Journal of Geometry and Physics 11 (1993) 337-348
North-Holland

JOURNAL OF
GEOMETRY AND
PHYSICS

Superselection sectors in low dimensional quantum field theory

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The occurrence of braid group statistics in low dimensions is reviewed and an extension of the general theory to solitonic sectors is described where the DHR endomorphisms are replaced by homomorphisms between different extensions of the algebra of observables.

1. Introduction

One of the characteristic features of quantum field theory is the decomposition of the space of physical states into superselection sectors between which relative phases cannot be observed. A typical example is the theory of a Majorana field ψ where the physical Hilbert space is generated from the vacuum by polynomials of ψ. The field ψ itself is not observable due to its anticommutativity at spacelike distances (and/or its nontrivial transformation property under full rotations). Only even functions of ψ can be observable. Hence the Hilbert space decomposes into a direct sum of the even and the odd part, and matrix elements of observables between even and odd states vanish identically.

A similar situation may occur for a Hermitian scalar field ϕ provided only even functions of ϕ are observable. However, here the vacuum is not necessarily invariant under the symmetry $\phi \mapsto -\phi$. If it is noninvariant, i.e. the symmetry is spontaneously broken, ϕ can be approximated by even functions of ϕ, and the Hilbert space generated by the even functions coincides with the Hilbert space generated by all polynomials.

The general structure of superselection sectors can best be analyzed in the framework of algebraic quantum field theory [1]. There one starts from a family of von Neumann algebras $\mathcal{A} = \mathcal{A}(\mathcal{O})_{\mathcal{O}\in\mathcal{K}}$ where $\mathcal{K}$ denotes the set of double cones in Minkowski space. $\mathcal{A}(\mathcal{O})$ is interpreted as the algebra generated by all observables which can be measured within the space–time region $\mathcal{O}$. In models, one typically constructs this algebra in terms of all observable Wightman fields $\phi(x)$ with $x \in \mathcal{O}$. There are some technical problems in this construction which I will not discuss (see ref. [2] and references therein). $\mathcal{A}$ is supposed to satisfy the Haag–Kastler axioms [3].

$$\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \qquad \text{(isotony)}, \tag{1}$$

0393-0440/93/$ 06.00

$$\mathcal{O}_1 \subset \mathcal{O}_2' \Rightarrow \mathcal{A}(O_1) \subset \mathcal{A}(O_2)' \qquad \text{(locality)} \tag{2}$$

($\mathcal{O}_2'$ is the spacelike complement of $\mathcal{O}_2$ and $\mathcal{A}(\mathcal{O}_2)'$ is the commutant of $\mathcal{A}(\mathcal{O}_2)$.) Translations a are represented by automorphisms of $\mathcal{A}$ such that

$$\alpha_a(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O} + a) \qquad \text{(covariance).} \tag{3}$$

For the purposes of particle physics one is mainly interested in positive energy representations (PER) of $\mathcal{A}$ [4]. These are representations π of $\mathcal{A}$ in some Hilbert space $\mathcal{H}_\pi$ together with a strongly continuous representation U of the translation group such that $\mathrm{Ad}\, U(a) \circ \pi = \pi \circ \alpha_a$, and the spectrum of U is contained in the closed forward light cone $\overline{V}_+$. A vacuum representation is a PER together with a unique (up to a phase) translation invariant unit vector Ω, and a one particle representation is a PER where the spectrum of U contains an isolated mass hyperboloid.

The investigation of PER's was initiated by Borchers. In his analysis localization properties of representations turned out to play an important role. This point of view was fully developed by Doplicher, Haag and Roberts [5,6] who analyzed representations π of $\mathcal{A}$ which were local excitations of some vacuum representation π_0 in the following sense:

$$\pi|_{\mathcal{A}(O')} \simeq \pi_0|_{\mathcal{A}(\mathcal{O}')} \quad \forall \mathcal{O} \in \mathcal{K} \qquad \text{(DHR criterion)} \tag{4}$$

Here $\mathcal{A}(\mathcal{O}')$ denotes the C*-subalgebra of $\mathcal{A}$ generated by all algebras $\mathcal{A}(\mathcal{O}_1)$ with $\mathcal{O}' \supset \mathcal{O}_1 \in \mathcal{K}$, and $\simeq$ means unitary equivalence.

