Determination of the running coupling in the SU(2) Yang-Mills theory from first principles

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The ideas underlying a recent computation of the running coupling in the SU(2) gauge theory are discussed and a summary of the results obtained is given.

1. INTRODUCTION

The running coupling in QCD, $\alpha_s(q)$, has been determined experimentally from jet production rates in e^+e^- collisions and from various other processes. At $q = 91\,\text{GeV}$ the result quoted is [1, 2]

$$\alpha_s(q) = 0.118 \pm 0.007. \qquad (1)$$

Perturbation theory predicts the evolution of the running coupling at high energies, but not its value at a given momentum q. The situation is different in lattice QCD, because here we may fix the parameters in the lagrangian at low energies, taking as experimental input the pion decay constant and the masses of the π, K, D and B mesons, for example. The theory then determines the physical amplitudes uniquely, at all energies, and so in particular the running coupling in the chosen scheme.

It would obviously be very interesting to compute $\alpha_s(q)$ along these lines, not only to compare with experiment, but also to find out at which energies the perturbative evolution of the coupling sets in. At present our technical abilities are not sufficiently developed to perform such a computation. We can, however, solve the analo-

gous problem in the SU(2) gauge theory without matter fields, and so we restrict attention to this case from now on.

In pure gauge theories the bare coupling g_0 is the only parameter in the lagrangian. Only one low-energy quantity is hence required as "experimental" input, the canonical choice being the string tension K. We may then ask what the value of the running coupling $\alpha_{\overline{\text{MS}}}(q)$ in the $\overline{\text{MS}}$ scheme of dimensional regularization is at say $q = 20 \times \sqrt{K}$. The important point to note is that the question has a unique answer. There is no free parameter left and it is just a matter of being able to solve the theory to a sufficient degree.

A straightforward approach to the problem is based on the force $F(r)$ between static quarks at distance r [3–5]. The idea is to calculate the running coupling

$$\alpha_{q\bar{q}}(q) = \tfrac{4}{3}r^2 F(r), \qquad q = 1/r, \qquad (2)$$

through numerical simulation and then to convert to the $\overline{\text{MS}}$ scheme using the one-loop formula [6, 7]

$$\alpha_{\overline{\text{MS}}} = \alpha_{q\bar{q}} - 0.055 \times (\alpha_{q\bar{q}})^2 + \dots \qquad (3)$$

Of course this expansion is only applicable when the coupling is sufficiently small, i.e. deep in the perturbative short distance regime.

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The basic difficulty encountered in this calculation is that at present it is practically impossible to reach momenta q greater than a few GeV. This is so because one needs a lattice with physical size L greater than about 1.5 fm (to avoid finite volume effects) and a spacing a much smaller than r (to avoid cutoff effects). It is already compromising if we allow r to be as small as $3a$, and we then end up with the bound

$$q\,[\text{GeV}] \leq 0.044 \times (L/a). \tag{4}$$

The largest lattice simulated so far has 56×48^3 points [8] and so even in this case is one limited to momenta $q \leq 2\,\text{GeV}$.

El-Khadra et al. [9] have recently put forward another idea of how to compute the running coupling. Their starting point is the perturbation expansion [10]

$$\alpha_{\overline{\text{MS}}}(\pi/a) = \tilde{\alpha}_0 - 0.206 \times (\tilde{\alpha}_0)^2 + \dots, \tag{5}$$

which relates the $\overline{\text{MS}}$ coupling to Parisi's improved bare coupling [11–13]

$$\tilde{\alpha}_0 = \alpha_0/P, \qquad \alpha_0 = g_0^2/4\pi. \tag{6}$$

The plaquette expectation value P occurring here is a function of g_0 and is evaluated through numerical simulation. Eq.(5) allows us to determine the renormalized coupling at scale $q = \pi/a$ for any given small value of the bare coupling. To be able to express q in physical units, we must, at the same value of g_0, also compute some low-energy scale such as the string tension or, as El-Khadra et al. propose, the mass difference Δm between the 1P and 1S (quenched) charmonium levels. Δm is an attractive quantity, because it is meaningful beyond the quenched approximation and because it is experimentally measurable (in contrast to the string tension).

Whichever low-energy quantity one decides to compute, it is clear that this method is also subject to the basic limitation discussed above, since all relevant scales must fit onto a single lattice. A further difficulty is that one has no check on the reliability of the perturbation expansion (5) in the accessible range of lattice spacings.

