# Quantum mechanics inversion for symmetry scattering

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Symmetry scattering is related to the scattering by an effective local potential in the Schrödinger frame using the quantum mechanics inversion techniques of Marchenko, Gel'fand, and Levitan. Exact solutions or exponentially convergent approximations are found.

## I. INTRODUCTION

In symmetry scattering,[1] systems possessing symmetry are considered. These systems are described by differential operators which are invariant under certain symmetry transformations. Usually these operators are second-order Casimir operators of Lie groups, or more precise, of noncompact Riemannian symmetric spaces. Symmetry scattering is obtained by comparing asymptotically the systems having the whole symmetry with those obeying only a more trivial one which results by letting some of the symmetry operations commute. From this comparison, a scattering operator related to the interaction due to the difference of the symmetries can be introduced. Thanks to the well developed mathematics in this area, the relevant quantities for the scattering theory turn out to be given by explicit formulas depending only on the symmetry, i.e., on the parameters characterizing the noncompact Riemannian symmetric spaces.

Symmetry scattering thus provides us with a series of scattering matrices corresponding to the abstract symmetry interaction.

The abstract character of the interaction can be visualized as potential scattering in the Schrödinger frame by applying similarity transformations to the eigenvalue equation of the Casimir operators to transform them into corresponding Schrödinger equations with certain potentials.[2] In this manner for instance, symmetry scattering for the $SO(2,1)/SO(2)$ space is seen to correspond to the scattering by a radial Pöschl–Teller potential.

This approach, however, is not general and for each operator it must individually be determined.

We associate here, the abstract symmetry scattering with the scattering by local potentials in the Schrödinger frame using the quantum mechanics inversion techniques of Marchenko and Gel'fand–Levitan.

In this paper we are mainly concerned with the development of a convenient set up for the inversion to deal with symmetry scattering, we hence illustrated this approach in the most simple case of the rank one spaces. The inversion applied to higher rank spaces can in principle be treated using the same approach as for rank one spaces but requires additionally the use of frame Hamiltonian techniques.[3] This inversion together with its large number of applications, will be treat in a forthcoming publication.

The paper is organized into four sections. After the Introduction, in Sec. II, the quantum mechanics inversion techniques of Marchenko and Gel'fand–Levitan are presented in a manner that suits the needs of symmetry scattering. In Sec. III the inversion technique is applied to all rank one spaces and the results are compared with the ones available in the literature. A conclusion completes the paper.

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## II. GENERAL THEORY

To relate scattering data presented by the $S$-matrix with the scattering by a local potential, Marchenko's equation is well suited. Assuming that the scattering data results from the scattering of a system described by the wave function $\Psi$, where $\Psi$ obeys the radial Schrödinger equation ($\ell=0$) with unknown potential $q(x)$, $x\in[0,\infty)$,

$$-\frac{d^2\Psi(x)}{dx^2}+q(x)\Psi(x)=k^2\Psi(x),$$

the potential is obtained from

$$q(x)=-2\frac{d}{dx}A_x(x), \tag{1}$$

where the transformation kernel $A_x(y)$ obeys Marchenko's integral equation[4]

$$A_x(y)=F(x+y)+\int_x^\infty A_x(t)F(y+t)dt=0, \quad y>x, \tag{2}$$

with

$$F(\xi)=\frac{1}{2\pi i}\int_{-\infty}^\infty e^{ik\xi}[S(k)-1]dk, \tag{3}$$

and $S(k)$ the $S$-matrix.

The scattering data resulting from symmetry scattering is such that the following general properties are valid for the input kernel $F(\xi)$: (1) $F(\xi)\in L^2\cap L^1$ on infinite interval $[x_0,\infty)$ for any $x_0>0$; (2) $F(\xi)$ decreases exponentially as $\xi\to\infty$, i.e., $F(\xi)=o(\xi^{-N})$ for any $N\geqslant 0$; (3) $F(\xi)\in\mathbf{R}$.

