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Running gauge coupling in three dimensions and the electroweak phase transition

M. Reuter
DESY, Notkestrasse 85, W-2000 Hamburg 52, Germany

C. Wetterich
Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, W-6900 Heidelberg, Germany

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We present a method which allows us to deal with the strong infrared effects in three-dimensional gauge theories. In particular, we compute the three-dimensional running of the gauge coupling. Applying these results to the electroweak phase transition in the standard model, we conclude that the transition cannot be of second order. It is either a first-order transition or a smooth cross-over.

1. Introduction

The high temperature phase transition in gauge theories [1] is of interest both for the QCD phase transition and the electroweak phase transition in the early universe. Recently, a lively discussion on the order of the electroweak phase transition in the standard model has developed [2], mainly triggered by the observation that the baryon asymmetry in the universe may be created during this transition [3]. Unfortunately, the methods of high temperature perturbation theory [4] are not reliable in all regions of interest. This is due to strong infrared effects [5] in the effective three-dimensional theory for the modes with momenta much smaller than the temperature.

One effect which is usually neglected in high temperature perturbation theory is the running of the gauge coupling. We will show in this paper that this running is an important effect in the symmetric phase and also for small expectation values of the scalar field in the phase with spontaneous symmetry breaking. Let us consider the one-loop graph of fig. 1a which contributes a correction to the gauge coupling in a nonabelian gauge theory. If the momenta of the external gauge bosons vanish and the gauge boson circulating in the loop is massless, this graph gives an infrared

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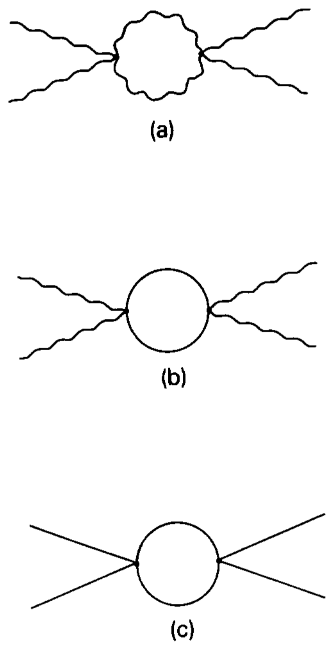


Fig. 1. Infrared divergent graphs for massless particles in three dimensions.

divergent contribution $\sim \int d^d q\, q^{-4}$. In four dimensions we recognise the familiar logarithmic infrared divergence. It is regularized in practice by some physical infrared cutoff k. Usually the infrared scale k is determined either by nonvanishing external momenta or by a nonvanishing mass of the gauge bosons if the symmetry is spontaneously broken, or else by the scale of confinement for a nonabelian gauge theory in the confinement phase. For the four-dimensional theory of electroweak interactions at vanishing temperature the contribution of fig. 1a is $\sim g^4 \ln(\Lambda/k)$ (with Λ some ultraviolet cutoff) and describes the slow running of the small electroweak gauge coupling *. The smallness of this effect at zero temperature has led many authors to assume that the running of the gauge coupling also gives a small effect at nonvanishing temperature and can therefore be neglected.

In less than four dimensions the infrared divergence becomes a power divergence instead of a logarithmic divergence. In particular, one finds in three dimensions a correction to the three-dimensional gauge coupling g_3,

$$\Delta g_3^2 \sim g_3^4/k. \tag{1.1}$$

(We note that g_3^2 has dimension of mass.) For sufficiently small values of k this correction becomes as big as g_3^2 itself and perturbation theory breaks down. This

* In this case only external momenta and mass are relevant infrared cutoffs k.

effect persists in renormalization group-improved perturbation theory since (1.1) implies a fast running of $g_3^2(k)$ for small k,

$$k\frac{\partial}{\partial k}g_3^2 \sim -\frac{g_3^4}{k}. \tag{1.2}$$

It is well known that field theories at high temperature behave effectively as three-dimensional theories if one considers momenta smaller than the temperature, $q^2 \ll T^2$. Indeed, the integration over the "time" component q_0 in a euclidean field theory is replaced by a discrete sum with $q_0 \to 2\pi mT$, $m \in \mathbb{N}$. For scales below T the contribution of the modes with $m=0$ dominates all infrared sensitive quantities and the momentum integrals are therefore reduced to three-dimensional integrals. One expects that the space components of the gauge fields behave as a three-dimensional gauge theory if the temperature is high enough compared to momenta. Associating as usual the three-dimensional gauge coupling g_3^2 with the four-dimensional gauge coupling g_4^2 by

$$g_3^2 = g_4^2 T \tag{1.3}$$

one finds for the high temperature correction to the four-dimensional gauge coupling

$$\Delta g_4^2 \sim g_4^4 T/k, \tag{1.4}$$

$$\frac{\partial}{\partial t}g_4^2 \sim -\frac{g_4^4 T}{k}. \tag{1.5}$$

Eq. (1.4) gives a rough estimate on the infrared scale $\hat{k}$, where the running of the gauge coupling becomes an important effect, namely

$$\hat{k} \sim g_4^2 T. \tag{1.6}$$

Only if the physical infrared cutoff is sufficiently larger than $\hat{k}$ the running of the gauge coupling can be neglected for high temperature. We observe that $\hat{k}$ is of the same order as the "magnetic mass" of the transversal gauge bosons [1,2]. We conclude that a careful study of corrections to the gauge coupling is necessary for all effects involving transversal gauge bosons. This is the subject of this paper.