In the analysis a maximality property of local algebras in the vacuum representation first proposed in ref. [7] and verified by Araki [8,9] for the free field was used in a crucial way:

$$\pi_0(\mathcal{A}(\mathcal{O}'))' = \pi_0(\mathcal{A}(\mathcal{O})) \qquad \text{(Haag duality).} \tag{5}$$

Roberts later showed [10] (see also ref. [11]) that Haag duality is violated in the case of spontaneous breakdown of symmetry. A weaker property, essential duality, holds under more general circumstances due to results of Bisognano and Wichmann [12], and leads, in more than two space–time dimensions to the same structure of DHR superselection sectors.

A DHR representation π in general does not satisfy Haag duality [5]. The deviation from Haag duality can be measured by a number $d(\pi) \in [1, \infty]$ called statistical dimension. $d(\pi)$ equals one if and only if π satisfies Haag duality. As Longo recently showed [13], $d(\pi)^2$ can be interpreted as the Jones index [14] of the inclusion $\pi(\mathcal{A}(\mathcal{O}'))' \supset \pi(\mathcal{A}(\mathcal{O}))$ (which is independent of $\mathcal{O} \in \mathcal{K}$). The class of DHR representations with finite statistical dimensions in $D \geq 3$ dimensional space–time is now completely understood by the work of Doplicher and Roberts [15] . There are always a compact group G (the group of internal symmetries) and embeddings of the local algebras $\mathcal{A}(\mathcal{O})$ into algebras $\mathcal{F}(\mathcal{O}) \subset \mathcal{F}$ on which G acts by automorphisms such that $\mathcal{A}(\mathcal{O})$ is the set of fixed points in $\mathcal{F}(\mathcal{O})$. The algebra $\mathcal{F}(\mathcal{O})$ has a $\mathbb{Z}_2$ grading related to the presence of Fermi fields such that the system of algebras $\mathcal{F}(\mathcal{O})_{\mathcal{O}\in K}$ satisfies the Haag–Kastler axioms with a graded version of locality. The restriction of the vacuum representation of $\mathcal{F}$ to $\mathcal{A}$ is a direct sum of irreducible DHR representations with multiplicities given by the statistical dimensions which at the same time are dimensions of irreducible G-modules in $\mathcal{F}$ whose elements induce transitions to the vacuum sector.

The proposal of Borchers to derive localization properties from the spectrum condition was further pursued in the work of Buchholz and myself [16]. We could show that one particle representations automatically satisfy a version of the DHR criterion (4) where the double cone $\mathcal{O}$ is replaced by an infinitely extended spacelike cone. Moreover, it turned out that one particle representations always have finite statistical dimensions [17]. The DHR analysis could be generalized to this class of representations and yielded the same structure in $D \geq 4$ dimensions [16,15].

It was known since long time that in low dimensions the structure of sectors is more complicated, see e.g. ref. [18]. But only after the progress in the representation theory of the braid group due to Jones [14] and the discovery of new interesting models of conformal field theory in two dimensions by Belavin, Polyakov and Zamolodchikov [19] which exhibit a rich superselection structure, a detailed analysis of superselection sectors in low dimensional space–time was performed, mainly by Buchholz, Mack and Todorov [20], by Fröhlich, Gabbiani and Marchetti [21–24], by Longo [13] and by Rehren, Schroer and myself [25,26]. In this note I want to treat the following cases: DHR sectors in 2D (sectoin 2), sectors with localization in spacelike cones in 3d (section 3) and soliton sectors in 2d (section 4). The first two items were already discussed in detail in the literature, but the last item is essentially new. It goes back to a proposal which I made in [27]. Earlier observations of Fröhlich [28] and of Schroer and Swieca [29] on models and a proposal of Fröhlich abstracted from models turn out to be special cases of the general framework which is presented here.

