W at time $x^0 = 0$ to the final configuration W' at $x^0 = L$.

2.2. BACKGROUND FIELD

At small couplings g_0, the integral (2.4) is dominated by the field configurations with least action. In the cases considered below, there is only one such configuration (modulo gauge transformations). Through the boundary conditions, we have thus forced a background field into the system.

As indicated at the beginning of this section, we are interested in generating a constant colour-electric background field. This can be achieved by choosing the

boundary values W and W' to be constant diagonal matrices. More precisely, we set

$$W(\boldsymbol{x}, k) = \exp\{aC_k\}, \quad C_k = \eta \frac{\tau_3}{iL}, \tag{2.5}$$

$$W'(\boldsymbol{x}, k) = \exp\{aC'_k\}, \quad C'_k = (\pi - \eta) \frac{\tau_3}{iL}, \tag{2.6}$$

where η is a real parameter, to be fixed later, and τ_3 the third Pauli matrix. The induced background field is then given by

$$V(x, \mu) = \exp\{aB_\mu(x)\}, \tag{2.7}$$

with

$$B_0 = 0, \qquad B_k = \left[x^0 C'_k + (L - x^0)C_k\right]/L. \tag{2.8}$$

As shown in ref. [3], this is indeed a configuration with least action for the specified boundary values, provided

$$0 < \eta < \pi \quad \text{and} \quad L/a \geqslant 4. \tag{2.9}$$

From the linear time dependence of B, it is evident that V corresponds to a constant colour-electric field.

2.3. DEFINITION OF $\bar{g}^2(L)$

The effective action of the background field (2.7) is defined by

$$\Gamma = -\ln \mathcal{Z}. \tag{2.10}$$

To leading order in perturbation theory, Γ is simply equal to the classical action

$$S[V] = \frac{6}{g_0^2} \left\{ \frac{2L^2}{a^2} \sin\left[\frac{a^2}{2L^2} (\pi - 2\eta) \right] \right\}^2, \tag{2.11}$$

and the higher-order corrections may be worked out by expanding the Schrödinger functional about the background field.

A crucial observation now is that the continuum limit of the effective action exists, provided the bare coupling is renormalized in the usual way. As discussed in ref. [3], this follows from power counting and gauge invariance. The statement has, furthermore, been checked explicitly to one-loop order.

We are thus led to define a renormalized coupling $\bar{g}^2(L)$ through

$$\frac{\partial \Gamma}{\partial \eta} = \frac{k}{\bar{g}^2(L)}, \tag{2.12}$$

where the proportionality constant k is adjusted such that $\bar{g}^2(L)$ is equal to g_0^2 to lowest order. From eq. (2.11) we deduce

$$k = -24\frac{L^2}{a^2}\sin\left[\frac{a^2}{L^2}(\pi - 2\eta)\right]. \tag{2.13}$$

A derivative with respect to η is taken, because expectation values are much easier to compute than partition functions (cf. sect. 3).

To complete the definition of $\bar{g}^2(L)$ we must finally pick some value for the background field parameter η. We decided to take

$$\eta = \pi/4, \tag{2.14}$$

which is half-way between the zero action point $\eta = \pi/2$ and the boundary of the stability interval (2.9).

2.4. RELATION TO OTHER SCHEMES

In the continuum limit, and at sufficiently high energies (small L), perturbation theory may be used to relate different running couplings. In particular, the connection between the $\overline{\text{MS}}$ scheme of dimensional regularization [6] and our finite volume scheme is [3]

$$\alpha_{\overline{\text{MS}}} = \alpha + k_1\alpha^2 + \ldots, \quad k_1 = 0.94327(5) \tag{2.15}$$

[cf. eq. (2.1); both couplings are at the same momentum q].

Another coupling in infinite volume, $\alpha_{q\bar{q}}(q)$, is defined by

$$\alpha_{q\bar{q}}(q) = \tfrac{4}{3}r^2F(r), \quad q = 1/r, \tag{2.16}$$

where $F(r)$ denotes the force between static quarks at distance r. Combining the expansion above with the one-loop results of refs. [7,8], one finds

$$\alpha_{q\bar{q}} = \alpha + h_1\alpha^2 + \ldots, \quad h_1 = 0.99802(5), \tag{2.17}$$

i.e. to this order, there is practically no difference between $\alpha_{q\bar{q}}$ and $\alpha_{\overline{\text{MS}}}$.