Our aim is to develop concepts and methods for a solution of the infrared problem in nonabelian gauge theories at high temperatures. As a first step, we address in this paper the somewhat simpler problem in the abelian gauge theory. The effective three-dimensional running (1.2) and (1.5) is of a similar nature as for the abelian case, except for the opposite sign of the β-function. The insights learned from the abelian theory are therefore generalized in the conclusions to the nonabelian theory, with implications for the high temperature behaviour of the electroweak interactions. The infrared problem in the abelian Higgs model arises from fluctuations of a massless charged scalar field as shown in fig. 1b. We will introduce an effective infrared cutoff k which is independent of masses and momenta. For masses and momenta smaller than k simple dimensional analysis leads to a running of the three-dimensional gauge coupling ($t = \ln k$),

$$\frac{\partial}{\partial t} e_3^2 \sim \frac{e_3^4}{k}, \tag{1.7}$$

in analogy to (1.2). As we have discussed above, an understanding of the infrared problem in the four-dimensional theory at high temperature requires an understanding of the three-dimensional infrared problem encoded in (1.7).

The corresponding problem for massless scalar field theories in three (or two) dimensions has already been solved along similar lines [6]. One introduces a variable infrared cutoff k independent of mass and momenta and takes the limit $k \to 0$ at the end of the calculation. For $k > 0$ the diagram shown in fig. 1c is then infrared regulated. This yields in lowest order for the k-dependence of the quartic scalar coupling λ_3 at zero momentum [6]

$$\frac{\partial}{\partial t} \lambda_3 = \frac{l_2^3}{4\pi} \frac{\lambda_3^2}{k} \left(N - 1 + 9 s_2^3 \right). \tag{1.8}$$

Here we consider a scalar theory with N real components in the regime with spontaneous symmetry breaking. For a vacuum expectation value $\rho_0 = \frac{1}{2}\varphi_0^2$ the excitations consist of $N-1$ massless Goldstone bosons and the radial excitation with mass $m^2 = 2\lambda_3 \rho_0$. The "threshold function" s_2^3 is renormalized to one if $m^2 \ll k^2$ and vanishes rapidly for $m^2 \gg k^2$, and l_2^3 is a computable constant of order one. This equation is of the type (1.2), (1.7). For small values of $k \to 0$ it has the asymptotic solution ($N \geqslant 2$)

$$\lambda_3(k) \sim k. \tag{1.9}$$

In this case the coupling vanishes for $k \to 0$ and the infrared problem disappears. These features are most easily seen by the use of a dimensionless coupling

$$\lambda = \lambda_3 / k \tag{1.10}$$

for which the evolution equation takes the form

$$\frac{\partial \lambda}{\partial t} = -\lambda + \frac{l_2^3}{4\pi^2} \lambda^2 \left[N - 1 + 9 s_2^3 (2\lambda\kappa) \right]. \tag{1.11}$$

The scale k does not appear explicitly on the RHS if dimensionless quantities are used ($\kappa = \rho_0/k$) since the threshold function s_2^3 depends only on the ratio $(\text{mass}/k)^2$, i.e. $2\lambda_3\rho_0/k^2 = 2\lambda\kappa$.

The situation is completely analogous for the running of the abelian gauge coupling in three dimensions. One introduces the dimensionless renormalized gauge coupling

$$e^2 = e_3^2/k \tag{1.12}$$

and finds the evolution equation

$$\frac{\partial}{\partial t} e^2 = \beta_{e^2} = -e^2 + \frac{l_g^3}{6\pi^2} e^4 s_g^3. \tag{1.13}$$

Here l_g^3 is again a constant of order one. The threshold function s_g^3 depends on m^2/k^2 in the symmetric regime and on $2\lambda\kappa$ and $2e^2\kappa$ in the regime with spontaneous symmetry breaking. In the second case the quantity $2e^2\kappa$ correspond to the ratio $(\text{photon mass}/k)^2$. The threshold function is again normalized to one for small masses and vanishes rapidly for large arguments ($m^2/k^2 \gg 1$ or $2\lambda\kappa \gg 1$ or $2e^2\kappa \gg 1$). We note that the general form of (1.13) directly follows from (1.7) or simple dimension counting for massless fields plus a decoupling property for heavy mass fields (encoded in the threshold function).

The next sections are devoted to a derivation of the evolution equation (1.13). In the conclusions, which are essentially self-contained without using detailed properties of $l_g^3 s_g^3$, we discuss the physical consequences of this equation and its generalization to nonabelian gauge theories. The reader not interested in more technical developments may therefore jump immediately to the conclusions. Our discussion will be valid for arbitrary dimension d and we find the following general properties:

For the abelian Higgs model with a massless scalar field in $d < 4$ the gauge coupling runs to zero with a power of k, i.e. $e_R^2(\mathrm{k}) = e_*^2 k^{4-d}$. The infrared fixpoint e_*^2 of the dimensionless gauge coupling is computed for this case. We propose an upper bound for the physical gauge coupling e_R^2 in dependence on the mass of the scalar field m. In less than four dimensions this bound is proportional to a power of m. For the four-dimensional theory at high temperature T we formulate a criterion for which values of m/T the running of the gauge coupling becomes an important effect. If this condition is violated, the high temperature perturbation theory (which does not account for the running of the gauge coupling) may give a qualitatively wrong picture. We finally generalize our discussion to the nonabelian gauge theory of the standard model. We conclude that the electroweak phase transition cannot be of second order. If the three-dimensional theory is asymptotically free, the sign of the β-function is negative. As a result the nonabelian gauge coupling g_3 will grow fast and become large at the three-dimensional confinement scale $\Lambda_{\text{conf}}^{(3)}$. Since even for small g_3 the running is linear in k^{-1} and not logarithmic as for $d=4$, one expects $\Lambda_{\text{conf}}^{(3)}/T$ to be a constant not too far below one, in contrast to the exponentially small ratio $\Lambda_{\text{conf}}^{(4)}/\Lambda$ for $d=4$ and small $g_4(\Lambda)$. Three-dimensional confinement excludes massless particles and therefore a second-order transition. It remains open if the restoration or the electroweak symmetry at high temperatures is a first-order transition or rather a continuous cross-over.