The conclusion from all this is that some sort of many-lattice or renormalization group approach

is needed, if one desires to reach momenta q much greater than a few GeV. In his Cargèse lectures of 1979, Wilson [14] has already stressed the necessity of a recursive procedure. His proposition was to set up a "block spin" renormalization group transformation, an idea which was taken up by many authors later on (a nice review is ref.[15]).

Our own method is best referred to as a recursive finite-size technique [16–18]. It relies on a renormalized coupling $\bar{g}^2(L)$ which runs with the lattice size L. The renormalization step then amounts to matching lattices with the same renormalized coupling but different lattice spacings. We can also easily study the evolution of the coupling by changing L at fixed a. By combining these steps, we are able to pass to the continuum limit and to trace $\bar{g}^2(L)$ over a large range of box sizes L given in units of the string tension K.

Our aim in the following is to discuss the technique in some detail and to summarize the results that have been obtained so far.

2. RUNNING COUPLING AT FINITE L

The precise definition of the renormalized coupling $\bar{g}^2(L)$ that we use in our numerical work is complicated. To keep the discussion transparent, it is better to start with an alternative coupling, which is equally good from a theoretical point of view and which is straightforward to define. We shall return to our particular choice of coupling in sect. 5, after the basic ideas of the calculation have been made clear.

Let us choose a hyper-cubic lattice with spatial size L and time-like extent T much larger than L. Zero modes and other technical complications in small volumes [19, 20] can be avoided by taking twisted periodic boundary conditions in the space directions [21–23]. We then consider the force $F(r, L)$ between static quarks at distance r and define a renormalized coupling through

$$\bar{g}^2(L) = k\,\{r^2 F(r, L)\}_{r=L/4}, \tag{7}$$

where the proportionality constant k is chosen such that $\bar{g}^2(L) = g_0^2 + \text{O}(g_0^4)$. Since r/L is held fixed, the coupling depends on L alone and is, therefore, a running coupling.
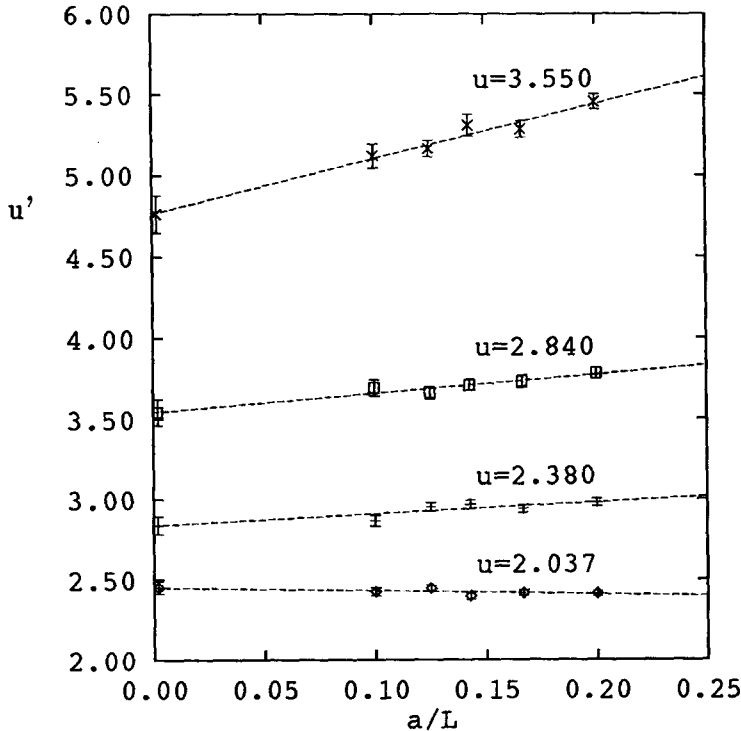
Figure 1. Computation of the step scaling function $\sigma(2, u)$. The continuum limit (left-most points) is reached by linear extrapolation.

The heavy quark force can be worked out in perturbation theory, either on the lattice or directly in the continuum limit using dimensional regularization. If we set

$$\alpha(q) = \bar{g}^2(L)/4\pi, \qquad q = 1/L, \tag{8}$$

the result obtained in the continuum theory may be written in the form of a series

$$\alpha_{\overline{MS}} = \alpha + k_1\alpha^2 + \dots \tag{9}$$

with purely numerical coefficients. Note that the only L–dependence on the l.h.s. of this equation arises through the argument $q = 1/L$. The $\overline{MS}$ coupling is defined as usual without reference to a finite volume.