2.5. LOW-ENERGY REGIME

When L is greater than the confinement scale, the behaviour of the running coupling is determined by non-perturbative effects. An important observation at this point is that the boundary fields W and W' are locally pure gauge configurations. Any dependence of the effective action Γ on the background field parameter η is hence associated with correlations "around the world". Since these are exponentially suppressed in a massive theory, we conclude that $\bar{g}^2(L) \propto \mathrm{e}^{\mu L}$ at large L.

The mass μ occurring here is characteristic for the dynamics of the gauge fields close to the boundaries of the lattice. In particular, its relation to the bulk correlation length is not obvious. To make contact with the physical scales, some extra work will therefore be needed (cf. sect. 4).

As a reference energy scale in infinite volume, we shall take the string tension K which is defined by

$$K = \lim_{r \to \infty} F(r). \tag{2.18}$$

There is no fundamental reason for choosing this particular quantity. Alternatives would be the energy splitting between the 1P and 1S (quenched) charmonium levels [4], or the distance r at which $r^2F(r) = 5$, for example. (The force between physical heavy quarks is known to be approximately equal to 1 GeV/fm at $r = 1$ fm so that $r^2F(r) = 5$ at this point.)

2.6. RENORMALIZATION GROUP *

We again assume that the continuum limit has been taken and define the Callan–Symanzik β-function through

$$\beta(\bar{g}) = -L\frac{\partial \bar{g}}{\partial L}. \tag{2.19}$$

From eq. (2.15) and the known perturbation expansion of the β-function in the $\overline{\mathrm{MS}}$ scheme, we infer that

$$\beta(g) \underset{g \to 0}{\sim} -g^3 \sum_{n=0}^{\infty} b_n g^{2n} \tag{2.20}$$

with [9–12]

$$b_0 = \tfrac{11}{3}(8\pi^2)^{-1}, \qquad b_1 = \tfrac{34}{3}(8\pi^2)^{-2}. \tag{2.21}$$

* Our notation here is slightly different from the one employed in ref. [2]. The reason for this is that the conventional normalization of the Λ-parameter is unfortunately not the same in gauge theories and two-dimensional non-linear σ-models.

The 3-loop coefficient b_2 depends on our choice of running coupling and is presently not available.

When integrated towards short distances, eq. (2.19) yields the asymptotic expression

$$\bar{g}^2(L) = \frac{1}{b_0 t} - \frac{b_1 \ln t}{b_0^3 t^2} + O\big(t^{-3}(\ln t)^2\big), \quad t = -\ln(\Lambda L)^2, \tag{2.22}$$

where Λ is an integration constant, the Λ-parameter.

In the following a key role is played by the step scaling function $\sigma(s, u)$. For any given scale factor s and initial value $u = \bar{g}^2(L)$, the coupling $u' = \bar{g}^2(sL)$ may be computed by integrating the renormalization group equation (2.19) (assuming the β-function is known). u' is a well-determined function of s and u, and so we may define

$$\sigma(s, u) = u'. \tag{2.23}$$

In other words, the step scaling function is an integrated form of the β-function, which tells us what happens to the coupling if the box size is changed by a factor s.

It is possible to calculate the step scaling function through numerical simulation of the lattice theory. To this end one chooses some value for the bare coupling g_0^2 and simulates two lattices with size L and $L' = sL$. The coupling $u' = \bar{g}^2(L')$ then provides an approximation to $\sigma(s, u)$ at $u = \bar{g}^2(L)$.

To understand how good the approximation is, we first note that for all lattice spacings a functional dependence

$$u' = \Sigma(s, u, a/L) \tag{2.24}$$

exists, which is obtained by eliminating g_0^2 in favour of u. From the discussion of the cutoff dependence of the effective action in ref. [3], we expect that $\Sigma(s, u, a/L)$ converges to $\sigma(s, u)$ in the continuum limit $a \to 0$ with a rate roughly proportional to a/L.

A more quantitative impression on the size of the cutoff effects may be obtained in perturbation theory. In particular, the relative deviation

$$\delta(u, a/L) = \frac{\Sigma(2, u, a/L) - \sigma(2, u)}{\sigma(2, u)} = \delta_1(a/L)u + \delta_2(a/L)u^2 + \ldots, \tag{2.25}$$

TABLE 1
One-loop coefficient $\delta_1(a/L)$ in the expansion (2.25)

L/a	δ_1	L/a	δ_1
6	0.00623	12	0.00396
8	0.00540	14	0.00347
10	0.00459	16	0.00309

turns out to be quite small at one-loop order (see table 1). It will nevertheless be necessary to extrapolate the numerical data to the continuum limit by simulating a sequence of lattice pairs with decreasing lattice spacings (and fixed coupling u).