Under these conditions the operator $K_x$,

$$(K_xf)(y)\overset{\text{def}}{=}\int_x^\infty F(y+t)f(t)dt, \tag{4}$$

is compact on the interval $[x,\infty)$. Its spectrum is hence discrete and since $K_x = K_x^*$ it is self-adjoint on $L^2[x,\infty)$. We can therefore expand the function $A_x(y)$ using the orthogonal set of its eigenfunctions,

$$A_x(y)=\sum_m a_m(x)f_m(x,y), \quad y>x. \tag{5}$$

Here $x$ is a parameter and

$$K_xf_m(x,y)=\lambda_m(x)f_m(x,y). \tag{6}$$

Substituting Eq. (5) into Eq. (2) we obtain

$$a_m(x)=-\frac{\langle F(x+y),f_m\rangle}{1+\lambda_m},$$

where $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2[x,\infty)$. Now, since $\langle F(x+y),f_m\rangle=(K_xf_m)(x)$ $=\lambda_m(x)f_m(x,x)$, we have

$$A_x(x) = -\sum_m \frac{\lambda_m(x) f_m^2(x,x)}{1+\lambda_m(x)}. \tag{7}$$

Next, let us consider the eigenvalue problem (6) for the operator $K_x$. Calculating the derivative of Eq. (6) with respect to the parameter $x$ (denoted by prime) we have

$$-F(x+y) f_m(x,x) + \int_x^\infty F(y+t) f_m'(x,t) dt = \lambda_m'(x) f_m(x,y) + \lambda_m(x) f_m'(x,y).$$

Using the self-adjointness of $K_x$ to calculate in this equality the inner product with $f_m(x,y)$ it follows,

$$f_m^2(x,x) = -\frac{\lambda_m'(x)}{\lambda_m(x)}. \tag{8}$$

Equalities (7) and (8) imply

$$A_x(x) = \sum_m \frac{\lambda_m'(x)}{1+\lambda_m(x)} \tag{9}$$

which is a direct consequence of the well-known formula[5]

$$q(x) = -2 \frac{d^2}{dx^2} \ln \det(K_x + I). \tag{10}$$

The simple proof of the equality (9) has been given above for the sake of completeness.

Equations (9) or (10) are used in this paper to obtain the potential $q(x)$. For this purpose, some preparation is required. We note that in symmetry scattering the Jost functions are just products of gamma functions;[1] hence using a contour integration the analytic structure of the gamma functions imply that the kernel of the operator $K_x$ can be represented in the form of finite or infinite sum of separable terms,

$$F(\xi) = \sum_{n \geq 1} \tau_n e^{-b_n \xi}, \tag{11}$$

where $b_n > an > 0$, $a \in \mathbf{R}$, and $\tau_n$ increase slower than $e^{\alpha n}$ for $\alpha > 0$. Now, using this expansion, we rewrite the eigenvalue problem (6) as

$$\sum_{n \geq 1} \tau_n e^{-b_n y} \int_x^\infty e^{-b_n t} f(x,t) dt = \lambda f(x,y). \tag{12}$$

Calculating the inner product with the function $e^{-b_k y}$ and using the notation

$$\varphi_n(f) \stackrel{\text{def}}{=} \langle f, e^{-b_n y} \rangle = \int_x^\infty f(x,y) e^{-b_n y}\, dy,$$

we have

$$\sum_{n \geq 1} \frac{\tau_n}{b_n + b_k} e^{-b_n x} e^{-b_k x} \varphi_n = \lambda \varphi_k.$$

We have thus transformed the original eigenvalue problem for the operator $K_x$ into an eigenvalue problem in the $l_+^2$ space, namely,

$$T\varphi=\lambda\varphi, \tag{13}$$

where

$$T_{k,n}(x)=\frac{\tau_n}{b_n+b_k}e^{-(b_n+b_k)x} \quad \text{and} \quad \varphi=\{\varphi_n\}_{n\geq 1}\in l_+^2. \tag{14}$$

Note that the operator $T$ is in general not self-adjoint because the system $\{e^{-b_n y}\}$ is not orthogonal in $L^2[x_0,\infty)$. This is however not essential here, since the function $A_x(x)$ is gained from Eq. (9) which only uses the spectrum and not the eigenfunctions of $T$.

If the sum in the representation (11) is finite, the input kernel is separable and thus the potential $q(x)$ can be calculated in a totally analytic way. When the sum is infinite we can only approximate the potential by truncating the sum. To estimate the accuracy of such an approximation we use Lidsky's theorem[6] which states that for a normal non-negative compact operator of trace class, its trace is equal to the sum of its eigenvalues. The operator $T$ satisfies indeed all these requirements. The normality and compactness are clear. To see that it is non-negative, we note that Eq. (7) implies that the eigenvalues $\lambda_m(x)$ as function of $x$ do not cross the real axis for any $m$. Thus, eventually choosing $-T$ this condition is also fulfilled. To calculate the trace of the operator $T$ we use the basis $\{e^i\}_{i\in \mathrm{N}}$, where $e^i$ is the sequence having a one at the $i$th position and zeros elsewhere. We obtain

$$\mathrm{Tr}\, T=\sum_l T_{ll}=\frac{1}{2}\sum_l \frac{\tau_l}{b_l}e^{-2b_l x}, \tag{15}$$

where $b_l$ and $\tau_l$ are defined in Eq. (11). From the properties of $b_l$ and $t_l$ it follows $\mathrm{Tr}\, T<\infty$.

