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An exponentially increasing semiclassical spectral form factor for a class of chaotic systems

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Abstract. The spectral form factor $K(\tau)$ plays a crucial role in the understanding of the statistical properties of quantal energy spectra of strongly chaotic systems in terms of periodic orbits. It allows the semiclassical computation of those statistics that are bilinear in the spectral density $d(E)$, like the spectral rigidity $\Delta_3(L)$ and the number variance $\Sigma^2(L)$. Since Berry's work on the semiclassical approximation of the spectral rigidity in terms of periodic orbits, it is generally assumed that the periodic-orbit expression for the spectral form factor universally obeys $K(\tau) = 1$ for $\tau \gg 1$. Here we show that for a wide class of strongly chaotic systems, including billiards with Neumann boundary conditions and the motion on some Riemann surfaces, the asymptotic behaviour of the semiclassical spectral form factor $K(\tau)$ depends very sensitively on the averaging employed. A Gaussian averaging is preferable from a theoretical as well as from a numerical point of view to, for example, a rectangular averaging. However, we show in this paper that the Gaussian averaging leads in some cases to an asymptotic behaviour like $K(\tau) \sim e^{c\tau}$, where $c > 0$ depends only on the energy E at which the statistic is considered.

1. Introduction

A main issue in quantum chaology has been the study of statistical properties of quantal energy spectra of classically chaotic systems. Based on the examination of a variety of systems, it is now widely accepted that statistics measuring short- and medium-range correlations are universal and agree with the results of random-matrix theory [1–3]. Long-range correlations are non-universal and differ from system to system. A theoretical explanation for universal, as well as for non-universal, behaviour was given by Berry by examining semiclassical approximations for statistical quantities measuring two-point correlations [4, 5]. These approximations were obtained using the semiclassical approximation for the level density according to periodic-orbit theory [3].

The semiclassical results are conveniently described in terms of the spectral form factor $K(\tau)$ [5]. It is defined as the Fourier transform of the two-point correlation function of the oscillatory part $d_{osc}(E)$ of the spectral density $d(E) = \bar{d}(E) + d_{osc}(E)$, where $\bar{d}(E)$ is the mean level density.

For the semiclassical approximation of $K(\tau)$ the following results have been obtained. For small values of τ, $K(\tau)$ has peaks at times corresponding to the shortest periodic orbits and is thus non-universal. In a slightly higher range of τ these peaks become very dense,

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and smoothing over them yields a linearly increasing $K(\tau)$ in agreement with the results of random-matrix theory. For large values of τ, the smoothed $K(\tau)$ approaches the value 1.

Results for other frequently used statistical measures, like the spectral rigidity $\Delta_3(L)$ or the number variance $\Sigma^2(L)$, can be obtained from $K(\tau)$ by an integral transform. In this way one obtains a universal behaviour for $\Delta_3(L)$ and $\Sigma^2(L)$ in the lower L-range and a non-universal saturation for large values of L [4, 6]. This saturation also has consequences for many other statistics like the higher level spacings $P(k, s)$ [7].

The semiclassical results for $K(\tau)$, however, were obtained under certain assumptions on the convergence of the periodic-orbit sums as well as on the validity of the semiclassical approximation. In the present paper we demonstrate that for the case where all periodic orbits have the same phase factor χ_n, e.g. certain billiards with Neumann boundary conditions, these assumptions are not necessarily satisfied, and the behaviour of the semiclassical form factor $K(\tau)$ depends sensitively on the energy (and time) averaging used for its evaluation. For this reason, we have to choose carefully the way in which the semiclassical form factor is evaluated. A convenient choice is a Gaussian averaging, which leads to a periodic-orbit sum which allows an efficient summation of periodic orbits. In addition, the Gaussian averaging leads to an absolutely convergent periodic-orbit sum so that all problems concerning an analytical continuation of the periodic-orbit expression are absent. We will show, however, that in this case the semiclassical approximation to the spectral form factor increases exponentially for large values of τ if evaluated at fixed energy and with fixed smoothing parameters. We give an explanation of the origin of this exponential increase on the example of systems where Gutzwiller's trace formula is exact rather than only a semiclassical approximation. By using the trace formula, we give a rigorous proof of the exponential increase of the semiclassical $K(\tau)$ for these systems. We further show why the semiclassical sum rule of Berry, which was used for the derivation of the result $K(\tau) \to 1$ for large values of τ, cannot be applied in this case. At the end of the paper we describe a rectangular smoothing by which the exponential increase of the semiclassical $K(\tau)$ can be avoided.

2. The spectral form factor with Gaussian smoothing

$K(\tau)$ is defined as

$$K(\tau) := \frac{1}{\bar{d}(E)} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \, \langle d_{\mathrm{osc}}(E - \epsilon/2) d_{\mathrm{osc}}(E + \epsilon/2) \rangle \, \mathrm{e}^{2\pi \mathrm{i} \bar{d}(E) \epsilon \tau} \tag{1}$$

where the brackets $\langle \ldots \rangle$ denote a local averaging over an energy interval ΔE around energy E. The variable τ is the time measured in units of the Heisenberg time $T_{\mathrm{H}} = 2\pi\hbar\bar{d}(E)$. Universal behaviour is expected when $K(\tau)$ is evaluated in the semiclassical regime with $\bar{d}(E)^{-1} \ll \Delta E \ll E$, i.e. the energy interval ΔE contains many energy levels but is classically small and τ is large enough so that the corresponding classical motion fills the phase space uniformly.