2. DHR sectors in two dimensions

Let π_0 be a fixed vacuum representation satisfying Haag duality, and let π be a representation which fulfils the DHR criterion. Choose some $\mathcal{O}_0 \in \mathcal{K}$. From (4) there is a unitary $V: \mathcal{H}_{\pi_0} \mapsto \mathcal{H}_\pi$ such that

$$V\pi_0(A) = \pi(A)\, V \tag{6}$$

for $A \in \mathcal{A}(\mathcal{O}_0')$. Then one can define the representation $\pi_V = \mathrm{Ad}\, V^{-1} \circ \pi$ in $\mathcal{H}_{\pi_0}$. The representation π_V is unitarily equivalent to π and coincides with π_0 on $\mathcal{A}(\mathcal{O}_0')$. Now let $\mathcal{O}_1 \in \mathcal{K}$ be arbitrary. There is some $\mathcal{O} \in \mathcal{K}$ which contains both $\mathcal{O}_0$ and $\mathcal{O}_1$. Using isotony, locality and the coincidence of π_V and π_0 on $\mathcal{A}(\mathcal{O}') \subset \mathcal{A}(\mathcal{O}'_0)$ we obtain

$$\pi_V(\mathcal{A}(\mathcal{O}_1)) \subset \pi_V(\mathcal{A}(\mathcal{O})) \subset \pi_V(\mathcal{A}(\mathcal{O}'))' = \pi_0(\mathcal{A}(\mathcal{O}'))'. \tag{7}$$

Using Haag duality we finally get

$$\pi_V(\mathcal{A}(\mathcal{O}_1)) \subset \pi_0(\mathcal{A}(\mathcal{O})), \tag{8}$$

hence the algebra $\pi_V(\mathcal{A})$ is a subalgebra of $\pi_0(\mathcal{A})$. Since by (4) the kernels of π and π_0 coincide we may without loss of generality assume that π_0 is faithful. We then consider the endomorphism $\rho = \pi_0^{-1} \circ \pi_V$ of $\mathcal{A}$. ρ acts trivially on $\mathcal{A}(\mathcal{O}'_0)$. We refer to this fact by saying that ρ is localized in $\mathcal{O}_0$. We may identify $\mathcal{A}$ by its image in the vacuum representation and consider the endomorphism ρ as a representation. Now let ρ and ρ' be endomorphisms localized in $\mathcal{O} \in \mathcal{K}$ which are unitarily equivalent as representations. Then, by Haag duality, the corresponding unitary is an element of $\mathcal{A}(\mathcal{O})$, and the endomorphisms are related by an inner automorphism. The latter fact implies that the multiplication of endomorphisms induces a composition law of equivalence classes of representations. It also follows that endomorphisms localized in spacelike separated regions commute so that the composition law is commutative on the level of equivalence classes. This is the basis for an intrinsic definition of statistics.

Let $\rho_i = \mathrm{Ad}\, U_i \circ \rho$, $i = 1, 2$ be localized in mutually spacelike regions $\mathcal{O}_i$, $i = 1, 2$. Then

$$\rho^2 \simeq \rho_1\rho_2 = \rho_2\rho_1 \simeq \rho^2 \tag{9}$$

with a unitary intertwiner

$$\varepsilon = \rho(U_1^{-1})\, U_2^{-1} U_1 \rho(U_2) \in \mathcal{A}(\mathcal{O}_0). \tag{10}$$

ε is called the statistics operator. ε turns out to depend only on the regions $\mathcal{O}_1$ and $\mathcal{O}_2$, not on the choice of intertwiners U_1 and U_2. Moreover, it is locally constant under changes of $\mathcal{O}_1$ and $\mathcal{O}_2$ provided the regions remain spacelike to each other. Hence in two dimensions ε can assume at most two values. It satisfies the relations

$$\varepsilon\rho^2(A) = \rho^2(A)\,\varepsilon, \quad A \in \mathcal{A} \tag{11}$$