We can therefore apply Lidsky's theorem to obtain

$$\mathrm{Tr}\, T=\sum_l T_{ll}=\sum_m \lambda_m. \tag{16}$$

To calculate the kernel $A_x(x)$ and hence the potential when the input kernel is given by an infinite exponential expansion, we represent the operator $T(x)$ in the form

$$T=T^{(N)}+\tilde{T}^{(N)}, \tag{17}$$

where

$$T_{kn}^{(N)}=\begin{cases} T_{kn}, & \text{if} \quad k,n\leqslant N \\ 0, & \text{if} \quad k>N \quad \text{or} \quad n>N. \end{cases} \tag{18}$$

The kernel $A_x^{(N)}(x)$ is obtained from the $N$-dimensional operator $T^{(N)}$ again in a purely analytic way using Eq. (10).

We now estimate the required dimension $N=N(\epsilon)$ of the operator $T^{(N)}$ sufficient to obtain the accuracy $\epsilon$ in the calculation of $A_x(x)$, i.e.,

$$|A_x(x)-A_x^{(N)}(x)|\leqslant\epsilon, \tag{19}$$

where

$$A_x^{(N)}(x)=\sum_{m=1}^{N}\frac{(\lambda_m^{(N)})'(x)}{1+\lambda_m^{(N)}(x)}.$$

For this purpose we introduce the notation $\Delta_m^{(N)}=\lambda_m-\lambda_m^{(N)}$. We have

$$\frac{\lambda'_m}{1+\lambda_m}-\frac{(\lambda_m^{(N)})'}{1+\lambda_m^{(N)}}=\frac{\lambda'_m-(\lambda_m^{(N)})'}{1+\lambda_m^{(N)}}-\Delta_m^{(N)}\frac{\lambda'_m}{(1+\lambda_m)(1+\lambda_m-\Delta_m^{(N)})}.$$

Using the following inequality,

$$|\lambda_m-\lambda_m^{(N)}|\leqslant\|\widetilde{T}^{(N)}\|_{l^2_+}\leqslant\text{Tr }\widetilde{T}^{(N)}=\sum_{l>N}T_{ll},$$

and supposing that $N$ is large enough so that

$$\sum_{l>N}T_{ll}<\frac{1}{2}, \tag{20}$$

it follows,

$$\left|\sum_m\frac{\lambda'_m}{1+\lambda_m}-\frac{(\lambda_m^{(N)})'}{1+\lambda_m^{(N)}}\right|<\left|\sum_m(\lambda'_m-(\lambda_m^{(N)})')\right|+\left|\sum_m(\lambda'_m-(\lambda^{(N)})'_m)\frac{\lambda_m^{(N)}}{1+\lambda_m^{(N)}}\right|$$

$$+2\text{ Tr }\widetilde{T}^{(N)}\sum_m|\lambda'_m|.$$

Obviously $\Sigma_m\lambda'_m=(\text{Tr }T)'=\Sigma_l T'_{ll}$, hence,

$$\left|\sum_m\left(\frac{\lambda'_m}{1+\lambda_m}-\frac{(\lambda_m^{(N)})'}{1+\lambda_m^{(N)}}\right)\right|<\left|\sum_{l>N}T'_{ll}\right|+2\sum_m\lambda_m^{(N)}|\lambda'_m-(\lambda_m^{(N)})'|+2\sum_{l>N}T_{ll}\left|\sum_{m\geqslant 1}T'_{mm}\right|. \tag{21}$$

The values $|\lambda'_m-(\lambda_m^{(N)})'|$ are now estimated through the norm of the perturbation operator and of its derivative, $\|\widetilde{T}^{(N)}\|_{l^2_+}\leqslant\Sigma_{l>N}T_{ll}$ and $\|\widetilde{T}^{(N)\prime}\|_{l^2_+}\leqslant\Sigma_{l>N}T'_{ll}$.