In the following we restrict ourselves to the most transparent case of strongly chaotic systems, i.e. two-dimensional billiards with a positive Kolmogorov–Sinai entropy and periodic orbits which are all unstable and isolated. The periodic-orbit expression for $d_{\mathrm{osc}}(E)$ is given by [3]

$$d_{\mathrm{osc}}(E) = \frac{m}{2\pi\hbar p} \sum_{\{l_n\}} \sum_{k=-\infty}^{\infty}{}' A_{n,k} \exp\left(\frac{\mathrm{i}}{\hbar} p k l_n\right) \qquad A_{n,k} = \frac{g_n l_n \chi_n^k}{\sqrt{|2 - \mathrm{Tr}\, M_n^{|k|}|}} \tag{2}$$

where l_n, g_n, $\chi_n \in \{\pm 1, \pm \mathrm{i}\}$ and M_n denote the length of the periodic orbit, its multiplicity, phase factor and monodromy matrix, respectively. The momentum is $p = \sqrt{2mE}$. The phase factor $\chi_n = \exp(-\mathrm{i}\pi\nu_n/2)$ is determined by ν_n. The classical quantity ν_n is the maximal number of conjugate points along a primitive periodic orbit plus twice the number of reflections on those parts of the boundary where Dirichlet boundary conditions are required [3]. The prime at the k-sum counting positive and negative traversals denotes that $k = 0$ is excluded. In general, the periodic-orbit sum in (2) is not convergent. A semiclassical approximation for $K(\tau)$ is obtained by inserting (2) into (1). The ϵ-integration is then carried out by a stationary-phase approximation in which the actions in the exponent $pkl_n =: S_{n,k}(E)$ are expanded up to first order in energy. This leads to

$$K_{\mathrm{sc}}(\tau) = \frac{m}{2\pi\hbar p\bar{d}(E)} \bigg\langle \sum_{\{l_n\}} \sum_{k=-\infty}^{\infty}{}' \sum_{\{l_{n'}\}} \sum_{k'=-\infty}^{\infty}{}' A_{n,k} A^*_{n',k'} \times \exp\left(\frac{\mathrm{i}}{\hbar} p(kl_n - k'l_{n'})\right) \delta(\ell - \tfrac{1}{2}(kl_n + k'l_{n'})) \bigg\rangle \qquad (3)$$

where

$$\ell := \frac{2\pi\hbar p\bar{d}(E)}{m}\tau. \qquad (4)$$

The spectral form factor $K(\tau)$ is a quantity with huge fluctuations about its mean value. In its semiclassical approximation this is exemplified by the presence of the Dirac delta-functions. In order to get a useful quantity, $K(\tau)$ has to be smoothed in the variable τ, which we do with a Gaussian weight. This is equivalent to replacing the Dirac delta-functions in the above equation by a Gaussian $\delta(x) \to (\sqrt{2\pi}\Delta l)^{-1} \exp(-x^2/(2(\Delta l)^2))$. For the local average $\langle \ldots \rangle$ in (3) we also choose a Gaussian averaging with variance $(\Delta p)^2$ centred at momentum p. This has the advantage that for the special choice $\Delta l = \hbar/(2\Delta p)$ the periodic-orbit double sum can be reduced to the absolute square of a single sum [8]

$$K_{\mathrm{sc}}(\tau) = \frac{m\Delta p}{\sqrt{2\pi}\pi\hbar^2 p\bar{d}(E)} \left| \sum_{\{l_n\}} \sum_{k=-\infty}^{\infty}{}' A_{n,k} \exp\left(\frac{\mathrm{i}}{\hbar} pkl_n\right) \exp\left(-\frac{\Delta p^2}{\hbar^2}(\ell - kl_n)^2\right) \right|^2. \qquad (5)$$

Evaluations of the semiclassical spectral form factor have been carried out with this formula with good results [9]. The Gaussian smoothing has the additional advantage that the periodic-orbit sum in equation (5) is absolutely convergent since the exponential proliferation of the periodic orbits, being typical for chaotic systems, is suppressed by the Gaussian factor. We would also like to note that the derivation leading to (5) can be performed by dealing only with absolutely convergent sums in each step. This can be done by using a Gaussian-smoothed level density instead of the unsmoothed level density of equation (2), i.e. by applying a third Gaussian smoothing. This corresponds to a replacement of the amplitudes $A_{n,k}$ by $A_{n,k} \exp(-(\eta k l_n/2\hbar)^2)$ in the periodic-orbit sum [10, 11]. Equation (5) is then obtained in the limit $\eta \to 0$.

Now, let us consider the periodic-orbit sum in (5) in the case where all phase factors obey $\chi_n = +1$. In this case the average contribution of long orbits to the periodic-orbit sum can be estimated by using the asymptotic behaviour of the periodic orbits. Assume that the length spectrum $\{l_n\}$ is completely known up to a cut-off length $\mathcal{L}$. The periodic-orbit sum in (5) is then evaluated by summing over all periodic orbits with length $l_n \leqslant \mathcal{L}$ and taking into account the average contribution of the periodic orbits with length $l_n > \mathcal{L}$ by a remainder term $R(\mathcal{L}, p)$ [12]. This remainder term is obtained from the periodic-orbit sum in the following way. The denominator

$$\sqrt{|2 - \operatorname{Tr} M_n^{|k|}|}$$

is replaced by its asymptotic value $\exp(|k|u_n/2)$, where u_n is the stability exponent. The stability exponent $u_n = \lambda_n l_n$ can be approximated by $u_n \sim \overline{\lambda} l_n$, where $\overline{\lambda}$ is the metric entropy defined as the asymptotic mean of the Lyapunov exponents λ_n. Furthermore, only the $k = 1$ contribution of the k-sum is important asymptotically. Finally, the sum over primitive periodic orbits is replaced by an integral, taking into account that the number of periodic orbits with length shorter than l increases asymptotically as $N(l) \sim (e^{tl}/tl)$, where t is the topological entropy. One thus obtains for the remainder term

$$\sum_{l_n > \mathcal{L}} \frac{g_n l_n}{e^{u_n/2}} \exp\left(\frac{i}{\hbar} p l_n\right) \exp\left(-\frac{\Delta p^2}{\hbar^2}(\ell - l_n)^2\right)$$
$$\simeq \int_{\mathcal{L}}^{\infty} dl \, \exp\left(\left(t - \frac{\overline{\lambda}}{2}\right) l + \frac{i}{\hbar} p l - \frac{\Delta p^2}{\hbar^2}(\ell - l)^2\right) := R(\mathcal{L}, p) \tag{6}$$