In order to handle the infrared problems we will use in this paper the method of the average action as formulated in ref. [7]. The average action Γ_k is an effective action for averages of fields. The average is taken over a volume $\sim k^{-d}$ such that all degrees of freedom with momenta $q^2 > k^2$ are effectively integrated out. The average action is formulated in continuous space and is the analogue of the block spin action [8] proposed earlier on a lattice. The average action for gauge theories has been developed in ref. [9]. Due to the use of a gauge-invariant constraint (which determines the averaging procedure) Γ_k is manifestly gauge-invariant. In ref. [9] the effective scalar potential and the scalar kinetic term appearing in Γ_k were calculated for the abelian Higgs model.

Here we complete the definition of Γ_k for the pure gauge sector, concentrating on the case of abelian symmetry. The formal properties of Γ_k are established in sects. 2–4. In particular, the average action defines a gauge theory with a covariant momentum cutoff $\sim k$. This means that Green functions for low momenta can be computed from Γ_k and this calculation is explicitly ultraviolet finite.

We then proceed (sects. 5–7) to compute the running in dependence on k for the abelian gauge coupling in arbitrary dimension d. The computation is performed directly in the relevant dimension without invoking expansions in the parameter $\epsilon = 4 - d$. This will later allow a reliable quantitative description of the effective transition from four-dimensional ($k \gg T$) to three-dimensional ($k \ll T$) running in the four-dimensional theory at high temperature T. (The corresponding treatment of the pure scalar theory [10] has produced a good quantitative description of the second-order phase transition.) We emphasize that our formalism is well adapted to deal even with massless particles which are relevant if the phase transition between the spontaneously broken phase and the symmetric phase is second-order. The average action allows to handle the notorious infrared divergences of massless theories in three and two dimensions.

2. Formulation of the average action: normalization and integration measure

We consider scalar electrodynamics in d dimensions. The euclidean action of the microscopic fields $\chi(x)$ and $a_\mu(x)$ is given by

$$S[\chi, a_\mu] = \int \mathrm{d}^d x \left[\tfrac{1}{4} f_{\mu\nu} f^{\mu\nu} + \left| D_\mu(a)\chi \right|^2 + V(\chi^* \chi) \right]. \tag{2.1}$$

Here $f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu$ and $D_\mu(a) \equiv \partial_\mu + i\bar{e}a_\mu$ where $\bar{e}$ denotes the bare gauge coupling. The potential reads

$$V(\chi^*\chi) = \overline{m}^2\chi^*\chi + \tfrac{1}{2}\overline{\lambda}(\chi^*\chi)^2. \tag{2.2}$$

We first study the symmetric regime where $\overline{m}^2 > 0$. Following ref. [9], the average action for this model is obtained by functional integration over the microscopic variables with a constraint

$$\exp\left(-\Gamma_k^{un}[\varphi, A_\mu]\right) = \int \mathrm{D}\chi \mathrm{D}a_\mu \exp\left(-S_{\mathrm{constr}}[\varphi, A_\mu; \chi, a_\mu] - S[\chi, a_\mu]\right), \tag{2.3}$$

where *

$$\begin{aligned} S_{\mathrm{constr}} = \int \mathrm{d}^d x \Bigg(& \tfrac{1}{4}\left[F_{\mu\nu} - f_k(-\partial^2) f_{\mu\nu}\right] \frac{1}{1 - f_k^2(-\partial^2)} \left[F^{\mu\nu} - f_k(-\partial^2) f^{\mu\nu}\right] \\ & + \frac{1}{2\alpha} \partial_\mu (A^\mu - a^\mu) \frac{1}{1 - f_k^2(-\partial^2)} \partial_\nu (A^\nu - a^\nu) \\ & + \left[\varphi - f_k(-D^2(A))\chi\right]^* \frac{-D^2(A) + \overline{m}^2}{1 - f_k^2(-D^2(A))} \left[\varphi - f_k(-D^2(A))\chi\right] \Bigg), \end{aligned}$$

$$f_k(x) = \exp\left[-a(x/k^2)^\beta\right]. \tag{2.4}$$

The functional Γ_k^{un} is invariant under gauge transformations acting on the macroscopic fields φ and A. As already noted in ref. [9], no gauge fixing is needed for the computation of Γ_k^{un} since the constraint already eliminates infinite factors from the volume of the group of gauge transformations. (A similar observation was made in the context of a lattice approach in ref. [11].) The constraint enforces suitable gauge-invariant averages of the microscopic fields χ and a_μ (or, more precisely, averages for the field strength $f_{\mu\nu}$) to coincide with the macroscopic field up to a certain degree of allowed fluctuations. The details of the averaging are specified by the function f_k.