2.7. STRATEGY

Our principal aim is to compute $\bar{g}^2(L)$ for a range of box sizes L connecting the low-energy domain with the perturbative scaling region. This is achieved by setting up a recursion

$$u_{i+1} = \sigma(s_i, u_i), \quad i = 0, 1, 2, \ldots, \tag{2.26}$$

starting from some initial value u_0 of the running coupling. By construction we have

$$u_i = \bar{g}^2(L_i), \qquad L_i = L_0 \prod_{j=0}^{i-1} s_j, \tag{2.27}$$

for some box size L_0. Our choice of initial value u_0 and scale factors s_i will be such that the recursion progresses from the perturbative small coupling regime towards larger box sizes.

Of course the statistical and extrapolation errors limit the number of iterations that can be done in practice. A careful discussion of the error propagation is certainly necessary, and we shall come back to this issue when we analyse our data in sect. 4.

After a certain number n of iterations, depending on the initial value u_0, the final box size L_n will be close to or even larger than the scale set by the string tension K. At this point it is possible to determine the dimensionless combination $L_n\sqrt{K}$. As a result one has calculated the running coupling at all L_i given in units of the string tension.

At the lower end of the range of L covered, where the coupling is small, we may finally apply perturbation theory to determine the value of any other coupling such as $\alpha_{\overline{\mathrm{MS}}}$ or $\alpha_{\mathrm{q\bar{q}}}$ (cf. subsect. 2.4). Note that all reference to a finite volume drops out in this last step. One simply gets $\alpha_{\overline{\mathrm{MS}}}$ at some large momenta q_i given in units of the string tension.

3. Numerical simulation

For fixed boundary values W and W', the system defined through the partition function (2.4) can be simulated by adapting any one of the known Monte Carlo algorithms for pure gauge theories on periodic lattices. The simulations that we

have performed are here described in some detail and a complete list of our data on the running coupling is given. These will be further analysed in sect. 4.

3.1. OBSERVABLES

From the definition (2.12) of the running coupling it follows that

$$\bar{g}^2(L) = k\langle \partial S/\partial \eta \rangle^{-1}. \tag{3.1}$$

The field variables integrated over do not depend on the background field parameter η and so only the plaquettes touching the boundary contribute to $\partial S/\partial \eta$. The observable we shall be concerned with in the following is thus given by

$$\frac{\partial S}{\partial \eta} = -\frac{2a^3}{g_0^2 L} \sum_{\mathbf{x}} \sum_{l=1}^{3} \{E_l'(\mathbf{x}) + E_l(\mathbf{x})\}, \tag{3.2}$$

where E_l' and E_l denote the τ_3 component of the colour-electric field at the top and bottom of the lattice. In particular,

$$E_l(\mathbf{x}) = \frac{1}{ia^2}\mathrm{tr}\Big\{\tau_3 W(\mathbf{x}, l)U(x + a\hat{l}, 0)U(x + a\hat{0}, l)^{-1}U(x, 0)^{-1}\Big\}_{x^0=0}, \tag{3.3}$$

and a similar expression is obtained for E_l'.

It is conceivable that other observables exist which have the same expectation value as $\partial S/\partial \eta$ but a significantly smaller variance. We were, however, not successful in our search for such an "improved" observable. In particular, the so-called multi-hit method [13], when applied to the dynamical link variables in eq. (3.3), did not result in any appreciable increase in efficiency. The data listed below have thus been generated taking $\partial S/\partial \eta$ as the observable.

3.2. MONTE CARLO ALGORITHM

The most efficient simulation algorithms for pure gauge theories known today involve the idea of over-relaxation in one form or the other [14–16]. Our algorithm is a hybrid one, with N exactly microcanonical sweeps through the lattice followed by 1 heatbath update pass. Geometrically the program is organized in time slices which are visited sequentially. At any given time, the lattice is divided into 2 or 4 sublattices, depending on whether L/a is even or odd. This is done in such a way that the link variables in a fixed sublattice are decoupled and so can be updated in a vector mode.