(this is nothing than the intertwining property of ε) and

$$\varepsilon\rho(\varepsilon)\,\varepsilon = \rho(\varepsilon)\,\varepsilon\rho(\varepsilon) \tag{12}$$

which follows by the following calculation (we choose $U_1 = 1$ and $U_2 = U$ in the definition of ε):

$$\varepsilon\rho(\varepsilon) = U^{-1}\rho^2(U) \tag{13}$$

hence

$$\varepsilon\rho(\varepsilon)\,\varepsilon = U^{-1}\rho^2(U)\varepsilon = U^{-1}\varepsilon\rho^2(U) \tag{14}$$

by (11) and

$$\rho(\varepsilon)\varepsilon\rho(\varepsilon) = \rho(\varepsilon)U^{-1}\rho^2(U) = U^{-1}\,\mathrm{Ad}\, U \circ \rho(\varepsilon)\rho^2(U) \tag{15}$$

hence (12) follows from the fact that $\mathrm{Ad}\, U \circ \rho$ is localized spacelike to $\mathcal{O}_0$ and therefore acts trivially on $\varepsilon \in \mathcal{A}(\mathcal{O}_0)$. The relations (11) and (12) now lead, by setting

$$\varepsilon^{(\rho)}(\sigma_i) = \rho^{i-1}(\varepsilon), \tag{16}$$

to a unitary representation of the braid group B_∞, i.e. the group generated by generators σ_i, $i \in \mathbb{N}$ with the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| \geq 2 \tag{17}$$

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \tag{18}$$

(the first relation being respected because of (11) and the second because of (12)).

The occurring braid group representation describes an anomalous statistics of particles. It may be analyzed in terms of a so-called left inverse of the endomorphism ρ. One always finds a Markov trace (in the sense of Jones [14]) on the braid group and associated link and ribbon invariants as well as invariants of three-manifolds. A general classification has not yet been reached, but many interesting examples were found. For details see refs. [26,13,24].

3. Braid group statistics in three dimensions

DHR sectors in three space–time dimensions always have permutation group statistics. The reason is that the statistics operator is globally constant since any pair of spacelike separated double cones can be continuously deformed into each other pair such that the two regions remain always spacelike separated. Then $\varepsilon^2 = 1$, hence $\varepsilon^{(\rho)}$ factors through a representation of the permutation group.

There is an apparent conflict with the observation in quantum mechanics [30] that braid group statistics may occur in two spatial dimensions. The solution of this conflict comes from the fact that a PER does not necessarily satisfy the DHR criterion.

As mentioned in the Introduction, a general localization property of irreducible one particle representations was found by Buchholz and myself. Extending an older result of Swieca [31] we showed that there always exists a dense subspace in the representation space of a one particle representation π such that for any unit vector Φ in that subspace the derivatives of expectation values

$$\partial_\mu(\Phi, \pi\alpha_a(A)\Phi) \tag{19}$$

are strongly decreasing for $|\boldsymbol{a}|-|a^0| \to \infty$ for all local observables A. It follows that the expectation value of A approaches a limit as a tends to spacelike infinity. The limit is independent of Φ and, in at least three space–time dimensions, independent of the direction in which one goes to spacelike infinity. Hence there is a state ω_0 (in the sense of expectation functionals on the algebra of observables)

$$\omega_0(A) = \lim_a (\Phi, \pi\alpha_a(A)\Phi), \tag{20}$$

which is translation invariant. By the GNS construction there is a Hilbert space $\mathcal{H}_0$, a representation π_0 and a cyclic vector $\Omega \in \mathcal{H}_0$ such that

$$\omega_0(A) = (\Omega, \pi_0(A)\Omega) \quad , \tag{21}$$

together with a unitary implementation of the translations

$$U_0(a)\pi_0(A)\Omega = \pi_0\alpha_a(A)\Omega \quad , \tag{22}$$

and one can show that U_0 is strongly continuous and satisfies the spectrum condition, hence π_0 is a vacuum representation which is uniquely associated to the one particle representation π.

In two space–time dimensions spacelike infinity is not connected, hence there may be different limits in (20) for right and left spacelike infinity leading to different vacuum representations π_+ and π_-, respectively. This case ("solitons") will be treated in the next section.