Let $\{\varphi^m\}$ and $\{\psi^m\}$ be biorthogonal sets of the eigenvectors of the operators $T$ and $T^*$, respectively:

$$T\varphi^m=\lambda_m\varphi^m,\quad T^*\psi^m=\lambda_m\psi^m,\quad(\varphi^m,\psi^n)=\delta_{mn}.$$

Calculating the derivative of the equation $T\varphi^m=\lambda_m\varphi^m$ with respect to the parameter $x$ and considering the inner product with $\psi^m$ in the space $l^2_+$ we obtain

$$(T'\psi^m,\varphi^m)=\lambda'_m.$$

In a similar way $((T^{(N)})'\psi^{(N)m},\varphi^{(N)m})=(\lambda_m^{(N)})'$. Hence,

$$|\lambda'_m-(\lambda_m^{(N)})'|=|(T'\psi^m,\varphi^m)-((T^{(N)})'\psi^{(N)m},\varphi^{(N)m})|\leqslant 2\|T'\|\,\|\varphi^m-\varphi^{(N)m}\|+\|\widetilde{T}^{(N)\prime}\|.$$

The Born series yields the estimate $\|\varphi^m-\varphi^{(N)m}\|\leqslant 2\|\widetilde{T}^{(N)}\|$, hence,

$$|\lambda'_m-(\lambda_m^{(N)})'|\leqslant 4\|T'\|\,\|\widetilde{T}^{(N)}\|+\|(\widetilde{T}^{(N)})'\|.$$

Therefore,

$$\sum_m \lambda_m^{(N)} |\lambda_m' - (\lambda_m^{(N)})'| \leqslant \sum_l T_{ll} \left( 4 \sum_{m>N} T_{mm} \left| \sum_n T_{nn}' \right| + \left| \sum_{m>N} T_{mm}' \right| \right). \tag{22}$$

Finally, relations (21) and (22) give

$$|A_x(x) - A_x^{(N)}(x)| \leqslant 2 \sum_{l>N} T_{ll} \left| \sum_{m\geqslant 1} T_{mm}' \right| \left( 1 + 4 \sum_{n\geqslant 1} T_{nn} \right) + \left| \sum_{l>N} T_{ll}' \right| \left( 1 + 2 \sum_{m\geqslant 1} T_{mm} \right). \tag{23}$$

From Eq. (14) we further have

$$T_{ll} = \frac{\tau_l}{2b_l} e^{-2b_l x}, \quad T_{ll}' = -\tau_l e^{-2b_l x};$$

and since in symmetry scattering the coefficients $\tau_l$ are bounded by $l$, i.e., $|\tau_l| \leqslant l$, and $b_l \geqslant al$, it follows,

$$T_{ll} \leqslant \frac{1}{2a} e^{-2alx}.$$

This leads to

$$\sum_{l>N} T_{ll} \leqslant \frac{1}{2a} e^{-2a(N+1)x} \sum_{l=0}^{\infty} e^{-2alx} = \frac{1}{2a} \frac{e^{-2a(N+1)x}}{1 - e^{-2ax}} = Q(x) e^{-2aNx}, \tag{24}$$

where

$$Q(x) \stackrel{\text{def}}{=} \frac{e^{-ax}}{a \sinh ax}. \tag{25}$$

For $N=0$ we then have

$$\sum_{l=1}^{\infty} T_{ll} \leqslant Q(x).$$

In the same manner we obtain

$$\sum_{l>N} (T_{ll})' \leqslant 4a^2 Q^2(x) e^{-2aNx}, \tag{26a}$$

$$\sum_{l=0}^{\infty} (T_{ll})' \leqslant 4a^2 Q^2(x). \tag{26b}$$

Formulas (23)–(26) finally imply

$$|A_x(x) - A_x^{(N)}(x)| \leqslant 4a^2 Q^2(x) (1 + 4Q(x) + 8Q^2(x)) e^{-2aNx}. \tag{27}$$

Thus, to obtain the kernel $A_x(x)$ in the point $x$ with the accuracy $\epsilon$ [see Eq. (19)] the finite-dimensional matrix $T^{(N)}$ of the dimension

$$N(\epsilon) \geqslant \frac{1}{2ax} \ln \frac{4a^2Q^2(x)(1+4Q(x)+8Q^2(x))}{\epsilon} \tag{28}$$

is required. Let us note that in symmetry scattering for rank one spaces, $a=2$ or $a=1$, further $\Sigma_{l>N} T_{ll} < 1/2$ for $N \geqslant 2$.