More precisely, the propagator for the gauge field fluctuation a_μ can be read off directly from the term quadratic in a in $S + S_{\mathrm{constr}}$. (It is, of course, the propagator in presence of the constraint which is relevant for a computation of Γ_k). For large momenta $q^2 \gg k^2$ the function f_k vanishes exponentially, yielding in this limit the quadratic term $\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + (1/2\alpha)(\partial_\mu a^\mu)^2$. For large momenta we

* We have modified the constraint for the scalar field of ref. [9] by adding a mass term. This is more appropriate for the symmetric regime. For simplicity, we also have omitted wave function renormalization factors in the formulation of the constraint.

therefore obtain the propagator of a usual gauge-fixed theory. A perturbative calculation of Γ_k^{un} is therefore well defined.

The functional $\Gamma_k^{\text{un}}[\varphi, A_\mu]$ is not yet normalized in such a way that it gives rise to the same partition function as $S[\chi, a_\mu]$. This can be remedied if we define

$$\exp\left(-\Gamma_k^{\text{norm}}[\varphi, A_\mu]\right) = \int \mathrm{D}\chi \mathrm{D}a_\mu \; P_k[\varphi, A_\mu; \chi, a_\mu] \exp\left(-S[\chi, a_\mu]\right), \quad (2.5)$$

with the "constraint operator" P_k normalized according to

$$\int \mathrm{D}\varphi \mathrm{D}A_\mu \; P_k[\varphi, A_\mu; \chi, a_\mu] = 1. \quad (2.6)$$

Then the partition functions obtained from Γ_k^{norm} and S coincide:

$$Z = \int \mathrm{D}\varphi \mathrm{D}A_\mu \exp\left(-\Gamma_k^{\text{norm}}[\varphi, A]\right). \quad (2.7)$$

Writing

$$P_k[\varphi, A_\mu; \chi, a_\mu] = N_k[A_\mu] \exp\left(-S_{\text{constr}}[\varphi, A_\mu; \chi, a_\mu]\right) \quad (2.8)$$

it follows from eq. (2.6) that the normalization factor N_k has to be chosen as

$$N_k[A_\mu] = \det\left[\frac{-D^2(A) + \bar{m}^2}{1 - f_k^2(-D^2(A))}\right] \equiv \det\left[H(-D^2(A))\right]. \quad (2.9)$$

Here the determinant cancels the inverse determinant which results from the integration over φ. The remaining integration over A_μ yields only a field independent constant which is omitted. (This will be different for the nonabelian case!) The two definitions (2.5) and (2.3) are therefore related by

$$\Gamma_k^{\text{norm}}[\varphi, A_\mu] = \Gamma_k^{\text{un}}[\varphi, A_\mu] - \ln\left(N_k[A_\mu]\right). \quad (2.10)$$

They differ by terms which only depend on the gauge field A_μ. At this point one might be tempted to identify the functional $\Gamma_k^{\text{norm}}[\varphi, A_\mu]$ with the effective action for the macroscopic fields φ and A_μ since it is obtained from the microscopic action $S[\chi, a_\mu]$ by "inserting a factor of unity" in the path integral (2.5). We will see, however, that it is more convenient to split off from Γ_k^{norm} a measure factor such that

$$Z = \int \mathrm{D}\varphi \mathrm{D}A \; \mu_k[A] \exp\left(-\Gamma_k[\varphi, A]\right), \quad (2.11)$$

$$\exp\left(-\Gamma_k^{\text{norm}}[\varphi, A]\right) = \mu_k[A] \exp\left(-\Gamma_k[\varphi]\right). \quad (2.12)$$

We will motivate this choice of the average action Γ_k in the following.

Let us fix a gauge field $A_\mu(x)$ and diagonalize $-D^2(A)$ for this gauge field. In terms of its eigenvalues $\{\lambda\}$ the normalization factor (2.8) becomes

$$N_k[A_\mu] = \prod_\lambda H(\lambda), \tag{2.13}$$

where the product runs over all eigenvalues. We will see that the eigenvalues with $\lambda < k^2$ and those with $\lambda > k^2$ play a very different role. We therefore split the determinant (2.8) in a product of two factors, $\det_>[H]$ and $\det_<[H]$, defined by

$$\det{}_>[H] = \prod_{\lambda > k^2} H(\lambda) \tag{2.14}$$

and similar for $\det_<[H]$. (For the sake of the argument we define $\det_>$ and $\det_<$ by a sharp cutoff at $\lambda = k^2$. Later on we shall employ a "smoothened" version of this definition.) First we recall from the general discussion of ref. [7] that a physically sensible average action should be obtained from the fundamental action S by integrating out only the modes with momenta larger than k. The modes with momenta much smaller than k should be left untouched. This means that for such modes the constraint operator $P_k \sim \exp(-S_{\mathrm{constr}})$ should reduce to an approximate δ-functional $\delta[\phi - \chi]$ if both $\chi(x)$ and $\phi(x)$ are slowly varying. More precisely, expanding χ and ϕ in terms of eigenfunctions of $-D^2(A)$, the constraint for the modes with $\lambda \ll k^2$ should lead to δ-functions. Because for such fields $f_k(\lambda)\chi \approx \chi$ and $f_k(\lambda)\phi \approx \phi$, the last term on the RHS of eq. (2.4) indeed gives rise to an approximate δ-functional. It is properly normalized if we multiply $\exp(-S_{\mathrm{constr}})$ by $\det_<[H]$. The other part of the normalization factor, $\det_>[H]$, is not needed for this purpose.