Eq.(9) thus establishes a connection between finite and infinite volume physics. At first sight one may be surprised that such a simple connection exists, but it is really just a consequence of the

fact that the theory has only one bare parameter. Any two different renormalized couplings can therefore be expressed in terms of each other, independently of the context in which they were defined.

3. RENORMALIZATION GROUP

In the continuum limit the Callan-Symanzik β–function appropriate to our finite-volume scheme is defined through

$$\beta(\bar{g}) = -L\frac{\partial \bar{g}}{\partial L}. \tag{10}$$

From eq.(9) and the known perturbation expansion of the β–function in the $\overline{MS}$ scheme, we infer that

$$\beta(\bar{g}) \underset{\bar{g}\to 0}{\sim} -\bar{g}^3 \sum_{n=0}^{\infty} b_n \bar{g}^{2n}, \tag{11}$$
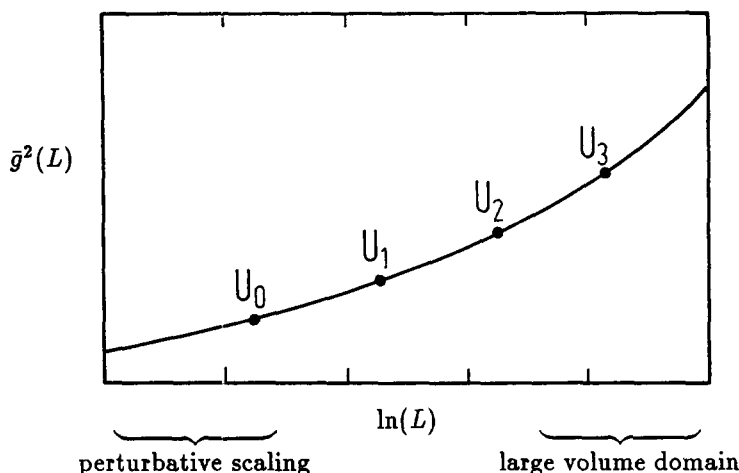
Figure 2. Qualitative behaviour of the running coupling as a function of the box size L.

where $b_0 = 11/24\pi^2$ and $b_1 = 17/96\pi^4$ are the usual universal coefficients.

More useful for our purposes is the step scaling function $\sigma(s, u)$, which tells us what happens to the coupling if the box size is changed by a factor s. So if we choose an initial value $u = \bar{g}^2(L)$ and integrate eq.(10) up to the scale sL, the resulting coupling is

$$\sigma(s, u) = \bar{g}^2(sL). \qquad (12)$$

In other words, the step scaling function is an integrated form of the β-function.

An important remark now is that $\sigma(s, u)$ is computable through numerical simulation of the lattice theory. For $s = 2$ the calculation proceeds as follows.

1. Choose u, the renormalized coupling at which the step scaling function is desired, and pick a lattice with L/a points on each spatial side.

2. Adjust the bare coupling g_0 until the renormalized coupling $\bar{g}^2(L)$ assumes the value u.

3. At the same value of g_0, simulate a lattice with twice the spatial size and compute $u' = $

$\bar{g}^2(2L)$. Following Symanzik it is possible to show that [17]

$$u' = \sigma(2, u) + O(a/L). \qquad (13)$$

4. Repeat the cycle (steps 1–3) for a range of L/a and determine $\sigma(2, u)$ by extrapolating u' to the continuum limit.

In fig. 1 some of our data are plotted to illustrate the procedure. At the larger values of u, the cut-off effects are quite significant, but the extrapolation to the continuum limit is smooth and does not cause any problem. It should be emphasized that no exceedingly large lattices are required in these calculations.

4. STRATEGY

The running coupling $\bar{g}^2(L)$ is expected to be a monotonically rising function of L as shown in fig. 2. At very small volumes the evolution of the coupling is accurately described by perturbation theory [eqs.(10),(11)]. Then there is a transition region and when L is larger than the fundamental correlation lengths in the theory, the behaviour of the coupling is determined by non-perturbative effects. Our aim is to compute $\bar{g}^2(L)$ for a range

of L connecting the perturbative scaling region with the low-energy regime.