leading to

$$R(\mathcal{L}, p) = \frac{\sqrt{\pi}\hbar}{2\Delta p} \exp\left(\left(\frac{\hbar\rho}{2\Delta p}\right)^2 + \rho\ell\right) \operatorname{erfc}\left(\frac{\Delta p}{\hbar}(\mathcal{L} - \ell) - \frac{\hbar\rho}{2\Delta p}\right) \tag{7}$$

with $\rho := t - (\overline{\lambda}/2) + i(p/\hbar)$. $R(\mathcal{L}, p)$, together with the length spectrum $\{l_n\}$ up to $\mathcal{L}$, allows the computation of $K_{sc}(\tau)$. For fixed parameters p, Δp and $\mathcal{L}$, the remainder term $R(\mathcal{L}, p)$ is a function of τ being related to the length ℓ by (4). For a given τ (or ℓ), only periodic orbits with lengths kl_n near ℓ contribute to the periodic-orbit sum in (5). This can be inferred from the Gaussian factor in the periodic-orbit sum having a width $\hbar/(\sqrt{2}\Delta p)$. This implies that the remainder term $R(\mathcal{L}, p)$ vanishes for $\tau \ll \hat{\tau}$ with

$$\hat{\tau} := \frac{m\mathcal{L}}{2\pi\hbar p \bar{d}(E)} \tag{8}$$

corresponding to lengths kl_n much smaller than the cut-off length $\mathcal{L}$. This behaviour is clearly visible in figure 1, where $R(\mathcal{L}, p)$ is shown as a function of τ for the system discussed below. In the neighbourhood of $\hat{\tau}$ the remainder term $R(\mathcal{L}, p)$ shows large oscillations compensating the large oscillations of the periodic-orbit sum due to the sharp cut-off [12]. For $\tau \gg \hat{\tau}$ the periodic-orbit sum is approximated by the remainder term alone since in this case the periodic orbits having lengths kl_n smaller than $\mathcal{L}$ yield no contribution.

In order to derive the mean behaviour of $K_{sc}(\tau)$ we now consider the remainder term (7) in the region $\tau \gg \hat{\tau}$ where it accounts for the full periodic-orbit sum. In this range of τ, the real part of the argument of the error function takes on large negative values so that the asymptotic behaviour of the error function can be used. One then obtains the result

$$R(\mathcal{L}, p) \simeq \frac{\sqrt{\pi}\hbar}{\Delta p} \exp\left(\left(\frac{\hbar\rho}{2\Delta p}\right)^2 + \rho\ell\right) \tag{9}$$

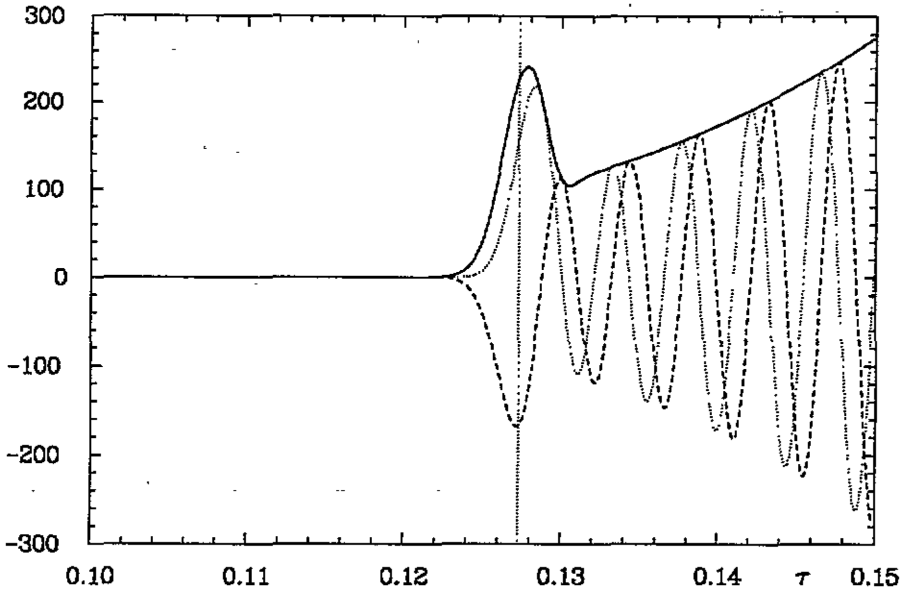

Figure 1. The remainder term (7) is shown for the values $p = 15$, $\Delta p = 5$ and $\mathcal{L} = 12$ as a function of τ related to ℓ by (4). The full curve represents the modulus, the dotted and the broken curves the real and imaginary parts, respectively. The dotted vertical line indicates the value of $\hat{\tau}$ corresponding to the cut-off length $\mathcal{L}$.

which is independent of the cut-off length $\mathcal{L}$. Since $\hat{\tau} \to 0$ in the semiclassical limit, $p \to \infty$ and the approximation (9) holds for all $\tau > 0$. With (9) the asymptotic behaviour of the spectral form factor $K_{sc}(\tau)$ is obtained

$$\overline{K_{sc}}(\tau) = C(p, \Delta p) \exp\left(2\left(t - \frac{\overline{\lambda}}{2}\right)\frac{2\pi\hbar p\bar{d}(E)}{m}\tau\right) \tag{10}$$

with

$$C(p, \Delta p) := \frac{m}{\sqrt{2\pi}\, p\Delta p\bar{d}(E)} \exp\left(\frac{\hbar^2(t - \overline{\lambda}/2)^2 - p^2}{2\Delta p^2}\right). \tag{11}$$

This is the main result of this paper which shows that the spectral form factor $K_{sc}(\tau)$ increases exponentially with τ if all phase factors χ_n are equal to one and $t > \overline{\lambda}/2$. This behaviour is absent when the phase factors χ_n in the periodic-orbit sum in (5) take on values from the set $\{\pm 1, \pm i\}$ (or $\{\pm 1\}$) with no correlation and with the same probability. Then neighbouring terms, with respect to a sorted-length spectrum, cancel. If all $\chi_n = +1$, no such cancellation can occur. The phases $\exp((i/\hbar)pkl_n)$ in (5) can only lead to a cancellation if neighbouring periodic orbits differ enough in length so that their phases can be considered random. However, because of the exponential proliferation of the number $N(l)$ of periodic orbits, this length difference decreases exponentially with increasing length, which implies that for long neighbouring orbits the phases are nearly identical which in turn leads to the behaviour (10).