Secondly, only scalar modes with $|-D^2(A)\varphi| < |k^2\varphi|$ and gauge field modes with $|-\partial^2 A| < |k^2 A|$ should contribute effectively to functional integrations of the type (2.7), (2.11) since the high momentum modes should already have been integrated out. For the average action of a theory containing only scalar fields this decoupling of the large momentum modes happens in the following way: For modes φ with momenta $q^2 \gg k^2$ the effective theory becomes approximately quadratic,

$$\Gamma_k[\varphi] \approx \int \mathrm{d}^d x\, \varphi^*(-\partial^2)\varphi. \tag{2.15}$$

In this way the integration over the large momentum modes yields only an irrelevant constant $\det_>[-\partial^2]$. Returning to the gauge theory, eq. (2.3) with (2.4) shows that for φ-modes with large eigenvalues $\lambda \gg k^2$

$$\Gamma_k^{\mathrm{un}}[\varphi, A_\mu] = \int \mathrm{d}^d x \left\{\mathcal{L}_{\mathrm{gauge}}(A) + \varphi^*\left[-D^2(A) + \bar{m}^2\right]\varphi\right\}. \tag{2.16}$$

If in addition A_μ has momentum $q^2 \gg k^2$, then $\mathscr{L}_{\text{gauge}} = \frac{1}{4}F_{\mu\nu}^2$. Clearly the situation is now more complicated because even for the large momentum modes Γ_k^{un} gives rise to an interacting theory. In particular, the low momentum modes of the gauge field do not decouple from the high momentum modes of the scalar field. The normalization of the constraint cures this disease in a very particular way: For the partition function

$$Z = \int \mathrm{D}\varphi \mathrm{D}A_\mu \, \det{}_>[H]\det{}_<[H] \exp\left(-\Gamma_k^{\text{un}}[\varphi, A_\mu]\right) \tag{2.17}$$

we observe that the integration over the φ-modes with $\lambda \gg k^2$ produces a factor which is the inverse of

$$\det{}_>[-D^2 + \overline{m}^2] \approx \det{}_>\left[\frac{-D^2 + \overline{m}^2}{1 - f_k^2(-D^2)}\right] \equiv \det{}_>[H]. \tag{2.18}$$

(Recall that $f_k(q^2) \approx 0$ for $q^2 \gg k^2$.) We conclude that the contribution from the high frequency modes is exactly cancelled by the $\det_>[H]$-part of the normalization factor. This guarantees that the high momentum modes indeed decouple from the low momentum modes. A similar argument applies to the n-point functions involving slowly varying fields. The disturbing determinant from the integration of high λ scalar modes is exactly cancelled by $\det_>[H]$. In consequence, the average action becomes an effective action for the low energy theory.

This observation has an important consequence for the correct definition of the average action Γ_k. We want to extract the low momentum physics from Γ_k directly, without any further functional integration of heavy modes. We therefore should not include $\det_>[H]$ in the definition of Γ_k. This factor only serves the purpose to guarantee the decoupling of the high momentum modes. It should be interpreted as an appropriate measure factor in the functional integration, which, together with the particular form of Γ_k for the high momentum modes, ensures decoupling. We therefore arrive at our final definition of the average action,

$$\exp\left(-\Gamma_k[\varphi, A_\mu]\right) = \det{}_<\left[H(-D^2(A))\right] \exp\left(-\Gamma_k^{\text{un}}[\varphi, A_\mu]\right), \tag{2.19}$$

$$\Gamma_k[\varphi, A_\mu] = \Gamma_k^{\text{un}}[\varphi, A_\mu] - \ln\left(N_k[A_\mu]\right) + \ln\left(\mu_k[A_\mu]\right), \tag{2.20}$$

$$\mu_k[A] = \det{}_>\left[H(-D^2(A))\right]. \tag{2.21}$$

For slowly varying fields and $k \to 0$ the low energy properties of the theory can be read off from Γ_k directly. Shortly speaking, Γ_k contains the physical vertices whereas μ_k only involves "regulator terms".

For practical calculations it is advantageous to define $\mu_k[A] = \det_> H$ slightly different from eq. (2.14). Instead of using a sharp cutoff at $\lambda = k^2$ we interpolate smoothly between the region $\lambda \gg k^2$, which has to contribute without any suppression factor, and the region $\lambda \ll k^2$, which should not contribute at all. Comparing two measures μ_{k_1}/μ_{k_2} at scales k_1 and $k_2 \gg k_1$, consistency requires that the ratio μ_{k_1}/μ_{k_2} receives contributions only from eigenvalues between k_1^2 and k_2^2:

$$\frac{\mu_{k_1}[A]}{\mu_{k_2}[A]} \approx \prod_{\lambda \in [k_1^2,\, k_2^2]} H(\lambda) \approx \prod_{\lambda \in [k_1^2,\, k_1^2]} (\lambda + \overline{m}^2). \tag{2.22}$$

In practice it is sufficient if the last relation in (2.22) holds in a central region $k_1^2 \ll \lambda \ll k_2^2$. (In general it will not be satisfied in the transition regions $\lambda \approx k_1^2$, k_2^2.) Also, field-independent constants due to the eigenvalues $\lambda \ll k_1^2$ and $\lambda \gg k_2^2$ may be ignored.