The basic idea is simple. We consider the theory in the continuum limit and construct a sequence of couplings $u_0, u_1, \ldots, u_n$ recursively, starting from some initial value u_0 in the perturbative domain. Each step $u_i \rightarrow u_{i+1}$ corresponds to a change of scale by a factor 2, i.e. the recursion is defined through

$$u_{i+1} = \sigma(2, u_i). \tag{14}$$

As discussed above the step scaling function is computable through numerical simulation. Given the initial value u_0, the couplings $u_1, u_2, \ldots, u_n$ can thus be determined successively.

By construction there exist box sizes L_i such that

$$u_i = \bar{g}^2(L_i), \qquad L_i = 2^i L_0. \tag{15}$$

Although it is well-defined, the initial scale L_0 is not really known at this point. By performing a sufficient number of renormalization steps, we may however arrange that the terminal box size L_n is in the large volume range. It is then possible to make contact with the low-energy scales of the theory and to express L_n in units of the string tension K. Once this is achieved, all box sizes L_i will be known in physical units.

Returning to the high-energy end of the sequence of couplings, we may finally convert to the $\overline{\text{MS}}$ scheme, using perturbation theory [eq.(9)]. An important point to note is that in this last step of the calculation all reference to a finite volume drops out. One simply gets $\alpha_{\overline{\text{MS}}}$ at some momenta $q_i = 1/L_i$ given in units of the string tension. The running coupling in finite volume may hence be regarded as merely a technical device, which is needed to connect the high-energy regime with the fundamental low-energy scales of the theory.

5. DEFINITION OF $\bar{g}^2(L)$ REVISITED

In practice the number of renormalization steps that can be performed is limited by the statistical and extrapolation errors which incur when the step scaling function is computed. These errors accumulate during the recursion and must be

carefully traced. Another source of error is the conversion from the finite volume to the $\overline{\text{MS}}$ coupling. Perturbation theory is used here and even though the conversion is only performed at the largest momenta reached, the neglected higher-order corrections may not be small.

For accurate results we hence require a renormalized coupling with the following properties.

1. $\bar{g}^2(L)$ is non-perturbatively defined.

2. On any given lattice in the scaling region (with say $L/a \leq 32$), $\bar{g}^2(L)$ can be accurately computed through numerical simulation.

3. The cutoff effects one encounters when calculating the step scaling function are small.

4. A perturbative computation of $\bar{g}^2(L)$ up to two-loop order is possible with a reasonable effort.

It is our experience that these conditions are hard to fulfill simultaneously. In particular, couplings defined through Wilson loop expectation values are unlikely to provide a satisfactory solution to the problem. This is so because expectation values of large loops are difficult to compute numerically — the signal-to-noise ratio is exponentially decreasing with the size of the loops — and because the perturbation expansion of these quantities to order g_0^6 may require an unacceptable amount of work (there is a scaring number of Feynman diagrams and standard momentum space techniques do not apply).

We were thus led to consider more exotic possibilities and finally came up with the idea to define $\bar{g}^2(L)$ through the response of the system to a constant colour-electric background field. The details of the definition are complicated, especially on the lattice, and we refer the reader to refs.[17, 18] for a full account. Our aim here is to explain in simple terms, using a continuum notation, how to generate a background field and how to extract a renormalized coupling from the effective action of the background field.

Let us consider the euclidean space-time manifold sketched in fig. 3. The time coordinate x^0
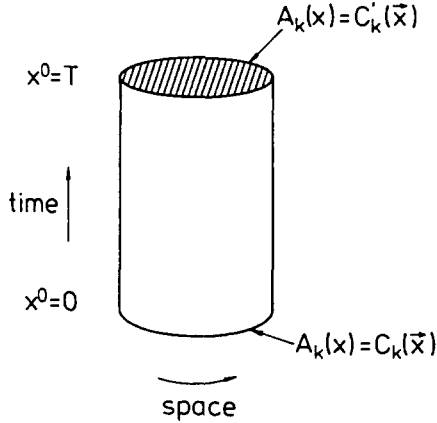
Figure 3. Space-time manifold and boundary conditions on the gauge field.

runs from 0 to T and space is assumed to be a torus of size L in all cartesian directions. Gauge fields are represented by periodic vector potentials $A_\mu(x)$ on $[0,T] \times \mathbb{R}^3$ (in the presence of background fields, zero modes and multiple minima of the action can be avoided and so there is no need to introduce twisted periodic boundary conditions here). At the boundaries of the space-time manifold we impose

$$A_k(x) = \begin{cases} C_k(\mathbf{x}) & \text{at } x^0 = 0, \\ C'_k(\mathbf{x}) & \text{at } x^0 = T, \end{cases} \qquad (16)$$

where C and C' are prescribed spatial gauge fields. They will be set to some particular values below.