The asymptotical exponential increase of $K_{sc}(\tau)$ sets a limit to the values of τ for which the semiclassical approximation $K_{sc}(\tau)$ can be expected to give a good approximation to the spectral form factor of the energy spectrum. In order that the exponential increase can be neglected one has to require that

$$\tau \ll \frac{mp}{(t - \overline{\lambda}/2)8\pi\hbar\bar{d}(E)\Delta p^2}. \tag{12}$$

One can perform the semiclassical limit $p \to \infty$ in such a way that the upper limit for τ in equation (12) goes to infinity. For that purpose one has to require that the width Δp satisfies $\Delta p/\sqrt{p} \to 0$ as $p \to \infty$.

However, for finite p and Δp it is also possible to extend the limit of validity of the semiclassical approximation. One can regularize the periodic-orbit sum by subtracting the mean behaviour equation (9) from the periodic-orbit sum in equation (5). In the next section we show that in the case of hyperbolic octagons this regularization yields the spectral form factor $K(\tau)$ from which the contribution of the zero-mode $E_0 = 0$ has been subtracted. Since this omission of a single quantal level does not change the statistical properties of the energy spectrum $\{E_n\}$, this regularized spectral form factor can be used for evaluating quantities like the number variance $\Sigma^2(L)$ or the spectral rigidity $\Delta_3(L)$.

3. The spectral form factor for hyperbolic octagons

We now consider an asymmetric hyperbolic octagon—a strongly chaotic system which describes the motion on a compact Riemann surface, of genus two, with constant negative curvature. (For an introduction to these systems see [3].) The only symmetry of an asymmetric octagon is the parity symmetry. The periodic-orbit theory can be formulated for each symmetry class separately [13], where the positive-parity class is the interesting case since all its phase factors are $\chi_n = +1$. For simplicity, we use from now on dimensionless units $\hbar = 2m = 1$. The Lyapunov exponents are given by $\lambda_n = 1$ and the metric entropy $\overline{\lambda}$ and the topological entropy t are both unity. In each symmetry class the mean level density is asymptotically $\bar{d}(E) = \frac{1}{2}$ and the quantal levels are distributed according to random-matrix theory with respect to short-range correlations [7].

For this system we compare numerically the asymptotic expression $\overline{K_{sc}}(\tau)$ (equation (10)) with the spectral form factor $K_{sc}(\tau)$ (computed from the length spectrum $\{l_n\}$ up to a cut-off length $\mathcal{L}$ including the remainder term (7)). Figure 2 shows $K_{sc}(\tau)$ for $p = 15$, $\Delta p = 5$ and $\mathcal{L} = 12$ corresponding to $\hat{\tau} = 0.1273\ldots$ (indicated as the dotted vertical line). The spectral form factor $K_{sc}(\tau)$ computed from the length spectrum up to $\mathcal{L}$ including the remainder term $R(\mathcal{L}, p)$ is shown as a full curve. The asymptotic mean (10) is displayed as a broken curve and good agreement is observed. Moreover, $K_{sc}(\tau)$ is significantly larger than the value already known for $\tau \simeq 0.11$. In order to avoid the impression that the exponential behaviour is enforced by the remainder term alone, an evaluation of (5) is shown (chain curve) where, in contrast to the correct evaluation (full curve), the periodic-orbit sum up to $\mathcal{L} = 12$ has been omitted, i.e. only the contribution of the remainder term $R(\mathcal{L}, p)$ is shown. For $\tau > 0.125$ one observes a non-vanishing contribution which comes, however, as no surprise since $R(\mathcal{L}, p)$ is practically zero for smaller values of τ, as can be seen in figure 1. Thus the exponential behaviour up to $\tau \simeq 0.125$ arises solely from the

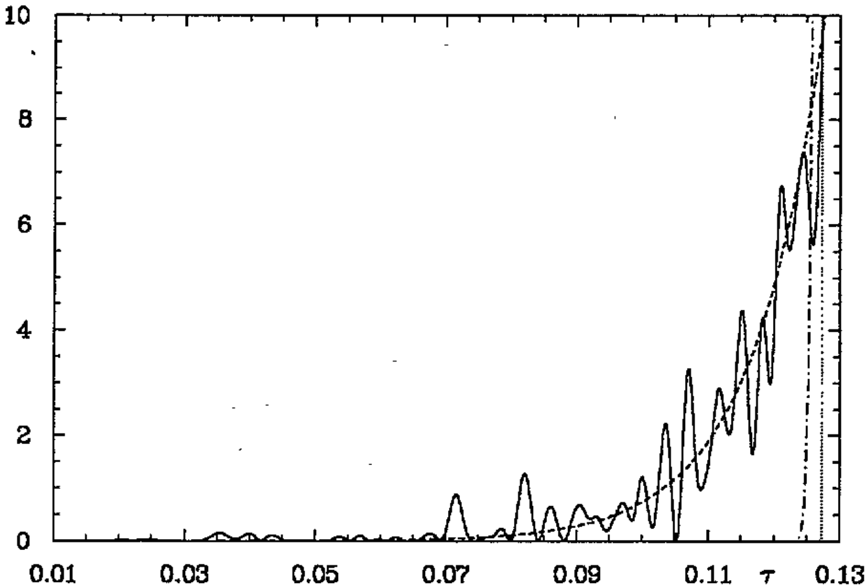

Figure 2. A comparison between the asymptotic behaviour (10) (broken curve) and the direct evaluation, based on the periodic-orbit expression (5) (full curve), is shown for the positive parity class of a hyperbolic octagon. The values $p = 15$, $\Delta p = 5$ and $\mathcal{L} = 12$ have been used. The dotted vertical line indicates the values of $\hat{\tau}$ corresponding to $\mathcal{L} = 12$. The chain curve is the result obtained for the above values but where the periodic-orbit sum up to $\mathcal{L} = 12$ is omitted.

contribution of the periodic orbits for the parameter values chosen above. This numerical example illustrates the exponential increase of $K_{sc}(\tau)$ well.