The following form of the measure meets these requirements:

$$\mu_k^{(\mathrm{I})}[A] = \det\left[-D^2(A) + \overline{m}^2 + k^2\right]. \tag{2.23}$$

In view of eq. (2.18) it is clear that the product of the eigenvalues $\lambda > k^2$ reproduces $\det_>[H]$. The eigenvalues $\lambda < k^2$ give rise to a factor $\det_<[\overline{m}^2 + k^2]$, which can be ignored because it does not depend on any field. For the massless theory, $\overline{m}^2 = 0$, we may also use the following measure:

$$\mu_k^{(\mathrm{II})}[A] = \det\left(\frac{-D^2(A)}{1 - f_k^2(-D^2(A))}\right)_{\beta=1}, \tag{2.24}$$

where the cutoff function $f_k(x) \equiv \exp(-ak^{-2\beta}x^\beta)$ must be taken with $\beta = 1$ so that $x[1 - f_k^2(x)]^{-1} \approx k^2/2a = \text{const.}$ for $x \ll k^2$. Only for $\beta = 1$ the eigenvalues $\lambda < k^2$ yield a constant factor $\det_<[k^2/2a]$, which we can discard, whereas for $\beta \neq 1$ one would obtain unwanted field-dependent contributions from the small eigenvalues. For $\lambda \gg k^2$ we recover $\det(-D^2) = \det H$.

With some obvious modifications the above discussion remains valid if we replace the scalar φ by a fermion. If the fermion is chiral, a naive mass term $\sim k^2$ as in $\mu_k^{(\mathrm{I})}$ is forbidden. In this situation we have to rely on the fermionic analogue of $\mu_k^{(\mathrm{II})}$. In the following sections we shall use both $\mu_k^{(\mathrm{I})}$ and $\mu_k^{(\mathrm{II})}$ and show that (in four dimensions) they lead to the same β-functions. Furthermore, we prove in Appendix A that for A_μ corresponding to a constant magnetic field the measure $\mu_k^{(\mathrm{I})}$ is exactly equivalent to the original definition (2.11) where we used a sharp cutoff:

$$\mu_k^{(III)}[A] = \exp \mathrm{Tr}\left[\Theta(-D^2(A) - k^2) \ln(-D^2(A) + \overline{m}^2)\right]. \tag{2.25}$$

More generally, the precise definition of $\mu_k[A]$ should not matter, provided that the measure factor fulfils the criteria discussed above.

3. Green functions from the average action

In this section we study the relation between the Green functions derived from $\Gamma_k[\varphi, A_\mu]$ and those obtained from the microscopic action $S[\chi, a_\mu]$. To start with, we first look at the simpler case of a theory containing only a complex scalar where [7]

$$\exp(-\Gamma_k[\varphi]) = \int \mathrm{D}\chi \exp\left[-\int \mathrm{d}^d x\left(|\partial_\mu \chi|^2 + V(\chi^*\chi) + [\varphi - f_k(-\partial^2)\chi]^* \frac{-\partial^2 + \overline{m}^2}{1 - f_k^2(-\partial^2)}[\varphi - f_k(-\partial^2)\chi]\right)\right]. \tag{3.1}$$

The connected Green functions of Γ_k are generated by the functional $W_k[\eta, \eta^*]$ defined by

$$\exp(W_k[\eta, \eta^*]) = \int \mathrm{D}\varphi \exp\left(-\Gamma_k[\varphi] + \int \mathrm{d}^d x(\eta^*\varphi + \varphi^*\eta)\right). \tag{3.2}$$

On the other hand, the microscopic action $S[\chi]$ gives rise to the following generating functional:

$$\exp(W[\eta, \eta^*]) = \int \mathrm{D}\chi \exp\left(-\int \mathrm{d}^d x\left(|\partial_\mu \chi|^2 + V(\chi^*\chi) - \eta^*\chi - \chi^*\eta\right)\right). \tag{3.3}$$

There exists a remarkably simple relation between W_k and W. Inserting (3.1) into (3.2) and performing the gaussian integration over φ we find

$$W_k[\eta, \eta^*] = W[f_k(-\partial^2)\eta, f_k(-\partial^2)\eta^*] + \int \mathrm{d}^d x\ \eta^*(x) \frac{1 - f_k^2(-\partial^2)}{-\partial^2 + \overline{m}^2} \eta(x). \tag{3.4}$$

We observe that W_k is obtained from W by averaging the source functions η and η^* and by adding an additional piece which contributes to the connected two-point function only. For sources which are rapidly varying at the scale of k^{-1} $(-\partial^2 \to \infty)$ we obtain a free theory: $W_k \sim \int \eta^*(-\partial^2 + \overline{m}^2)^{-1}\eta$. For slowly varying sources $(-\partial^2 \to 0)$ the second term on the RHS of eq. (3.4) vanishes and W_k and W become approximately equal. We conclude that for small external momenta Γ_k produces the same Green functions as S, but it describes a free theory at momenta large compared to k^2. Inverting the exact relation (3.4),

$$W[\eta, \eta^*] = W_k[f_k^{-1}(-\partial^2)\eta, f_k^{-1}(-\partial^2)\eta^*] - \int \mathrm{d}^d x\ \eta^*(x) f_k^{-2}(-\partial^2) \frac{1 - f_k^2(-\partial^2)}{-\partial^2 + \overline{m}^2} \eta(x), \tag{3.5}$$

gives a precise prescription how the true connected Green functions generated by W can be computed from the average action Γ_k.