The partition function of the system,

$$\mathcal{Z} = \int \mathrm{D}[A]\, e^{-S[A]}, \qquad (17)$$

involves the Yang-Mills action $S[A]$ and an integration over all gauge fields A with fixed boundary values C and C'. $\mathcal{Z}$ is a gauge invariant functional of the boundary fields and may be interpreted as the (euclidean) propagation kernel for going from the initial configuration C at time $x^0 = 0$ to the final configuration C' at $x^0 = T$.

In the weak coupling limit, which is also the semi-classical limit, the partition function is dominated by the field configuration B with least action

tion (we do not consider situations with several gauge inequivalent absolute minima). B is a solution of the Yang-Mills field equations and may be regarded as the classical background field induced by the boundary values at $x^0 = 0$ and $x^0 = T$.

If we choose

$$\begin{aligned} C_k(\mathbf{x}) &= \eta\, \tau_3 / iL, \\ C'_k(\mathbf{x}) &= (\pi - \eta)\, \tau_3 / iL, \end{aligned} \qquad (18)$$

where τ_3 is the third Pauli matrix and $0 < \eta < \pi$ a parameter, the induced background field is given by

$$\begin{aligned} B_0(x) &= 0, \\ B_k(x) &= \left[x^0 C'_k + (T - x^0) C_k \right] / T. \end{aligned} \qquad (19)$$

This field has the required boundary values and it is easy to verify that it satisfies the field equations. The absolute stability of B is guaranteed by a theorem proved in ref.[17].

The field tensor associated to B is independent of x and its magnetic components vanish, i.e. B is a constant colour-electric field. It is well-known that such fields are unstable in infinite volume [24]. This is perfectly consistent with our assertion above (that B is a minimal action configuration), because B converges to zero in the infinite volume limit.

The effective action of the background field, $\Gamma = -\ln \mathcal{Z}$, can be expanded in powers of the bare coupling g_0 by applying the saddle point integration method to the functional integral (17). The series has the form

$$\Gamma = g_0^{-2} \Gamma_0 + \Gamma_1 + g_0^2 \Gamma_2 + \dots \qquad (20)$$

with

$$\Gamma_0 = g_0^2 S[B] = 6(L/T)(\pi - 2\eta)^2. \qquad (21)$$

The one-loop contribution Γ_1 is a ratio of determinants of the fluctuation and the Faddeev-Popov operators, and at higher orders one has to evaluate Feynman diagrams with propagators and vertices that depend on the background field.

A detailed analysis now shows that the effective action (modulo a divergent additive constant) is power-counting renormalizable, without extra counterterms. Quantum field theories on

manifolds with boundaries in general require additional counterterms, constructed by integrating local operators over the boundary manifolds [25]. Such terms can be excluded here, because there are no candidate operators with the right symmetries and dimensions.

By differentiating with respect to the background field parameter η, any constant additive contribution to the effective action is removed and we conclude that

$$\Gamma' = \frac{\partial \Gamma}{\partial \eta} \tag{22}$$

is a renormalized quantity. Taking eqs.(20) and (21) into account, we are thus led to define a renormalized coupling through

$$\bar{g}^2 = \Gamma'_0 / \Gamma'. \tag{23}$$

We actually obtain a two-parameter family of couplings in this way, the parameters being T/L and η. For our numerical work we chose

$$T/L = 1 \quad \text{and} \quad \eta = \pi/4, \tag{24}$$

but there is no absolute necessity to stick to these values. At some point one may in fact be interested to consider different background fields to test the universality of the continuum limit.

6. RESULTS

Γ' is proportional to the expectation value of the colour-electric field operator at the boundaries of the space-time manifold. On the lattice this translates to an average of the time-like boundary plaquettes with an insertion of the Pauli matrix τ_3. It is straightforward to compute the required expectation value through numerical simulation, using a hybrid over-relaxed algorithm, for example.