For the heatbath part of the cycle, a modified Creutz algorithm [17] was implemented. The local heatbath SU(2) measure cannot be generated exactly but

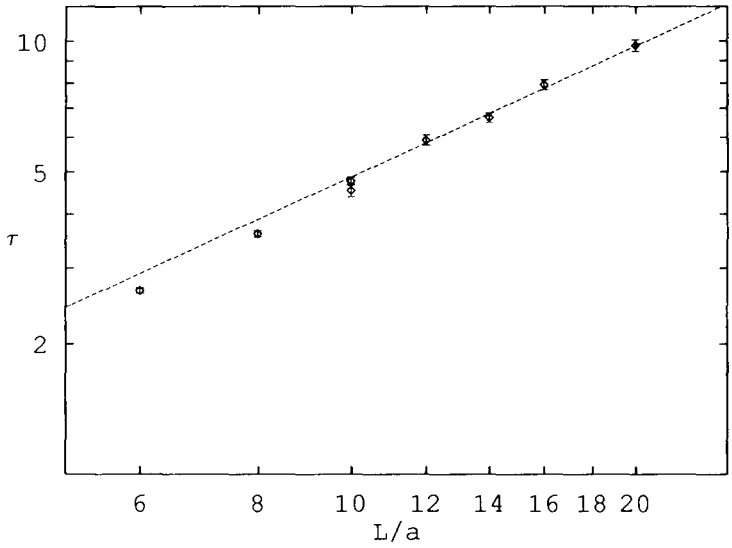

Fig. 1. Integrated autocorrelation times τ for the observable $\partial S/\partial\eta$, given in numbers of update sweeps through the lattice (counting overrelaxation and heatbath sweeps).

requires an accept/reject step. The observed acceptance rate increases towards smaller couplings and was always greater than 90%.

The parameter N generally has to grow as one approaches the continuum limit. In various models a tuning roughly inversely proportional to the lattice spacing proves to be optimal [18,19]. Some of the integrated autocorrelation times achieved in this way are shown in fig. 1. All points refer to approximately the same renormalized coupling $\bar{g}^2(L) \simeq 3.7$, i.e. the measurements are made at "constant physics". At large L/a the autocorrelation time is proportional to $(L/a)^z$ (dashed line). The fit gives $z = 1.0(1)$ for the dynamical critical exponent, which coincides with the expected best value for over-relaxed algorithms.

3.3. SIMULATION RESULTS

As explained in subsect. 2.6, the step scaling function $\sigma(s, u)$ may be computed by simulating pairs of lattices at the same bare coupling with sizes L and $L' = sL$. In our calculations we chose $s = 2$ throughout and produced enough data to be able to perform 4 renormalization steps in succession (see table 2). We have, furthermore, done an additional set of simulations at a fixed large value of the renormalized coupling (table 3). These data will be used in sect. 4 to make contact with the low-energy scales of the theory.

In each block of data listed in table 2, the renormalized coupling on the smaller lattices is constant within errors, i.e. the bare coupling has been tuned so as to

TABLE 2
Pairs of running couplings at fixed bare coupling $\beta = 4/g_0^2$

β	L/a	$\bar{g}^2(L)$	$\bar{g}^2(2L)$
3.4564	5	2.0371(32)	2.413(15)
3.5408	6	2.0369(52)	2.418(16)
3.6045	7	2.0370(55)	2.397(19)
3.6566	8	2.0370(63)	2.447(17)
3.7425	10	2.0369(83)	2.426(22)
3.1898	5	2.3800(43)	2.981(23)
3.2751	6	2.3801(67)	2.942(21)
3.3428	7	2.3799(67)	2.968(26)
3.4009	8	2.3801(79)	2.954(23)
3.5000	10	2.380(11)	2.870(30)
2.9568	5	2.8401(56)	3.783(33)
3.0379	6	2.8401(91)	3.731(35)
3.0961	7	2.840(10)	3.709(31)
3.1564	8	2.840(11)	3.663(34)
3.2433	10	2.841(16)	3.695(43)
2.7124	5	3.550(10)	5.456(40)
2.7938	6	3.550(14)	5.287(43)
2.8598	7	3.550(15)	5.310(58)
2.9115	8	3.550(16)	5.168(38)
3.0071	10	3.550(23)	5.122(58)

achieve this. A reweighting technique was employed in this step, as discussed in subsect. 4.1 of ref. [2].

The statistical errors were estimated both by jackknife binning (using several 100 bins) and, when no reweighting was required, by summing the autocorrelation function over an appropriate time interval. In the first three series of table 2 about 35 hours of CPU time were spent for the lattices with $L'/a \leqslant 14$, 60 hours for $L'/a = 16$ and 140 hours for the largest system with $L'/a = 20$. All times were roughly doubled in the last series (where $\bar{g}^2(L) = 3.55$). The typical speed achieved by our program is 210 Mflop/s. Speed and CPU times refer to a single CRAY YMP processor.