In the case of hyperbolic octagons, the periodic-orbit theory is exact and not only a semiclassical approximation since it is identical to the Selberg trace formula. This allows a further rigorous proof and an interpretation of the exponential behaviour of $K_{sc}(\tau)$ which we would like to discuss now. In [13] the Selberg trace formula has been derived for a given parity class which reads for a hyperbolic octagon (genus $g = 2$)

$$\sum_{n=0}^{\infty} h(p_n^{\pm}) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}p\, p \tanh(\pi p) h(p) \pm \frac{3}{4}\int_{-\infty}^{\infty} \mathrm{d}p\, \frac{h(p)}{\cosh \pi p} + \sum_{\{l_n\}} \sum_{k=1}^{\infty} \frac{g_n \chi_n^{\pm k} l_n}{2\sinh(kl_n/2)} g(kl_n) \tag{13}$$

where units $\hbar = 2m = 1$ are used. Here $p_n^{\pm}$ denotes the momenta corresponding to the energy eigenvalues $E_n^{\pm} = p_n^{\pm 2} + \frac{1}{4}$ of the two parity classes. The function $h(p)$ has to be even and holomorphic in a strip $|\,\mathrm{Im}\, p| \leqslant \frac{1}{2} + \epsilon$, $\epsilon > 0$. Furthermore, $h(p)$ has to decrease faster than $|p|^{-2}$ for $|p| \to \infty$, but otherwise it is arbitrary. Under these assumptions all sums and integrals involved are absolutely convergent. The Fourier transform of $h(p)$ is denoted by

$$g(x) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}px} h(p).$$

In the following we will consider only the positive-parity class for which $\chi_n^+ = +1$ and we will drop the superscript $+$.

The trace formula (13) can be used in order to express the periodic-orbit sum in (5) by a sum over the quantal energies E_n. To achieve this we choose

$$h(p') = \frac{\sqrt{\pi}}{\Delta p}\left\{\exp\left(-\frac{1}{4\Delta p^2}(p'-p)^2 - \mathrm{i}(p'-p)\ell\right) + \exp\left(-\frac{1}{4\Delta p^2}(p'+p)^2 + \mathrm{i}(p'+p)\ell\right)\right\} \tag{14}$$

which fulfils all conditions stated after (13). The Fourier transform of this $h(p')$ is given by

$$g(x) = \exp(-\Delta p^2(\ell - x)^2 + \mathrm{i}px) + \exp(-\Delta p^2(\ell + x)^2 - \mathrm{i}px). \tag{15}$$

With this function, the periodic-orbit sum in (13) is identical to the periodic-orbit sum in (5) and thus the Selberg trace formula can be used in order to express the periodic-orbit sum in terms of the quantal energy spectrum. In this way one obtains the following exact expression for the semiclassical form factor $K_{sc}(\tau)$ of equation (5)

$$K_{sc}(\tau) = \frac{\Delta p}{\sqrt{2\pi}\pi p}\left|\sum_{n=0}^{\infty} h(p_n) - \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}p'\, p' \tanh(\pi p') h(p') - \frac{3}{4}\int_{-\infty}^{\infty}\mathrm{d}p'\,\frac{h(p')}{\cosh \pi p'}\right|^2 \tag{16}$$

where dimensionless units have been used. The asymptotic behaviour of equation (10) can now also be obtained from (16). First one observes that the integrals in (16) are bounded for real ℓ so that they are not responsible for the exponential behaviour. Furthermore, the sum over the quantal energies is also bounded if the momenta p_n are real which is the case for $E_n \geqslant \frac{1}{4}$ because of $E_n = p_n^2 + \frac{1}{4}$. For the zero-mode $E_0 = 0$ one obtains the momentum $p_0 = \mathrm{i}/2$. This yields the crucial contribution to the exponential behaviour for $\ell \to \infty$

$$K_{sc}(\tau) \sim \frac{\Delta p}{\sqrt{2\pi}\pi p}|h(p_0)|^2 \simeq \frac{\mathrm{e}^{((1/4)-p^2)/2\Delta p^2}}{\sqrt{2\pi}\, p\Delta p}\mathrm{e}^{\ell} \tag{17}$$

which is identical to the result (10) for the asymmetric octagon. This derivation shows clearly that the exponential behaviour of $K_{sc}(\tau)$ is enforced by the term arising from the zero-mode $E_0 = 0$. The zero-mode is due to the constant eigenfunction having positive parity and thus belonging to the case where all phase factors are $\chi_n = +1$. Such a zero-mode is also always present in two-dimensional billiards with Neumann boundary conditions. The fact that a single level ($E_0 = 0$) has such a dramatic effect may appear very surprising, but this is not the first time that the zero-mode has played an important role in the context of the Selberg trace formula. A further example is the asymptotic behaviour of the weighted classical staircase $\widehat{N}(l) := \sum_{l_n \leqslant l} g_n \chi_n$, which is identical to $N(l)$ in the case $\chi_n = +1$. In the case $\chi_n = +1$, the zero-mode is responsible for the exponential increase of $\widehat{N}(l) = N(l) \sim \mathrm{e}^l/l$, as can be seen in the periodic-orbit formula which expresses $\widehat{N}(l)$ in terms of the quantal energies [14]. A periodic-orbit formula for $N(l)$ does not exist if the phase factor χ_n is not restricted to the value $+1$. In the general case, where the phase factors χ_n behave randomly, one expects $\widehat{N}(l)$ to fluctuate around zero instead of increasing exponentially. Of course, in this case $N(L)$ also increases exponentially due to the non-vanishing topological entropy.