Our results on the step scaling function $\sigma(2, u)$ are listed in table 1. As will become clear in the following, the couplings u (first column) are in the range of interest, approximately as $u_0, \ldots, u_3$ in fig. 2. For comparison the step scaling function as computed in perturbation theory, using the two-loop approximation for the β–function, is included in table 1. The agreement with the numerical data is almost perfect, except at the

Table 1
Values of the step scaling function

u	$\sigma(2, u)$	$\sigma(2, u)_{2-\text{loop}}$
2.037	2.45(4)	2.38
2.380	2.84(6)	2.86
2.840	3.54(8)	3.58
3.550	4.76(12)	4.83

lowest value of the coupling, where a 2σ deviation is observed. In a set of 4 independent measurements, this is not an unlikely event, and so it may well be that the discrepancy disappears when the statistics is enlarged.

Table 2
Running coupling as determined from table 1

L/L_{max}	$\bar{g}^2(L)$
1.000	4.765
0.499	3.55(6)
0.247	2.84(6)
0.123	2.38(5)
0.069	2.04(5)

Apart from small mismatches, the couplings in the first column of table 1 together with $u_4 = 4.765$ form a sequence $u_0, \ldots, u_4$ as described in sect. 4. In particular, between $u_0 = 2.037$ and u_4 there is a difference in scale of roughly a factor 16. A more precise analysis, tracing all errors and taking mismatches into account, yields the numbers quoted in table 2. As a unit of scale we here decided to take the box size L_{max} at which $\bar{g}^2 = u_4$. Note that the coupling is decreasing by more than a factor of 2 in the range of distances covered.

To relate L_{max} to the physical scales in infinite volume, we choose some fixed value of the bare coupling $\beta = 4/g_0^2$ where the string tension K is already known from studies on large lattices (see table 3; the data in the second column are from refs.[26, 27]). For a range of lattice sizes we then compute the renormalized coupling
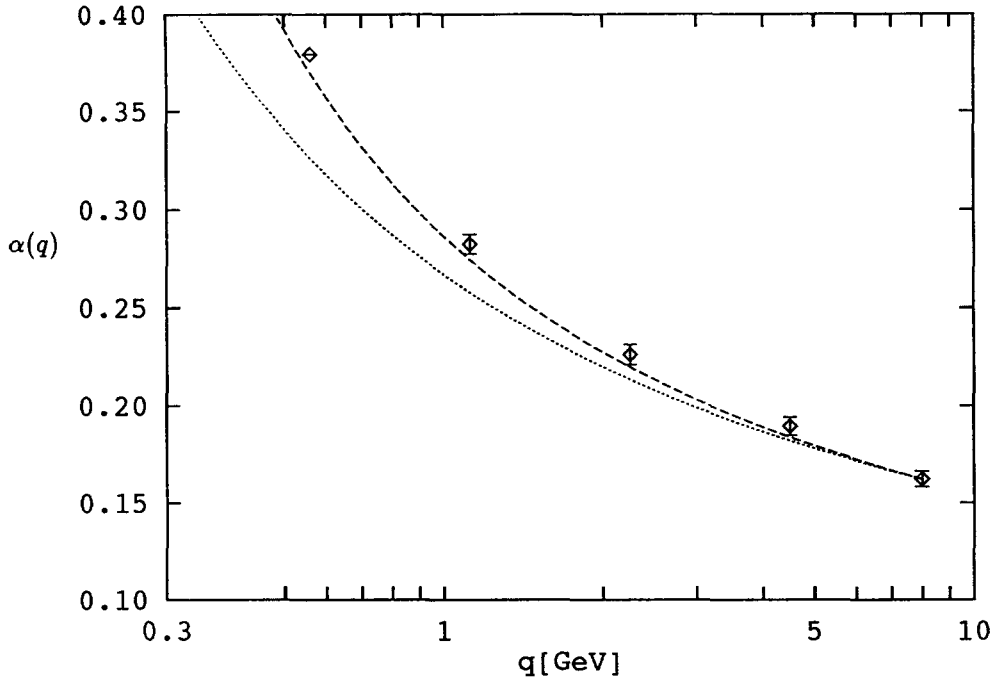
Figure 4. Comparison of numerically computed values of the running coupling (data points) with perturbation theory. The dashed (dotted) curve is obtained by integrating the evolution equation (10), starting at the right-most point and using the 2–loop (1–loop) formula for the β–function.