TABLE 3
Bare coupling vs. lattice size at fixed $\bar{g}^2(L) = 4.765$

L/a	β	L/a	β
6	2.5752(28)	10	2.7824(22)
7	2.6376(20)	12	2.8485(32)
8	2.6957(21)	14	2.9102(62)

4. Data analysis and results

We now follow the strategy sketched in subsect. 2.7 and determine the running coupling in the continuum theory for a range of box sizes L given in physical units.

4.1. STEP SCALING FUNCTION

The simulation results listed above allow us to compute the step scaling function $\sigma(2, u)$ at 4 values of u. This involves an extrapolation of the lattice data to the continuum limit $a/L \to 0$ (see fig. 2). The cutoff effects that we observe are quite small and linearly decreasing with the lattice spacing. This is the theoretically expected behaviour. At the lower values of the coupling, the size of the effect is compatible with what one predicts from perturbation theory (cf. subsect. 2.6). For a detailed comparison a higher statistical precision would however be required.

To perform the extrapolation to the continuum limit, the error on the argument u has been traded for an additional error on Σ, using an approximate value for $\partial\Sigma/\partial u$. A straightforward linear fit then yields the values and errors quoted in

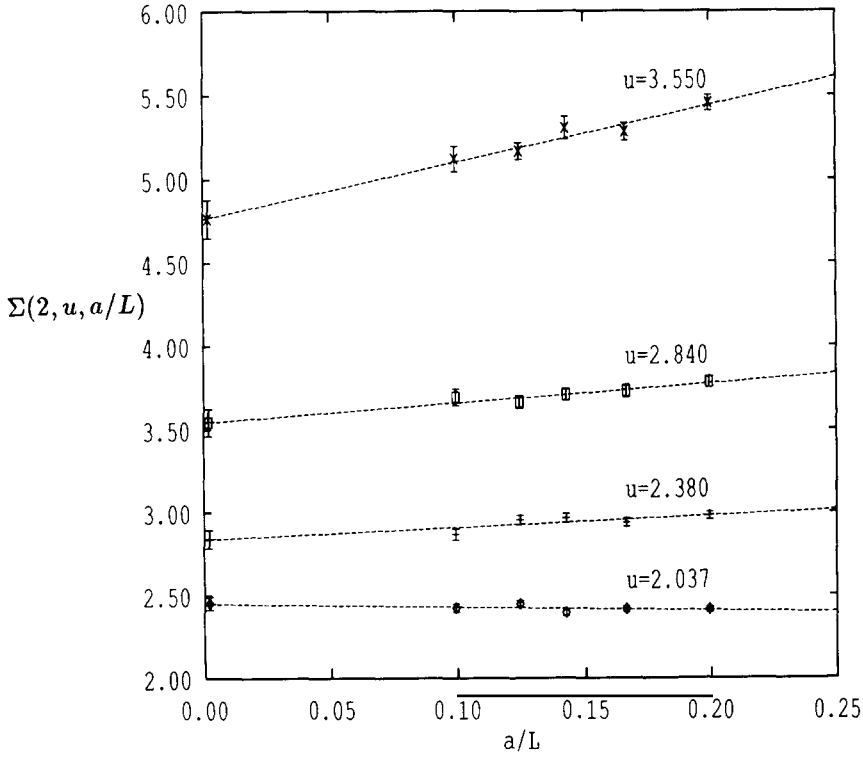

Fig. 2. Extrapolation of the lattice step scaling function $\Sigma(2, u, a/L)$ to the continuum limit. The left-most points represent the extrapolated values as given in table 4.

TABLE 4
Values of the step scaling function

u	$\sigma(2, u)$	$\sigma(2, u)_{2\text{-loop}}$
2.037	2.45(4)	2.38
2.380	2.84(6)	2.86
2.840	3.54(8)	3.58
3.550	4.76(12)	4.83

table 4. These can be compared with what one obtains by integrating the renormalization group equation (2.19), taking u as the initial value and the 2-loop formula for the Callan–Symanzik β-function (third column in table 4). The agreement is perfect, except for a 2σ deviation at the lowest value of u. In a set of 4 independent measurements, this is not an unlikely event, however.