In general, the estimation (28) is not effective for small $x$. In this region is better to calculate the asymptotical behavior of the kernel $A_x(x)$ [and consequently of the potential $q(x)$] directly from Marchenko's equation (2) or use Gelfand–Levitan's equation[7] for the function $B_x(y)$:

$$B_x(y) + G(x,y) + \int_0^x B_x(t)G(y,t)dt = 0, \quad y < x, \tag{29}$$

where $G(y,t) = G(t,y) \stackrel{\text{def}}{=} H(y-t) - H(y+t)$, and $H(\xi) = H(-\xi)$ is the Fourier cosine transformation of the spectral measure. Equation (29) can be solved in $L^2[0,x]$ if $H(\xi) \in L^2$; otherwise we can consider it on the interval $[x_0,x]$, $x_0 > 0$. The potential is given by Eq. (3), where the function $A_x(x)$ is substituted by $B_x(x)$.

Since $G(y,t) = H(y-t) - H(y+t)$; $0 \leqslant y,t \leqslant x$, the function $H(\xi)$ should be treated at least on the interval $-x \leqslant \xi \leqslant 2x$. For symmetry considerations, however, it is better to expand it over the cosines (as it is even) on the wider interval $[-2x,2x]$,

$$H(\xi) = \sum_{n=0}^{\infty} h_n \cos \frac{\pi n}{2x} \xi, \tag{30}$$

$$h_n = \sqrt{\frac{2}{\pi}} \int_0^{2x} H(\xi) \cos \frac{\pi n}{2x} \xi \, d\xi. \tag{31}$$

Then

$$G(y,t) = 2 \sum_n h_n \sin \frac{\pi n}{2x} y \sin \frac{\pi n}{2x} t. \tag{32}$$

Using the notation

$$\beta_k(x) = \int_0^x B_x(y) \sin \frac{\pi k}{2x} y \, dy, \tag{33}$$

we obtain from Eq. (29) the following relation for the set of coefficients $\beta_k$, $k \geqslant 1$

$$\beta_k + \frac{2x}{\pi} \sum_{m=0}^{\infty} h_{2m+1}(-1)^m R_{k,2m+1} + \frac{2x}{\pi} \sum_{n=1}^{\infty} h_n R_{k,n} \beta_n = 0, \tag{34}$$

where

$$R_{k,n} = \frac{1}{n-k} \sin \frac{\pi(n-k)}{2} - \frac{1}{n+k} \sin \frac{\pi(n+k)}{2}.$$

For even and odd $k$ we have the coupled system of equations

TABLE I. Parameters required by symmetry scattering for all rank one symmetric spaces of the noncompact type. $K$ stands for a maximal compact subgroup of the corresponding group.

| | Table of parameters | | |
|---|---|---|---|
| Space | $K$ | $m_\alpha$ | $m_{2\alpha}$ |
| $SO(n,1)/K$ | $SO(n)$ | $n-1$ | 0 |
| $SU(n,1)/K$ | $SU(n)\otimes U(1)$ | $2n-2$ | 1 |
| $Sp(n,1)/K$ | $Sp(n)\otimes Sp(1)$ | $4n-4$ | 3 |
| $F_4/K$ | $SO(9)$ | 8 | 7 |

$$(1+xh_{2p})\beta_{2p}+\frac{4x}{\pi}\sum_{m=0}^{\infty}\frac{2m+1}{4p^2-(2m+1)^2}(-1)^{p+m}h_{2m+1}+\frac{4x}{\pi}\sum_{q=0}^{\infty}\frac{2q+1}{4p^2-(2q+1)^2}$$
$$\times(-1)^{p+q+1}h_{2q+1}\beta_{2q+1}=0,$$
$$(1+xh_{2p+1})\beta_{2p+1}+\frac{8x}{\pi}\sum_{q=1}^{\infty}\frac{q^2}{(2p+1)^2-4q^2}(-1)^{p+q+1}h_{2q}\beta_{2q}=0, \tag{35}$$

which serve as the starting base for the calculation of the potential $q(x)$ near the origin.

## III. CALCULATIONS

The Jost functions in symmetry scattering can be written as[1]

$$c(k)=\frac{\Gamma(\frac{1}{2}ik)}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha+1+ik))}\frac{\Gamma(\frac{1}{2}ik+\frac{1}{2})}{\Gamma(\frac{1}{2}(\frac{1}{2}m_\alpha+m_{2\alpha}+ik))}. \tag{36}$$

The values for $m_\alpha$ and for $m_{2\alpha}$ are listed in Table I.