Returning to $D \geq 3$ space–time dimensions we may now ask whether local fields exist which interpolate between π_0 and π. One finds that for any spacelike cone

$$S = a + \bigcup_{\lambda>0} \lambda\mathcal{O}, \tag{23}$$

where $\mathcal{O}$ is a double cone spanned by two spacelike unit vectors with timelike difference the following modified DHR criterion holds:

$$\pi|_{\mathcal{A}(S')} \simeq \pi_0|_{\mathcal{A}(S')}. \tag{24}$$

This result may be understood as an abstract version of Mandelstam's heuristic formula for gauge invariant charged fields in gauge theories

$$\psi(x)\,\mathrm{P}\,\mathrm{e}^{\mathrm{i}\int_C A}, \tag{25}$$

where C is a path to infinity contained in some spacelike cone S and P denotes path ordering.

In view of this result one may redo the DHR analysis for representations π satisfying (24) for a fixed vacuum representation π_0. Let $\mathcal{S}$ denote the set of spacelike cones. In close analogy to DHR one chooses some $S_0 \in \mathcal{S}$ and exploits the unitary equivalence of π and π_0 on $\mathcal{A}(S_0')$ in order to replace π by an equivalent representation π_V on $\mathcal{H}_0$ such that π and π_0 coincide on $\mathcal{A}(S')$. Again π_V defines a faithful representation ρ of $\pi_0(\mathcal{A})$ in $\mathcal{H}_0$, the image of ρ, however, is not contained in $\pi_0(\mathcal{A})$, in general. If one assumes Haag duality for spacelike cones,

$$\pi_0(\mathcal{A}(S'))' = \pi_0(\mathcal{A}(S))'', \tag{26}$$

one finds for $S_0 \subset S_1 \in \mathcal{S}$

$$\rho(\pi_0(\mathcal{A}(S_1))) \subset \pi_0(\mathcal{A}(S_1))'', \tag{27}$$

hence the image of ρ may contain nonlocal operators which only belong to the weak closure of the algebra of a spacelike cone.

In $D \geq 4$ dimensions the problem can be circumvented [16,15], and one obtains the same structure of sectors as in the DHR theory. In $D = 3$ dimensions a new situation occurs which is essentially identical to the situation found in two dimensional chiral conformal field theory. A treatment of sectors where this point of view is emphasized may be found in ref. [26, II]. Here I want to describe the original method of ref. [16]; a modification of this method will then be used in the two dimensional case.

The idea is the choice of a forbidden spacelike direction r which replaces spacelike infinity in the case of bounded regions. Let $\mathcal{S}(r)$ denote the set of spacelike cones which contain λr for all sufficiently large values of λ. Then we introduce the C*-algebra

$$\mathcal{A}^r = \overline{\bigcup_{S \in \mathcal{S}(r)} \pi_0(\mathcal{A}(S'))''} \tag{28}$$

One finds that for each r, ρ has a unique extension ρ^r to $\mathcal{A}^r$ which is weakly continuous on each algebra $\pi(\mathcal{A}(S'))''$, $S \in \mathcal{S}(r)$. Moreover, if ρ is localized in some spacelike cone S which is spacelike to r (i.e. there is some $S_1 \in \mathcal{S}(r)$ such that $S \subset S_1'$) then ρ^r is an endomorphism of $\mathcal{A}^r$. In general, the extensions ρ^r and $\rho^{r'}$ for different spacelike directions do not coincide on the intersection of their domain of definition. They coincide, however, on the image of $\mathcal{A}$ under an endomorphism, so products of representations are well defined as in the DHR theory. It is at the level of intertwiners that the differences become visible.