We shall soon discover that the couplings occurring in table 4 correspond to a range of box sizes L from about 0.023 fm to 0.33 fm. It is thus rather surprising that the step scaling function is so accurately reproduced by perturbation theory.

4.2. RUNNING COUPLING

For the error analysis it proves useful to set up the recursion (2.26) in a logically reversed manner, where one first specifies the sequence of couplings u_i, $i = 0, 1, \ldots$, and then computes the associated scale factors s_i. The couplings are defined by

$$\begin{aligned} u_0 &= 2.037, \qquad u_1 = \sigma(2, u_0), \\ u_2 &= 2.380, \qquad u_3 = \sigma(2, u_2), \\ u_4 &= 2.840, \qquad u_5 = \sigma(2, u_4), \\ u_6 &= 3.550, \qquad u_7 = \sigma(2, u_6), \\ u_8 &= 4.765. \end{aligned} \tag{4.1}$$

u_0, u_2, u_4 and u_6 coincide with the numbers listed in the first column of table 4. To the precision stated, u_1, u_3, u_5 and u_7 are thus given by the second column in this table.

By definition, the scale factors s_i (as determined through eq. (2.26)) are exactly equal to 2 for $i = 0$, 2, 4 and 6. In all other cases, s_i is a number close to 1, which may be computed by evolving the coupling from u_i to u_{i+1} using the 2-loop formula for the β-function. Since the step scaling function is well reproduced by perturbation theory, we estimate that the systematic errors incurring at this point

TABLE 5
Running coupling at scales given in units of L_8

L/L_8	$\bar{g}^2(L)$
1.000	4.765
0.500(23)	3.550
0.249(19)	2.840
0.124(13)	2.380
0.070(8)	2.037

are negligible compared to the statistical errors in table 4 (which translate to errors on s_1, s_3, s_5 and s_7).

The box sizes L_i at which $\bar{g}^2(L_i) = u_i$ can now be computed straightforwardly through eq. (2.27). The result is given in table 5 and will be discussed below, after converting to more physical units. In this computation the errors on the scale factors s_i have been added in quadrature, because they arise from independent simulations.

4.3. PHYSICAL SCALES AND COMPUTATION OF $\alpha_{\overline{\rm MS}}(q)$

At the largest coupling in the recursion (4.1), it is possible to make contact with the low-energy scales of the theory in infinite volume. As already mentioned in sect. 2, we decided to take the string tension K as a reference energy scale in this regime. K has been determined on large lattices at $\beta = 2.70$ [20] and more recently at $\beta = 2.85$ [21] (second column of table 6; the errors quoted there are statistical only) *.

To determine the dimensionless combination $L_8\sqrt{K}$, we also need L_8 in lattice units at the same values of the bare coupling. This information can be extracted easily by interpolating the data listed in table 3 (see fig. 3). The outcome of the calculation is given in the third and fourth columns of table 6.

The differences between the values of $L_8\sqrt{K}$ obtained at $\beta = 2.70$ and $\beta = 2.85$ can tentatively be interpreted as a cutoff effect. In principle, $L_8\sqrt{K}$ should be extrapolated to the continuum limit in the same way as the step scaling function. We, however, are hesitating to do this, because we have only two data points and since it is not certain that the observed variation of $L_8\sqrt{K}$ is a pure cutoff effect. In particular, systematic errors on the string tension values quoted in table 6 of the order of 5–10% cannot be excluded at present [22].

In the following we take $L_8\sqrt{K} = 0.713$ and keep in mind that the total error on this number could be as large as 10%. If we set $\sqrt{K} = 425$ MeV to convert to more

* The numbers published in refs. [20,21] are actually higher by a factor of 1.09 and 1.24, respectively. The string tension has meanwhile been reevaluated, the results being as given here. We thank C. Michael and the UKQCD collaboration for communicating these revisions to us.

TABLE 6
Values of the string tension and the box size L_8

β	a^2K	L_8/a	$L_8\sqrt{K}$
2.70	0.0103(2)	8.08(4)	0.820(9)
2.85	0.00354(26)	11.98(7)	0.713(26)

physical units, we then deduce that $L_8 = 0.33$ fm. The lower end of the range of box sizes covered by table 5 is hence roughly equal to 0.023 fm.

As shown in fig. 4 the evolution of the running coupling $\alpha(q)$ is well described by perturbation theory, down to very low energies. The error bars in this plot only represent the statistical errors as given in table 5, but not the overall scale uncertainty discussed above. It should be emphasized that the latter amounts to a multiplication of the energy scale by a *constant* factor and so has no bearing on the scaling properties of the coupling.