We calculate now the input kernel $F(\xi)$ using Eq. (3) for all rank one symmetric spaces of the noncompact type. We obtain the following expressions: *Group SO(n,1)*

$$S(k)=\frac{\Gamma(ik)}{\Gamma(-ik)}\frac{\Gamma(-ik+(n-1)/2)}{\Gamma(ik+(n-1)/2)}. \tag{37a}$$

*Group SO(2m,1)*

$$F(\xi)=\sum_{j=1}^{\infty}\frac{(-1)^j}{j!(j-1)!}\left(\prod_{l=1}^{2j}(m+j-l-1/2)\right)e^{-j\xi}. \tag{37b}$$

*Group SO(2m+1,1)*

$$F(\xi)=\sum_{j=1}^{m-1}\frac{(-1)^j}{j!(j-1)!}\left(\prod_{l=1}^{2j}(m+j-l)\right)e^{-j\xi}. \tag{37c}$$

*Group SU(n,1)*

$$S(2k)=\frac{\Gamma(ik)}{\Gamma(-ik)}\frac{\Gamma(ik+1/2)}{\Gamma(-ik+1/2)}\frac{[\Gamma(-ik+n/2)]^2}{[\Gamma(ik+n/2)]^2}. \tag{38a}$$

*Group SU(2m,1)*

$$F(\xi)=\sum_{j=1}^{m-1}\frac{(-1)^j}{j!(j-1)!}\left(\prod_{l=1}^{2j}\frac{(m+j-l)^2}{j-l+1/2}\right)e^{-2j\xi}+\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{(j-1)!^2}\left(\prod_{l=1}^{2j-1}\frac{(m+j-1/2-l)^2}{j-1/2-l}\right)e^{2(-j+1/2)\xi}. \tag{38b}$$

*Group SU(2m+1,1)*

$$F(\xi)=\sum_{j=1}^{\infty}\frac{(-1)^j}{j!(j-1)!}\left(\prod_{l=1}^{2j}\frac{(m+j+1/2-l)^2}{j-l+1/2}\right)e^{-2j\xi}+\sum_{j=1}^{m}\frac{(-1)^{j+1}}{(j-1)!^2}\left(\prod_{l=1}^{2j-1}\frac{(m+j-l)^2}{j-1/2-l}\right)e^{2(-j+1/2)\xi}. \tag{38c}$$

*Group Sp(n,1)*

$$S(2k)=\frac{\Gamma(ik)}{\Gamma(-ik)}\frac{\Gamma(ik+1/2)}{\Gamma(-ik+1/2)}\frac{[\Gamma(-ik+n+1/2)]^2}{[\Gamma(ik+n+1/2)]^2}\frac{(ik+n-1/2)}{(-ik+n-1/2)}. \tag{39a}$$

$$F(\xi)=\sum_{j=1}^{\infty}\frac{(-1)^j}{j!(j-1)!}\left(\prod_{l=1}^{2j}\frac{(n+j+1/2-l)^2}{j-l+1/2}\right)\frac{n-1/2-j}{n-1/2+j}e^{-2j\xi}+\sum_{j=1}^{n}\frac{(-1)^{j+1}}{(j-1)!^2}\left(\prod_{l=1}^{2j-1}\frac{(n+j-l)^2}{j-1/2-l}\right)\frac{n-j}{n-1+j}e^{2(-j+1/2)\xi}. \tag{39b}$$

*Group $F_4$*

$$S(2k)=\frac{\Gamma(ik)}{\Gamma(-ik)}\frac{\Gamma(-ik+11/2)}{\Gamma(ik+11/2)}\frac{(3/2-ik)}{(3/2+ik)}\frac{(1/2-ik)}{(1/2+ik)}. \tag{40a}$$

$$F(\xi)=\sum_{j=1}^{\infty}\frac{(-1)^j}{j!(j-1)!}\left[\prod_{l=1}^{2j}\left(\frac{11}{2}+j-l\right)\right]\frac{(3/2+l)(1/2+l)}{(3/2-l)(1/2-l)}e^{-2j\xi}-1920e^{-3\xi}-20e^{-\xi}. \tag{40b}$$

To calculate the inversion potentials for all rank one spaces we use Eq. (10),

$$q(x)=-2\frac{d^2}{dx^2}\ln\det(T^{(N)}+I),$$

with the operator $T^{(N)}$ defined in Eq. (18).