Let us return to the gauge theory now. According to the discussion of sect. 2 the functional integral for the generating function obtained from $\Gamma_k[\varphi, A_\mu]$ contains an explicit measure factor of $\mu_k[A]$:

$$\exp\big(W_k[\eta, \eta^*, j_\mu]\big)$$
$$= \int \mathrm{D}\varphi \mathrm{D}A_\mu\ \mu_k[A] \exp\left(-\Gamma_k[\varphi, A] + \int \mathrm{d}^d x\big(\eta^* \varphi + \varphi^* \eta + j^\mu A_\mu\big)\right). \quad (3.6)$$

The microscopic action $S[\chi, a_\mu]$ gives rise to the conventional generating functional

$$\exp\big(W[\eta, \eta^*, j_\mu]\big) = \int \mathrm{D}\chi \mathrm{D}a_\mu \exp\left(-S[\chi, a_\mu] + \int \mathrm{d}^d x\big(\eta^* \chi + \chi^* \eta + j^\mu a_\mu\big)\right). \quad (3.7)$$

In order to establish a relation between W_k and W we insert the functional integral defining Γ_k into eq. (3.6). Using eq. (2.20) we see that $\mu_k \exp(-\Gamma_k) = N_k \exp(-\Gamma_k^{\mathrm{un}})$, so that the form of the integrand follows from (2.9) and (2.3) with (2.4), respectively. The integration over A_μ is gaussian only if $\eta = \eta^* = 0$. In this case we obtain

$$W_k[0, 0, j_\mu] = W\big[0, 0, [P_{\mathrm{L}} + f_k(-\partial^2) P_{\mathrm{T}}]_{\mu\nu} j^\nu\big]$$
$$+ \tfrac{1}{2} \int \mathrm{d}^d x\, j_\mu \frac{f_k^2(-\partial^2) - 1}{\partial^2} \left(\delta^{\mu\nu} + (\alpha - 1)\frac{\partial^\mu \partial^\nu}{\partial^2}\right) j_\nu, \quad (3.8)$$

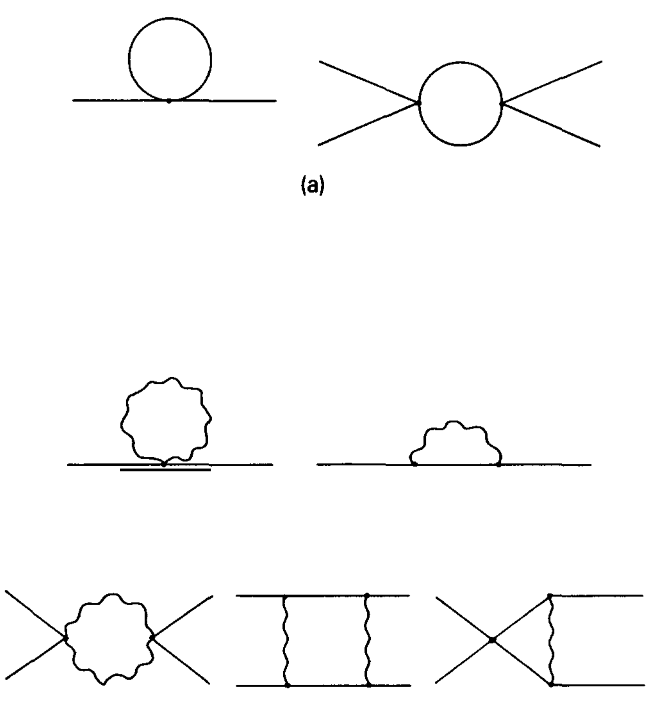
where $(P_{\mathrm{L}})_{\mu\nu} \equiv \partial_\mu \partial_\nu / \partial^2$ and $(P_{\mathrm{T}})_{\mu\nu} \equiv \delta_{\mu\nu} - \partial_\mu \partial_\nu / \partial^2$ are the usual projection operators on longitudinal and transversal modes. For conserved sources, $\partial_\mu j^\mu = 0$, which are rapidly varying on the scale of k^{-1} $(-\partial^2 \to \infty)$, we find a free theory: $W_k \sim j_\mu (\partial^2)^{-1} j^\mu$. In the opposite limit of small momenta $(-\partial^2 \to 0)$ the second term on the RHS of (3.8) vanishes $\sim (-\partial^2/k^2)^{\beta - 1}$ and W_k and W are approximately equal.

If η, $\eta^* \neq 0$ no simple relation between W and W_k can be written down. However, the general features found above still persist: for small momenta W_k is approximately equal to W, whereas the large momentum modes in W_k decouple. This point will be further discussed in the following section.

4. Gauge theory with covariant momentum cutoff

If the average action Γ_k is defined properly, all short-distance modes are already integrated out. The short-distance modes should therefore not contribute any more if Green functions are computed from Γ_k. Otherwise speaking, the functional integral (3.6) (or corresponding derivatives of W_k with respect to the sources) should not involve ultraviolet divergences. The ultraviolet behaviour of the original theory is already incorporated in Γ_k. The average action, together with the appropriate functional measure μ_k should exhibit an effective short-distance cutoff. In this section we sketch briefly how this is realized.

We first describe how the ultraviolet finiteness of the one-loop integrals is realized. The two- and four-point functions for external scalar fields with small momentum $Q^2 \ll k^2$ involve Feynman diagrams with either scalars or gauge bosons circulating in the loop (figs. 2a, b). (We specialize here for simplicity to an expansion around the symmetric point $\varphi = 0$ and give only the 1PI graphs.) The first class of diagrams (fig. 2a) is ultraviolet finite because of the momentum dependence of the four-point vertex in Γ_k. As we will argue below this four-point vertex vanishes exponentially for a large loop momentum $q^2 \gg k^2$, similar as in



(a)

(b)

Fig. 2. One-loop contributions with two or four external scalar fields.

the pure scalar theory discussed in ref. [7]. For the second class of diagrams (fig. 2b) one has to distinguish between the physical transverse gauge bosons and the gauge degrees of freedom (longitudinal gauge bosons) in the loop. We will see that the vertices in Γ_k which involve high momentum transverse gauge bosons and low momentum scalars are again suppressed for large loop momentum. The diagrams with transverse gauge bosons in the loop therefore become ultraviolet finite. On the other hand, the longitudinal gauge bosons in the loop correspond to the functional integration over gauge degrees of freedom. The resulting infinite factor cancels in the Green functions as usual and may be treated by a proper gauge fixing.