At the highest energies reached, we can finally convert to the $\overline{\text{MS}}$ scheme of dimensional regularization using perturbation theory. A typical result is

$$\alpha_{\overline{\text{MS}}}(q) = 0.187 \pm 0.005 \pm 0.009 \quad \text{at} \quad q = 20 \times \sqrt{K}\,. \tag{4.2}$$

The first error here is statistical, as inferred from table 5, while the second is an estimate of the total systematic error arising from a possible order α^3 correction in eq. (2.15) and the 10% scale uncertainty mentioned above.

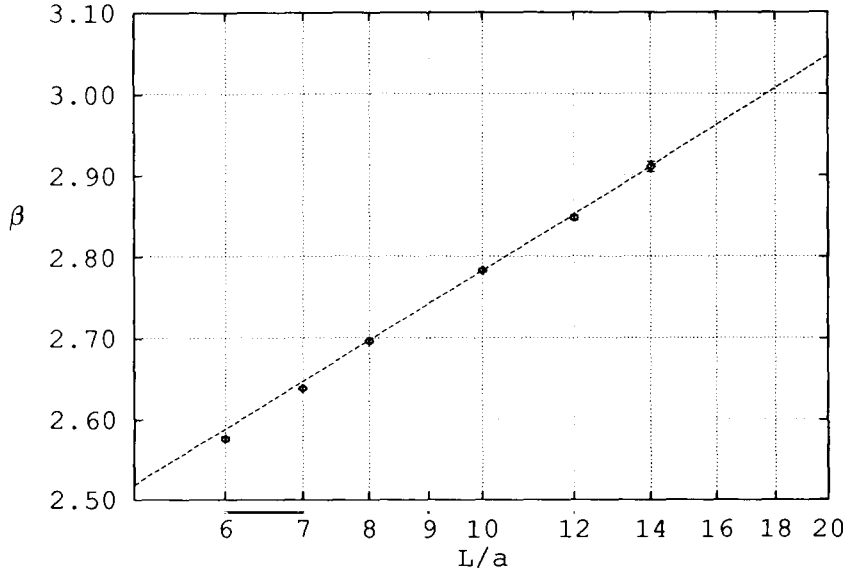

Fig. 3. Bare coupling $\beta = 4/g_0^2$ versus lattice size at fixed $\bar{g}^2(L) = 4.765$. The dashed curve is a fit, $\beta = 1.905 + 0.380\ \ln(L/a)$, to the data points with $L/a \geqslant 8$.

4.4. RELATION BETWEEN THE BARE AND THE RENORMALIZED COUPLING

Let us consider a large lattice, with bare coupling g_0^2 deep in the scaling region, and let us assume that the lattice spacing a is known in units of some low-energy scale. In the *continuum* theory, the running coupling $\bar{g}^2(a)$ then is a well-determined quantity, which may be related to the lattice coupling through an asymptotic series,

$$\bar{g}^2(a) = g_0^2 + c_1 g_0^4 + c_2 g_0^6 + \ldots, \tag{4.3}$$

with purely numerical coefficients. At present the 1-loop coefficient is known [3],

$$c_1 = 0.20235, \tag{4.4}$$

and an effort is being made to extend the calculation to the next order [28]. Whether the expansion applies in the accessible range of bare couplings (at $\beta = 2.85$ for example) is not known, however, and one may in fact have serious doubts that it does, because the first order correction is uncomfortably large.

Other renormalized couplings fare no better in this respect and one is thus led to suspect that g_0^2 is a "bad" expansion parameter [23–26]. On the basis of a mean field argument, Parisi [23] suggested many years ago that

$$\tilde{g}_0^2 = g_0^2/P, \qquad P = \tfrac{1}{2}\langle \mathrm{tr}\ U(p)\rangle, \tag{4.5}$$