For the group SO$(2m+1)$ this operator is finite dimensional and the calculated inversion potential is exact. For the other spaces the kernel is represented by an infinite series and we can only provide an approximation to the inversion potential. In Fig. 1 we present the inversion potentials for the spaces SO$(n,1)$ for several values of $n$ in either an exact form ($n$ odd) or as an approximation using $(6\times6)$ dimensional operators. To illustrate the fast convergence of our procedure we plot the potential corresponding to the space SO(2,1) using $T^{(N)}$ for different

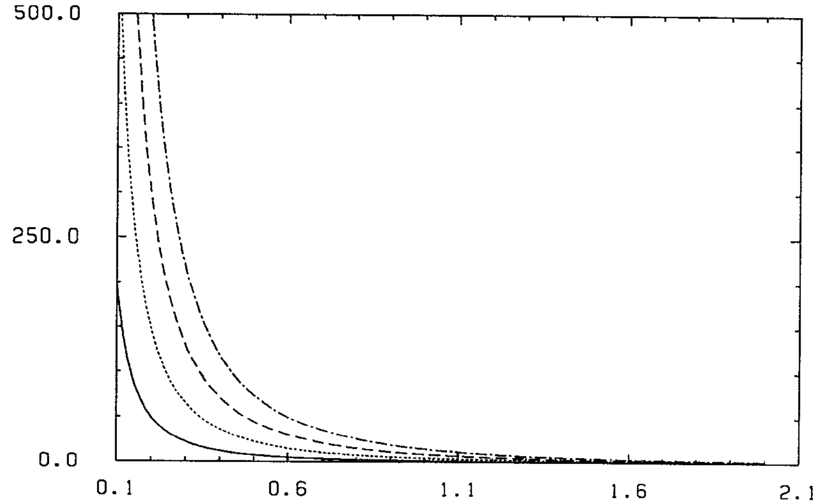


FIG. 1. The inverse scattering potentials for symmetry scattering related to the spaces $SO(n,1)/SO(n)$: $n=4$, full line; $n=5$, dotted line; $n=6$, dashed line and $n=7$, dashed–dotted line. The potentials for $n=5$ and $n=7$ are exact. For $n=4$ and $n=6$ a $(6\times6)$ dimensional operator is used. The axes are in arbitrary relative units.

values of $N$. From the approximation inequality (28) we see that for $N=2$ at $x=1$ the error is of the order of $10^{-2}$. For $N=6$ the error is smaller than $10^{-5}$ (Fig. 2).

The results obtained here through the inversion are in agreement with the ones that can be obtained as a direct problem. Let us discuss briefly how the symmetry scattering potentials for rank one spaces can be obtained in a direct way.

Symmetry scattering is related to the scattering described by the radial part of the eigenvalue equation for the Laplace Beltrami operator of the symmetric space:

$$-\Delta_\tau \Phi_k(\tau) = ((m_\alpha/2+m_{2\alpha})^2+k^2)\Phi_k(\tau). \tag{41a}$$

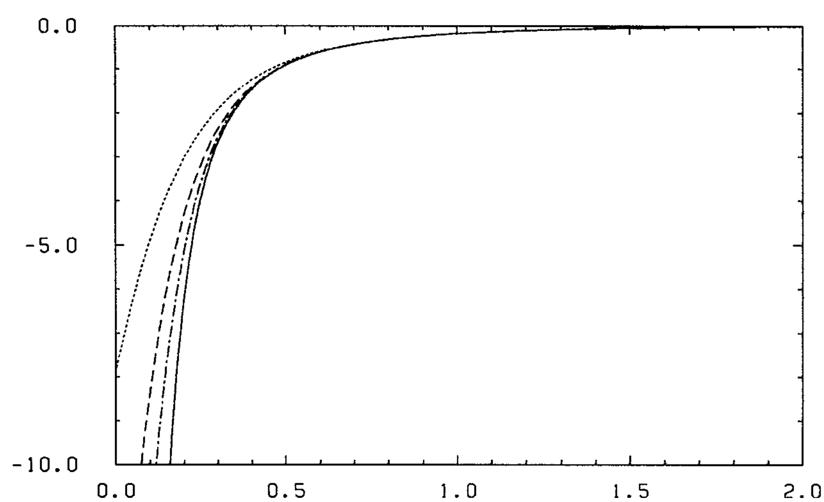


FIG. 2. Approximations to the inverse scattering potential for symmetry scattering related to the space $SO(2,1)/SO(2)$. The approximations use operators $T^{(N)}$ of different dimension $N$: $N=2$, dotted line; $N=4$, dashed line and $N=6$ dashed–dotted line. The full line is the exact potential found by other methods. The axes are in arbitrary relative units.