Let ρ_1, ρ_2 be localized in S, and let $\mathrm{Ad}\, U_2 \circ \rho_2$ be localized in $S_2 \subset S'$. Then for each spacelike direction r which is spacelike to S and to S_2 the statistics operator

$$\varepsilon^r(\rho_1, \rho_2) = U_2^{-1} \rho_1(U_2) \tag{29}$$

intertwines $\rho_1^r \rho_2$ and $\rho_2^r \rho_1$. But ε^r may be different from $\varepsilon^{r'}$ if the configuration S, r, S_2 cannot be continuously be deformed into the configuration S, r', S_2. In this case, $\varepsilon^{r'}(\rho_1, \rho_2)$ coincides with $\varepsilon^r(\rho_2, \rho_1)^{-1}$, hence the "monodromy operator" whose nontriviality shows the occurrence of braid group statistics

$$\varepsilon_M(\rho_1, \rho_2) := \varepsilon(\rho_2, \rho_1)\, \varepsilon(\rho_1, \rho_2) = \rho_1^{r'}(U_2)^{-1} \rho_1^r(U_2) \tag{30}$$

measures the difference of both extensions of ρ_1.

4. Soliton sectors in two dimensions

Let us first look at examples. In the sine-Gordon theory the field equation

$$\Box \varphi + \lambda \sin g\varphi = 0 \tag{31}$$

has the symmetry $\varphi \mapsto \varphi + 2\pi/g$. There are classical soliton solutions which are characterized by the topological charge

$$\varphi(\infty) - \varphi(-\infty) = \frac{2\pi n}{g}, \quad n \in \mathbb{Z}. \tag{32}$$

In quantum theory, the symmetry is spontaneously broken, there exist vacua $\langle \cdot \rangle_n$ with $\langle \varphi \rangle_n = 2\pi n/g$, $n \in \mathbb{Z}$ and interpolating states $\langle \cdot \rangle_{n_+,n_-}$

$$\langle \varphi(t,x) \rangle_{n_+,n_-} \to \frac{2\pi}{g} n_\pm, \quad x \to \pm\infty, \tag{33}$$

resembling very much the classical solitons.

In the $P(\varphi)_2$ models where P is a polynomial which is bounded from below and has global minima at points $v \in V$, one expects vacua $\langle \cdot \rangle_v$ with $\langle \varphi \rangle_v = v$ and interpolating states $\langle \cdot \rangle_{v_+,v_-}$ ("kinks") with

$$\langle \varphi(t,x) \rangle_{v_+,v_-} \to v_\pm, \quad x \to \pm\infty. \tag{34}$$

This phenomenon is not necessarily connected with spontaneous breakdown of symmetry.

In ref. [28] Fröhlich made a proposal for an extension of the theory of superselection sectors to sectors containing solitons. This proposal is restricted to theories where the set of vacua is a homogeneous space for some group of internal symmetries. The idea is that soliton sectors are generated by endomorphisms of the observable algebra which act trivially on some left half space and as an inner symmetry on some right half space (or vice versa). E.g. in the sine-Gordon theory he discusses the automorphism

$$\rho(\varphi(0,x)) = \varphi(0,x) + h(x), \tag{35}$$

$$\rho(\dot{\varphi}(0,x) = \dot{\varphi}(o,x) \tag{36}$$

with a smooth function $h(x)$ which vanishes for sufficiently negative values of x and equals $2\pi/g$ for sufficiently large x.

It is not clear how this idea can be generalized to situations where the vacua are not related by inner symmetries, and even in situations with the presence of symmetries it is not evident that one always can find endomorphisms with the prescribed properties.

For a discussion in the general case one may start from the result in ref. [16] on the localization of one particle representations which assumes in two dimensions the following form. Let π be a one particle representation. Then there exist vacuum representations $\pi_\pm$ such that

$$\pi|_{\mathcal{A}(W_\pm + x)} \simeq \pi_\pm|_{\mathcal{A}(W_\pm + x)}. \tag{37}$$