Table 3
Determination of L_{max} in physical units

β	$a\sqrt{K}$	L_{max}/a	$L_{max}\sqrt{K}$
2.70	0.101(1)	8.08(4)	0.820(9)
2.85	0.063(3)	11.98(7)	0.76(4)

and determine L_{max}/a by interpolation. When this number is multiplied with the string tension, the desired conversion factor $\rho = L_{max}\sqrt{K}$ is obtained.

As in the case of the step scaling function we expect that ρ approaches the continuum limit with a rate roughly proportional to a/L_{max}. We do in fact find a slight variation of ρ when β is increased from 2.70 to 2.85, but the effect is barely significant. The data are certainly insufficient to perform an extrapolation to the continuum limit. For the time being we shall, therefore,

take $\rho = 0.76$, the value closer to the continuum limit, and keep in mind that the total error on this number could be as large as 10%. If we set $\sqrt{K} = 425\,\mathrm{MeV}$ to convert to more physical units, we then deduce that $L_{max} = 0.35\,\mathrm{fm}$. The lower end of the range of box sizes covered by table 2 is hence roughly equal to 0.024 fm.

As shown in fig. 4 the evolution of the running coupling $\alpha(q)$ is well described by perturbation theory, down to very low energies. The error bars in this plot represent the statistical errors as given in table 2, but not the overall scale uncertainty discussed above. The latter amounts to a multiplication of the energy scale by a *constant* factor and so has no bearing on the scaling properties of the coupling.

At the highest energies reached, we can finally convert to the $\overline{\mathrm{MS}}$ scheme of dimensional regularization using the one-loop formula [17]

$$\alpha_{\overline{\mathrm{MS}}} = \alpha + 0.9433 \times \alpha^2 + \ldots \tag{25}$$

In particular, at $q = 20 \times \sqrt{K}$ we obtain

$$\alpha_{\overline{\mathrm{MS}}}(q) = 0.185 \pm 0.005 \pm 0.009. \tag{26}$$

The first error here is statistical, as inferred from table 2, while the second is an estimate of the total systematic error arising from a possible order α^3 correction in eq.(25) and the 10% scale uncertainty mentioned above.

With the presently available parallel computers the level of accuracy could be increased significantly at a reasonable cost. To keep the balance between statistical and systematic errors it is then necessary to extend the series (25) to the next order [28], and one should also use an O(a) improved action [17] to ease the extrapolation to the continuum limit.

7. CONCLUDING REMARKS

At first sight it seems unlikely that the strength of gluon interactions at high energies can be determined by studying the scaling behaviour of the theory in small and intermediate volumes. It is in fact well-known that the properties of the ground state change radically when the box size L becomes smaller than a fermi or so. One may thus be led to conclude that such studies are perhaps of technical interest, but have no relation to what is going on in real physics.

This argument overlooks two important facts. The first is that the renormalization of the theory does not depend on L. Renormalization constants, scaling functions and improvement coefficients may hence be computed in finite volume. This is obviously useful, since the corresponding calculations would be much more costly on physically large lattices. Asymptotic freedom is the other important property of the theory on which we rely. It implies that once the high-energy regime is reached, different renormalization schemes can be matched by perturbation theory. In particular, whether we probe the system by scattering gluons or through a finite volume eventually yields the same information, since the corresponding couplings can be expressed in terms of each other.

At some point contact with the fundamental scales in infinite volume must of course be made.

Lattices a few fermi wide are required here, but the renormalization group that we have set up saves us from requiring very small lattice spacings at the same time. Unmanageably large lattices are hence avoided.

Our study of the SU(2) theory reveals that the evolution of the renormalized coupling is accurately described by perturbation theory down to surprisingly small momenta (cf. fig. 4). To some extent this result is certainly dependent on the renormalization scheme employed. Nevertheless we may conclude that confinement, string formation and glueballs are in no way in contradiction with the high-energy regime, where weakly coupled gluons are the important degrees of freedom. One may have feared that there is a complicated transition region between the two regimes, but this is now definitely ruled out.

A similar study of the SU(3) Yang-Mills theory is currently under way. We estimate that about 3 times as much computer time is needed to attain the same statistical accuracy as in the SU(2) theory. We also believe that finite-size techniques of the type described here will be useful in QCD, not only to determine the running coupling, but also to study the renormalization of quark masses and the normalization of the axial currents (which is a non-trivial problem on the lattice).

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