The only approximation that was made in the derivation of the semiclassical $K_{sc}(\tau)$ for the octagon was the stationary-phase approximation of the integral in equation (1). This means that this stationary-phase approximation is the origin of the exponential increase since the quantum mechanical $K(\tau)$, which is evaluated from the quantal energies, does not have an exponential increase. The effect of the stationary-phase approximation can be illustrated further if one considers the Gaussian-smoothed spectral form factor for the *momentum* spectrum $\{p_n = \sqrt{E_n - \frac{1}{4}}\}$, which is defined by ($\Delta p\, \Delta\ell = \frac{1}{2}$)

$$K_p(\tau) := \lim_{\eta\to 0} \frac{1}{\bar{d}(p)} \int_{-\infty}^{\infty} \mathrm{d}l' \, \frac{1}{\sqrt{2\pi}\,\Delta l} \exp\left(-\frac{1}{2}\Delta l^2 (l' - \ell)^2\right) \times \int_{-\infty}^{\infty} \mathrm{d}p' \, \frac{1}{\sqrt{2\pi}\,\Delta p} \exp\left(-\frac{1}{2\Delta p^2}(p' - p)^2\right) \times \int_{-\infty}^{\infty} \mathrm{d}\epsilon \, d_{\mathrm{osc}}^{\eta}(p' - \epsilon/2) d_{\mathrm{osc}}^{\eta}(p' + \epsilon/2) \mathrm{e}^{\mathrm{i}\epsilon l'} \tag{18}$$

with $\ell = 2\pi\bar{d}(p)\tau$. $\bar{d}(p)$ is the mean momentum density which is asymptotically given by $\bar{d}(p) = p$. $d^{\eta}(p)$ is the Gaussian-smoothed momentum density. It is identical to the left-hand side of the Selberg trace formula (13) with the choice

$$\tilde{h}(p') = \frac{1}{\sqrt{\pi}\eta} \left[\exp\left\{ -\frac{(p - p')^2}{\eta^2} \right\} + \exp\left\{ -\frac{(p + p')^2}{\eta^2} \right\} \right]. \tag{19}$$

The Fourier transform of $\tilde{h}(p')$ is

$$\tilde{g}(x) = \frac{1}{\pi} \cos(px) \exp\left\{ -\frac{\eta^2 x^2}{4} \right\}. \tag{20}$$

Now, one expresses $d_{\mathrm{osc}}^{\eta}(p)$ in equation (18) by its periodic-orbit sum and evaluates the integrals, which can be done without approximation. One then finds that the spectral form factor for the momentum spectrum $K_p(\tau)$ is exactly identical to the semiclassical form factor $K_{sc}(\tau)$ of equation (5). That is, the effect of the stationary-phase approximation is to replace the form factor of the energy spectrum by the form factor of the momentum spectrum and since the momentum spectrum contains an imaginary eigenvalue this leads to the exponential increase of $K_{sc}(\tau)$ for large values of τ.

The explicit expression of $K_{sc}(\tau)$, in terms of the momentum eigenvalues in equation (16), gives the possibility to extend the application of the semiclassical approximation beyond the restriction (12). This can be done by subtracting the contribution $h(p_0) \simeq R(\mathcal{L}, p)$ (see equation (9)) of the zero mode from the periodic-orbit sum. In this way it is possible to evaluate the spectral form factor over the whole range of τ values. Preliminary numerical results indicate that the spectral rigidity $\Delta_3(L)$ can be computed from such a regularized spectral form factor $K(\tau)$ [15].

The imaginary momentum eigenvalue p_0 is also the reason why Berry's semiclassical sum rule cannot be applied in this case. This sum rule was used for the derivation of the asymptotic behaviour $K_{sc}(\tau) \to 1$ for $\tau \gg 1$. In the following we will shortly discuss the derivation of the semiclassical sum rule according to section 5.4. in [5].

Consider a Gaussian-smoothed delta peak

$$\delta_\eta(p) = \frac{1}{\sqrt{\pi}\eta} \exp\left\{-\frac{p^2}{\eta^2}\right\}. \tag{21}$$

The delta function is recovered by

$$\delta(p) = \lim_{\eta\to 0} \sqrt{2\pi}\eta\delta_\eta^2(p). \tag{22}$$

Assuming that the momentum spectrum has no degeneracies, one further obtains

$$\langle d(p)\rangle = \lim_{\eta\to 0} \sqrt{2\pi}\eta\langle d^\eta(p)^2\rangle. \tag{23}$$

Only the oscillatory contribution to the momentum density $d(p)$ will contribute on the right-hand side in the limit $\eta \to 0$. Substituting the periodic-orbit expression into equation (23) by using the Selberg trace formula with the functions $\tilde{h}(p')$ and $\tilde{g}(x)$ of equations (19) and (20), one obtains

$$\langle d(p)\rangle = \lim_{\eta\to 0} \frac{\eta}{\sqrt{2\pi}2\pi}\Bigg\langle \sum_{\{l_n\}} \sum_{k=-\infty}^{\infty}{}' \sum_{\{l_{n'}\}} \sum_{k'=-\infty}^{\infty}{}' A_{n,k}A_{n',k'} \exp\{\mathrm{i}p(kl_n - k'l_{n'})\}$$
$$\times \exp\left\{-\frac{\eta^2}{4}(k^2l_n^2 + k'^2l_{n'}^2)\right\}\Bigg\rangle \tag{24}$$

where $A_{n,k} = l_n(2\sinh(kl_n/2))^{-1}$. Equation (24) is the semiclassical sum rule. In order to obtain the asymptotic behaviour of $K_{\mathrm{sc}}(\tau)$ one approximates $k^2l_n^2 + k'^2l_{n'}^2$ by $\frac{1}{2}(kl_n + k'l_{n'})^2$ since the difference $\frac{1}{2}(kl_n - k'l_{n'})^2$ is assumed to be small for pairs of orbits which give a significant contribution to the double sum after averaging over momentum p. Then the double sum over periodic orbits can be expressed by an integral over the semiclassical form factor $K_{\mathrm{sc}}(\tau)$:

$$\langle d(p)\rangle \approx \lim_{\eta\to 0} \sqrt{2\pi}\eta\bar{d}(p)^2 \int_{-\infty}^{\infty} \mathrm{d}\tau\, K_{\mathrm{sc}}(\tau) \exp\{-2\eta^2\bar{d}(p)^2\tau^2\pi^2\}. \tag{25}$$

The asymptotic behaviour $K_{\mathrm{sc}}(\tau) \to 1$ for $\tau \gg 1$ then follows from this integral equation.