Here $W_\pm$ denote the right and left wedge, respectively,

$$W_\pm = \{x \in \mathbb{R}^2,\ |x^0| < \pm x^1\}. \tag{38}$$

In analogy to $D = 3$ dimensions one may introduce extensions of the algebra $\mathcal{A}$. These extended algebras depend not only on the spacelike direction (there are only two different possibilities in two dimensions) but also on a vacuum representation. Let π_0 be some vacuum representation. The set of trace class operators T in $\mathcal{H}_{\pi_0}$ induces via

$$\|A\|_T = |\operatorname{tr} T\pi_0(A)| \tag{39}$$

a family of seminorms on $\mathcal{A}(W_\pm + x)$. The associated completion $\mathcal{A}(W_\pm + x)_{\pi_0}$ is an abstract von Neumann algebra which is canonically isomorphic to the weak closure of $\pi_0(\mathcal{A}(W_\pm + x))$, the canonical isomorphism being the σ-weak extension of π_0. The inclusion $\mathcal{A}(W_\pm + x) \subset \mathcal{A}(W_\pm + y)$ for $W_\pm + x \subset W_\pm + y$ extends uniquely to the completions, hence we may define the C*-algebras

$$\mathcal{A}^\pm_{\pi_0} = \overline{\bigcup_x \mathcal{A}(W_\pm + x)_{\pi_0}} \quad . \tag{40}$$

By (37), π has unique extensions $\pi^\pm$ to $\mathcal{A}^\pm_{\pi_\pm}$ which are σ-weakly continuous on $\mathcal{A}(W_\pm + x)_{\pi_\pm}$ for all x.

We now assume that all vacuum representations π_0 satisfy Haag duality for wedges,

$$\pi_0(\mathcal{A}(W_\pm + x))' = \pi_0(\mathcal{A}(W_\mp + x))''. \tag{41}$$

We then may proceed as in the construction of DHR endomorphisms. Let $x \in \mathbb{R}^2$. By (37) there is a unitary $V: \mathcal{H}_{\pi_-} \to \mathcal{H}_\pi$ such that

$$V\pi_-(A) = \pi(A)V, \quad A \in \mathcal{A}(W_- + x). \tag{42}$$

The representation $\pi_V = \operatorname{Ad} V^{-1} \circ \pi$ then coincides with π_- on $\mathcal{A}(W_- + x)$. Now let $y \in \mathbb{R}^2$ be arbitrary. There is some $z \in \mathbb{R}^2$ such that

$$W_+ + z \supset W_+ + x \cup W_+ + y \quad . \tag{43}$$

Therefore

$$\pi_V^+(\mathcal{A}(W_+ + y)_{\pi_+}) \subset \pi_V(\mathcal{A}(W_- + y))' \subset \pi_V(\mathcal{A}(W_- + z))' \tag{44}$$

$$= \pi_-(\mathcal{A}(W_- + z))' = \pi_-(\mathcal{A}(W_+ + z))'' = \pi_-^+(\mathcal{A}(W_+ + z)_{\pi_-}) \tag{45}$$

hence we may define a homomorphism ρ from $\mathcal{A}^+_{\pi_+}$ to $\mathcal{A}^+_{\pi_-}$ by $\rho = (\pi_-^+)^{-1}\pi_V^+$. ρ is σ-weakly continuous on each subalgebra $\mathcal{A}(W_+ + u)_{\pi_+}$, $u \in \mathbb{R}^2$ and satisfies

$$\pi^+ \simeq \pi_-^+ \circ \rho, \tag{46}$$

$$\rho|_{\mathcal{A}(W_- + x)} = \operatorname{id}|_{\mathcal{A}(W_- + x)}. \tag{47}$$

These (right hand side localized) homomorphisms are the appropriate generalizations of the DHR endomorphisms. Their semigroupoid structure induces a natural composition rule for representations. Let π and π' be representations satisfying (37) for suitable vacuum representations $\pi_\pm$ and $\pi'_\pm$, respectively.

Then π and π' are composable provided $\pi_+ \simeq \pi'_-$. Let ρ and ρ' be corresponding right hand side localized homomorphisms. Then the product is defined up to equivalence by

$$\pi \times \pi' \simeq \pi_-^+ \circ \rho\rho'|_{\mathcal{A}} \tag{48}$$

One now has to show that the product does neither depend on the choice of the homomorphisms (this is easy since if ρ and $\widehat{\rho}$ both satisfy (46), (47) they differ by an inner automorphism of $\mathcal{A}_{\pi_-}^+$ induced by a unitary $U \in \mathcal{A}(W_+ + x)_{\pi_-}$) nor on the choice of the right hand side.