This equation takes the explicit form

$$-\left(\frac{d^2}{d\tau^2}+m_\alpha \coth \tau \frac{d}{d\tau}+2m_{2\alpha} \coth 2\tau \frac{d}{d\tau}\right)\Phi_k(\tau)=((m_\alpha/2+m_{2\alpha})^2+k^2)\Phi_k(\tau), \quad (41b)$$

and has as a solution

$$\Phi_k(\tau)=F(a,b,c,-\sinh^2 \tau). \quad (42)$$

Here

$$a=\frac{m_\alpha/2+m_{2\alpha}+ik}{2}, \quad b=\frac{m_\alpha/2+m_{2\alpha}-ik}{2}, \quad c=\frac{m_\alpha+m_{2\alpha}+1}{2}.$$

Now, the potential is gained by considering the function

$$\Psi_\lambda(\tau)=(2 \sinh \tau)^{m_\alpha/2}(2 \sinh 2\tau)^{m_{2\alpha}/2}\Phi_\lambda(\tau). \quad (43)$$

Equation (41b) becomes

$$\left(-\frac{d^2}{d\tau^2}+V(\tau)\right)\Psi_\lambda(\tau)=\lambda^2\Psi_\lambda(\tau),$$

where

$$V(\tau)=\frac{a}{\sinh^2 \tau}+\frac{b}{\sinh^2 2\tau},$$

with

$$a=\left[\left(\frac{m_\alpha+m_{2\alpha}-1}{2}\right)^2-\left(\frac{m_{2\alpha}-1}{2}\right)^2\right], \quad b=(m_{2\alpha}-1)^2-1. \quad (44)$$

## IV. CONCLUSION

The quantum mechanics inversion techniques have been applied to associate the abstract symmetry scattering with the scattering in the Schrödinger frame by local potentials. The potentials found for all rank one spaces are of the Pöschl–Teller type, and coincide with the results obtained in the literature by other methods.[8] Besides this confirmation, we found a remarkable fact concerning the spaces $S(n,1)/SO(n)$. The potentials gained for these spaces are similar, in fact they build the family of potentials

$$V(r)=\frac{(n-2)^2-1}{4 \sinh^2 (x)}.$$

However, analytically these spaces are different according to $n$ being even or odd $n$. This analytic difference is reflected, for instance, in the different character of their inversion. For $n$ odd, the input kernel is separable and the inversion is gained in a purely analytic manner. For $n$ even, the input kernel is no longer separable and the potential is obtained only as a fast convergent series. We thus point out that the same family of solvable potentials have substantially different scattering matrices.

The potentials for rank one spaces can also be obtained by other analytic methods. In this sense the results presented here have only complementary value. For spaces of higher rank, no other standard method exists that relates its abstract interaction with some effective local interaction. Thus, the discussed approach becomes of fundamental importance for the treatment of these spaces. The inversion potentials for spaces of higher rank will be handled in a forthcoming paper.

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[1] R. F. Wehrhahn, Phys. Rev. Lett. **65**, 1294 (1990); R. F. Wehrhahn, Yu. F. Smirnov, and A. M. Shirokov, J. Math. Phys. **32**, 3254 (1991).
[2] R. D. Levine and C. E. Wulfman, Chem. Phys. Lett. **60**, 327 (1979); Y. Alhassid, F. Gürsey, and F. Iachello, Ann. Phys. (N.Y.) **148**, 346 (1983); Chem. Phys. Lett. **99**, 27 (1983); A. Frank and K. B. Wolf, Phys. Rev. Lett. **52**, 1737 (1984); J. Math Phys. **26**, 973 (1985).
[3] Yu. A. Kuperin, P. B. Kurasov, Yu. B. Melnikov, and S. P. Merkuriev, Ann. Phys. (N.Y.) **205**, 330 (1991).
[4] K. Chadan and P. C. Sabatier, *Inverse problems in quantum scattering theory* (Springer-Verlag, New York, 1977).
[5] L. D. Faddeev, J. Sov. Math. **5**, 334 (1976).
[6] N. I. Achieser and I. M. Glasmann, *Theorie der linearen operatoren im Hilbert-Raum* (Akademie-Verlag, Berlin, 1977).
[7] B. M. Levitan, *Inverse Sturm–Liouville problems* (VNU Science, Utrecht, 1987).
[8] A. Perelomov, *Generalized Coherent States and Their Applications*, Texts and Monographs in Physics (Springer-Verlag, Berlin, 1985).