In the presence of an imaginary momentum eigenvalue this argumentation cannot, however, be applied because then the limit $\lim_{\eta\to 0}\sqrt{2\pi}\eta\langle d^\eta(p)^2\rangle$ becomes infinite, as will be shown in the following. Substituting the momentum spectrum into equation (23) and using again a Gaussian averaging for the averaging over momentum one obtains

$$\lim_{\eta\to 0}\sqrt{2\pi}\eta\langle d^\eta(p)^2\rangle = \lim_{\eta\to 0}\sqrt{2\pi}\eta \int_{-\infty}^{\infty} \mathrm{d}p'\, \frac{1}{\sqrt{2\pi}\Delta p} \exp\left\{-\frac{1}{2\Delta p^2}(p'-p)^2\right\}$$
$$\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s_n=\pm 1}\sum_{s_m=\pm 1} \frac{1}{\pi\eta^2} \exp\left\{-\frac{1}{\eta^2}(p'-s_np_n)^2\right\} \exp\left\{-\frac{1}{\eta^2}(p'-s_mp_m)^2\right\}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s_n=\pm 1}\sum_{s_m=\pm 1} \frac{1}{\sqrt{2\pi}\Delta p} \exp\left\{-\frac{1}{4\Delta p^2}(p-s_np_n)^2\right\}$$
$$\times \exp\left\{-\frac{1}{4\Delta p^2}(p-s_mp_m)^2\right\} \lim_{\eta\to 0} \exp\left\{-\frac{1}{\eta^2}\frac{2\Delta p^2}{\eta^2+4\Delta p^2}(s_np_n - s_mp_m)^2\right\}. \tag{26}$$

All terms in this multiple sum are positive and the terms with $s_n = 1$, $s_m = -1$, $p_n = p_m = \mathrm{i}/2$ and $s_n = -1$, $s_m = 1$, $p_n = p_m = \mathrm{i}/2$ go to infinity if $\eta \to \infty$. For this reason the sum rule cannot be applied.

4. The rectangular smoothing

It is possible to define a form factor for the momentum spectrum which does not have an exponential increase. For this purpose one has to use an averaging in p and a smoothing in τ that cuts off the contribution of the zero mode. This can be achieved, for example, by a rectangular smoothing in p and a cut-off of the Fourier transform over ϵ at $\pm\sigma$. This corresponds to a smoothing in ℓ (or τ) with a weight factor $\sin(\sigma(\ell - \ell'))/(\pi(\ell - \ell'))$. In this case, the form factor $K_p(\tau)$ has no contribution from the zero mode p_0 if it is evaluated in the semiclassical regime. However, one can show that for this choice of the smoothing, the periodic-orbit expression for $K_p(\tau)$ is divergent.

A solution to this problem is to work with a Gaussian-smoothed momentum density, for which the width at the peaks can be chosen to be arbitrarily small, but not equal to zero. It is, however, not possible to choose the same width of the Gaussian peaks η for all values of τ for the following reason: if τ is increased, the Fourier transform in the variable ϵ contains information about increasingly finer details of the momentum density. In order that the Fourier transform is not influenced by the finite width of the peaks at the momentum eigenvalues one has to decrease η while increasing τ. A proper choice is $\eta = \eta_0/\ell$, where η_0 is chosen to be very small. The second reason for choosing a τ-dependent parameter η is the fact that the Gaussian peak at the zero mode p_0 will give a small contribution to the spectral form factor, which eventually, for very large values of τ, will lead to an exponential increase of $K_p(\tau)$ if η is chosen to be constant. The choice $\eta = \eta_0/\ell$ suppresses the contribution of the zero mode for all values of τ.

In this way one defines a spectral form factor for the momentum density with rectangular smoothing as

$$K_p^r(\tau) := \frac{1}{\bar{d}(p)} \int_{-\sigma}^{\sigma} \mathrm{d}\epsilon\, \mathrm{e}^{2\pi\mathrm{i}\bar{d}(p)\epsilon\tau} \frac{1}{\Delta p} \int_{p-\Delta p/2}^{p+\Delta p/2} \mathrm{d}p'\, d_{\mathrm{osc}}^{\eta}\left(p' - \frac{\epsilon}{2}\right) d_{\mathrm{osc}}^{\eta}\left(p' + \frac{\epsilon}{2}\right) \tag{27}$$

where $\eta = \eta_0/\ell$ and $\ell = 2\pi\bar{d}(p)\tau$. $d^{\eta}(p)$ is the Gaussian-smoothed momentum density which corresponds to the left-hand side of the Selberg trace-formula equation (13), evaluated with the function $\tilde{h}(p')$ of equation (19). The variable σ is chosen to be of the same order of magnitude as Δp. As will be demonstrated later (equation (30)), this has the effect that the spectral form factor has only contributions from momentum eigenvalues which lie in an interval of the order of magnitude of Δp around p.