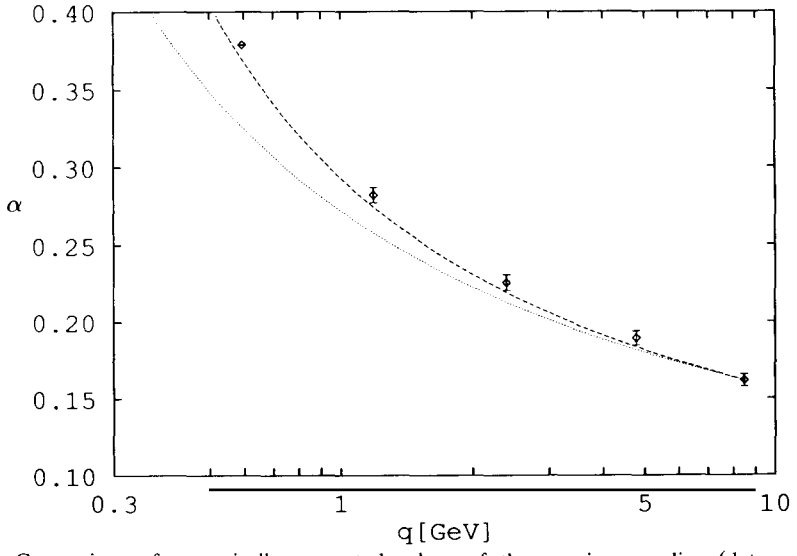

Fig. 4. Comparison of numerically computed values of the running coupling (data points) with perturbation theory. The dashed (dotted) curve is obtained by integrating the evolution equation (2.19), starting at the right-most point and using the 2-loop (1-loop) formula for the β-function.

would be a more natural choice of bare parameter for the lattice theory. The plaquette expectation value P is to be computed on an infinite lattice. At $\beta = 2.70$ and $\beta = 2.85$, it is equal to 0.68558 and 0.70577, respectively [27]. In terms of $\tilde{g}_0^2$, eq. (4.3) becomes

$$\bar{g}^2(a) = \tilde{g}_0^2 + \tilde{c}_1\tilde{g}_0^4 + \tilde{c}_2\tilde{g}_0^6 + \ldots, \tag{4.6}$$

with

$$\tilde{c}_1 = c_1 - \tfrac{3}{16} = 0.01485. \tag{4.7}$$

The corresponding expansion of the renormalized coupling in the $\overline{\text{MS}}$ scheme of dimensional regularization plays a key rôle in the work of El-Khadra et al. [4] and is further discussed in ref. [26].

While the 1-loop coefficient $\tilde{c}_1$ is much smaller than c_1, it is not guaranteed that the higher-order corrections are small and so it remains unclear whether the series (4.6) yields a reliable estimate for the renormalized coupling. Using the results obtained above we are now in a position to answer this question. At $\beta = 2.85$, for example, the lattice spacing in units of L_8 is equal to 0.0835(5), which is in the range covered by table 5. For the running coupling, the value $\bar{g}^2(a) = 2.11(5)$ is thus obtained. This is to be compared with the r.h.s. of eq. (4.6), which evaluates to $\bar{g}^2(a) = 2.05$ at 1-loop order. So we do confirm that the higher-order corrections are small and thus conclude that $\tilde{g}_0^2$ is a good expansion parameter at the scale of the cutoff.

5. Conclusions

Lattice gauge theories have been invented to study the properties of Yang–Mills theories and QCD at low energies. It proved to be difficult, however, to make contact with the perturbative regime, where weakly interacting quarks and gluons are the important degrees of freedom. The obstacle is that one cannot easily hold a wide range of physical scales on a single lattice, at least as long as numerical simulations are the only practical way to do non-perturbative computations in the scaling region.

Using a recursive finite-size technique, we have now been able to close this gap in the case of the pure SU(2) gauge theory. We found that the evolution of the renormalized coupling in the chosen scheme is well described by perturbation theory over the whole range of energies covered (cf. fig. 4). This is a bit surprising, but it should be noted that the coupling is defined through an off-shell amplitude and so is insensitive to threshold effects. In any case, our result proves that there is no complicated "intermediate" energy range before the coupling becomes small

and slowly decreasing according to the perturbative renormalization group. In this respect the situation is as in the two-dimensional non-linear σ-model [2].

Our method allows us to compute the running coupling in say the $\overline{\mathrm{MS}}$ scheme of dimensional regularization at energies far above the masses of the light particles in the theory. An example of such a result is given in eq. (4.2). It is certainly possible to achieve a higher precision in this calculation. Compared to the power of present day parallel computers, we have used an only small amount of CPU time. A refined study, with more statistics and an $O(a)$ improved action [3], is hence clearly feasible. It is then also necessary to extend the series (2.15) to the 2-loop level [28] to keep the balance between systematic and statistical errors.

We finally note that an extension of our work to the SU(3) Yang–Mills theory should not meet any fundamental difficulty. QCD requires more thought, however, because the scale dependence of the quark masses must be taken into account.

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