If we choose instead left hand side localized homomorphisms λ from $\mathcal{A}_{\pi_-}^-$ to $\mathcal{A}_{\pi_+}^-$ which satisfy (46), (47) with + and − exchanged we can define the composition of π and π' by

$$\pi \diamond \pi' \simeq \pi_+'^- \circ \lambda'\lambda|_{\mathcal{A}}. \tag{49}$$

We want to show that both composition laws are equivalent.

First we remark that

$$\pi \times \pi' \simeq \pi^+ \circ \rho_0' \simeq (\pi_0^- \circ \lambda_0)^+ \circ \rho_0' \tag{50}$$

and

$$\pi \diamond \pi' \simeq \pi'^- \circ \lambda_0 \simeq (\pi_0^+ \circ \rho_0')^- \circ \lambda_0, \tag{51}$$

where $\pi_0 \simeq \pi_+ \simeq \pi'_-$ and $\rho_0' = \rho'|_{\mathcal{A}}$ and $\lambda_0 = \lambda|_{\mathcal{A}}$. Hence what remains to be shown is essentially the commutativity of homomorphisms which are localized in spacelike separated regions. This can be done by an adaptation of the original argument in ref. [5].

Let $A \in \mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$. There are a right wedge $W_+ + x_+ \supset W_+ + x$ and a left wedge $W_- + x_- \supset W_- + x$ such that $\mathcal{O} \subset W_+ + x_+ \cup W_- + x_-$. Let $\widehat{\lambda}$ be localized in $W_- + x_-$ and $U_- \in \mathcal{A}(W_- + x)_{\pi_0}$ unitary such that

$$\lambda = \mathrm{Ad}\, U_- \circ \widehat{\lambda} \tag{52}$$

and let $\widehat{\rho'}$ be localized in $W_+ + x_+$ and $U_+ \in \mathcal{A}(W_+ + x)_{\pi_0}$ unitary such that

$$\rho' = \mathrm{Ad}\, U_+ \circ \widehat{\rho}'. \tag{53}$$

Then $\widehat{\lambda}(A) = A = \widehat{\rho}'(A)$. Since $\pi_0^- \circ \widehat{\lambda}_0$ and π_0 coincide on $\mathcal{A}(W_+ + x)$ they coincide also on the σ-weak closure with respect to π_0, hence

$$(\pi_0^- \circ \widehat{\lambda}_0)^+(U_+) = \pi_0^+(U_+) \in \pi_0(\mathcal{A}(W_+ + x))''$$

and by the corresponding argument

$$(\pi_0^+ \circ \widehat{\rho}'_0)^-(U_-) = \pi_0^-(U_-) \in \pi_0(\mathcal{A}(W_- + x))''$$

. We finally get

$$(\pi_0^- \circ \lambda_0)^+ \circ \rho_0'(A) = \mathrm{Ad}\, \pi_0^-(U_-)\pi_0^+(U_+)(A) \tag{54}$$

$$= \mathrm{Ad}\, \pi_0^+(U_+)\pi_0^-(U_-)(A) = (\pi_0^+ \circ \rho_0')^- \circ \lambda_0(A). \tag{55}$$

and we conclude that the composition laws $\times$ and $\diamond$ are equal.

It turns out that the endomorphisms discussed by Fröhlich fit nicely into the framework described above. Namely, let ρ be an endomorphism of $\mathcal{A}$ which acts trivially on $\mathcal{A}(W_- + x)$ and like the inner symmetry α on $\mathcal{A}(W_+ + y)$, $y - x \in W_+$. Then with π_0 also $\pi_0 \circ \alpha$ is a vacuum representation, and ρ has a unique extension ρ_{π_0}: $\mathcal{A}^+_{\pi_0} \to \mathcal{A}^+_{\pi_0\circ\alpha}$. Moreover, if ρ' is another such endomorphism we find for the product

$$(\rho'\rho)_{\pi_0} = \rho'_{\pi_0\circ\alpha}\rho_{\pi_0}, \tag{56}$$

hence the composition structure is the same.

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