The expression for $K_p^r(\tau)$ in terms of the momentum eigenvalues $\{p_n\}$ is obtained by replacing $d_{\mathrm{osc}}^{\eta}(p)$ by $d^{\eta}(p) - \bar{d}^{\eta}(p)$, inserting the Gaussian-smoothed momentum density and performing the integrals. The result is given by

$$K_p^r(\tau) \approx -2\pi\bar{d}(p)\frac{\sin(\sigma\ell)}{\pi\ell} + \frac{1}{4\bar{d}(p)\Delta p} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{s_n=\pm1}\sum_{s_m=\pm1} \exp\left\{-\mathrm{i}\ell(s_n p_n - s_m p_m) - \frac{\eta^2\ell^2}{2}\right\}$$
$$\times\left[\mathrm{erf}\left\{\frac{1}{\sqrt{2}\eta}(\sigma + s_n p_n - s_m p_m) - \frac{\mathrm{i}\eta\ell}{\sqrt{2}}\right\}\right.$$

$$+ \operatorname{erf}\left\{\frac{1}{\sqrt{2}\eta}(\sigma - s_n p_n + s_m p_m) + \frac{\mathrm{i}\eta\ell}{\sqrt{2}}\right\}\Bigg]$$

$$\times \left[\operatorname{erf}\left\{\frac{\sqrt{2}}{\eta}\left(\frac{\Delta p}{2} + p - \frac{s_n p_n + s_m p_m}{2}\right)\right\}\right.$$

$$\left. + \operatorname{erf}\left\{\frac{\sqrt{2}}{\eta}\left(\frac{\Delta p}{2} - p + \frac{s_n p_n + s_m p_m}{2}\right)\right\}\right]. \tag{28}$$

In this equation the quadruple sum is obtained from the integration over the term $d^\eta(p'-\epsilon/2)d^\eta(p'+\epsilon/2)$. The first term on the right-hand side of equation (28) is obtained by approximating

$$\frac{1}{\Delta p}\int_{p-\Delta p/2}^{p+\Delta p/2} \mathrm{d}p' \left[\bar{d}^\eta\left(p'-\frac{\epsilon}{2}\right)\bar{d}^\eta\left(p'+\frac{\epsilon}{2}\right) - d^\eta\left(p'-\frac{\epsilon}{2}\right)\bar{d}^\eta\left(p'+\frac{\epsilon}{2}\right)\right.$$
$$\left. - \bar{d}^\eta\left(p'-\frac{\epsilon}{2}\right)d^\eta\left(p'+\frac{\epsilon}{2}\right)\right] \approx -[\bar{d}(p)]^2. \tag{29}$$

For this approximation it was assumed that $\Delta p \ll p$ and $|\epsilon| \leqslant \sigma \ll p$. In equation (28) the magnitude of the arguments of the error functions is very large, since $\eta = \eta_0/\ell$ is very small (if ℓ is not too small). For this reason one can approximate the error functions by their asymptotic approximation $\operatorname{erf}(z) \approx 1 - \exp(z^2)/(\sqrt{\pi}z)$, $z \to \infty$, $|\arg z| < 3\pi/4$. One then finds that only the momentum eigenvalues which satisfy

$$p - \tfrac{1}{2}\Delta p < \tfrac{1}{2}(p_n + p_m) < p + \tfrac{1}{2}\Delta p \qquad \text{and} \qquad -\sigma < p_n - p_m < \sigma \tag{30}$$

give a significant contribution to the quadruple sum. All other contributions can be neglected. (We assume that there are no pairs of momentum eigenvalues whose difference is equal to $\pm\sigma$ or whose sum is equal to $2p \pm \Delta p$.) For that reason, $K_p^r(\tau)$ is given approximately by

$$K_p^r(\tau) \approx -2\pi\bar{d}(p)\frac{\sin(\sigma\ell)}{\pi\ell} + \frac{1}{\bar{d}(p)\Delta p}\sum_{m,n=0}^{\infty}{}^{R} \exp\{-\mathrm{i}\ell(s_n p_n - s_m p_m)\} \tag{31}$$

where the double sum is restricted to momentum eigenvalues which satisfy the conditions (30).

The periodic-orbit expression for $K_p^r(\tau)$ is obtained by inserting the periodic-orbit sum for $d_{\mathrm{osc}}^\eta(p)$ into equation (27). The result is

$$K_p^r(\tau) = \frac{1}{\bar{d}(p)\Delta p}\sum_{\{l_n\}}\sum_{k=-\infty}^{\infty}{}'\sum_{\{l_{n'}\}}\sum_{k'=-\infty}^{\infty}{}' A_{n,k}A_{n',k'}^{*}\mathrm{e}^{\mathrm{i}p(kl_n - k'l_{n'})}\frac{\sin[(\Delta p/2)(kl_n - k'l_{n'})]}{\pi(kl_n - k'l_{n'})}$$
$$\times \frac{\sin[\sigma(\ell - (1/2)(kl_n + k'l_{n'}))]}{\pi(\ell - (1/2)(kl_n + k'l_{n'}))}\exp\left\{-\frac{\eta^2}{4}(k^2l_n^2 + k'^2l_{n'}^2)\right\}. \tag{32}$$

In this expression the Gaussian term is necessary in order to make the periodic-orbit sums convergent.

A disadvantage of rectangular smoothing is the fact that long periodic orbits give a greater contribution to the periodic-orbit sums than in the case of Gaussian smoothing so that a much greater number of periodic orbits have to be taken into account. This effect becomes even more pronounced as τ is increased.

In summary, we have examined semiclassical approximations to the spectral form factor $K(\tau)$ for systems in which the periodic orbits contribute to Gutzwiller's trace formula with the same phase factor χ_n. We have shown that the semiclassical form factor is a sensitive quantity which has to be defined with care. In order to evaluate the form factor semiclassically, one has to specify the kind of averaging applied to energy and time and a very convenient choice is Gaussian averaging. By using asymptotic properties of periodic orbits, we have shown, however, that for this choice the semiclassical form factor increases exponentially for large values of τ. We have further considered asymmetric octagons for which Gutzwiller's trace formula is exact and is identical to the Selberg trace formula. We have demonstrated the exponential increase of the semiclassical $K(\tau)$ by numerical results and provided a proof of this asymptotic behaviour by using Selberg's trace formula. Finally, we have shown that the exponential increase can be avoided by using a rectangular smoothing.

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