In order to establish the exponential decrease with q^2 of the vertices mentioned above we first note that φ decouples from all other fields in (2.4) in the limit $f_k(-D^2(A)) \to 0$, $A \to 0$. This is exactly what happens for the φ-modes with large eigenvalues λ of the operator $-D^2(A)$, leading to the expressions (2.16) up to corrections of the order $\exp[-a(\lambda/k^2)^\beta]$. The exponential suppression of the four-scalar vertex with at least one large momentum follows by inserting $A=0$ since Γ_k becomes quadratic in φ up to corrections $\sim \exp[-a(q^2/k^2)^\beta]$.

The argument for the transverse gauge bosons with large momentum is similar. Let us consider $\varphi = \text{const.}$ and $A_{\mathrm{L}\mu}=0$ ($A_\mathrm{L}=P_\mathrm{L}A$, $A_\mathrm{T}=P_\mathrm{T}A$). The classical solution derived from $S_k = S + S_{\text{constr}}$ reads (up to corrections with more than two powers of $A_{\mathrm{T}\mu}$)

$$\chi=\varphi, \quad a_{\mathrm{L}\mu}=0, \quad a_{\mathrm{T}\mu}(q)=f_k(q)\frac{q^2}{q^2+2\bar{e}^2\varphi^2\left[1-f_k^2(q)\right]}A_{\mathrm{T}\mu}(q). \tag{4.1}$$

Inserting this solution into S_k gives the classical average action $\Gamma_k^{(0)}$. One finds that the vertex $\varphi^2 A_{\mathrm{T}\mu}(q)A_\mathrm{T}^\mu(-q)$ is exponentially suppressed for $q^2 \gg k^2$.

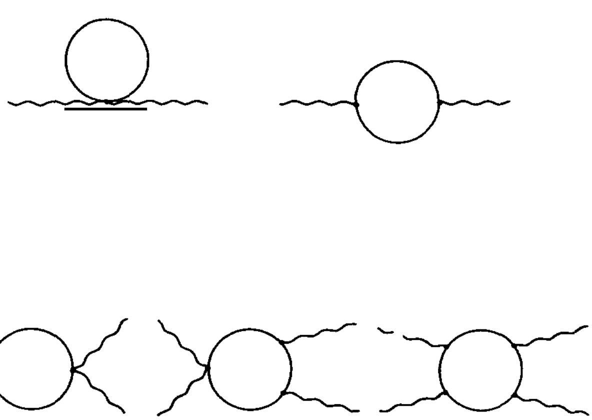


Fig. 3. One-loop contributions with two or four external gauge fields.

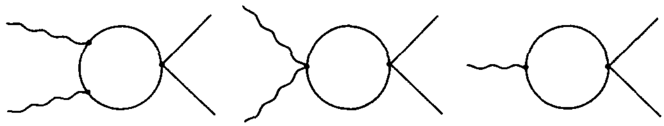


Fig. 4. One-loop contributions with external scalar and gauge fields.

We expect the suppression to persist after the inclusion of quantum corrections in the calculation of Γ_k. It should also hold for φ-modes with nonvanishing, but small momentum. Along the same lines one concludes the suppression of the cubic vertex involving one low momentum scalar, one high momentum scalar and one high momentum transverse gauge boson.

We next turn to the one-loop graphs contributing to the two- and four-point functions involving gauge bosons (fig. 3). In contrast to the graphs in fig. 2 a high momentum scalar in the loop does not decouple from the low momentum external gauge bosons. The interaction between these modes is exactly given by (2.16), up to exponentially small corrections. Standing alone, the graphs in fig. 2 would suggest an ultraviolet divergence. As we have discussed in the last section, however, the measure factor $\mu_k[A]$ exactly cancels the one-loop contributions involving scalar fields with high momentum. (Since Γ_k becomes quadratic in the large momentum modes of φ up to exponentially small corrections, only one-loop graphs have to be cancelled. Higher loops involving high momentum scalars are again suppressed and therefore ultraviolet finite.) Finally, the one-loop graphs involving external scalars and gauge fields (fig. 4) are ultraviolet finite due to the exponential suppression of the scalar four-point vertex, similar as for fig. 2a.

We did not specify yet how the original theory for the variables χ and a_μ (with action (2.1)) is regularized. In principle, we could formulate this theory on a lattice or use dimensional regularization. The previous discussion suggests that we can also use a continuum regularization in four dimensions involving an effective covariant momentum cutoff Λ. We require the following two properties for the gauge-invariant action:

(i) For large momenta of the gauge field ($q^2 \gg \Lambda^2$) the transverse gauge bosons should decouple such that

$$S = \int \mathrm{d}^d x \ \mathscr{L}, \quad \mathscr{L} = \tfrac{1}{4} f_{\mu\nu} f^{\mu\nu} + \mathscr{L}(\chi), \tag{4.2}$$

up to corrections which are exponentially suppressed for $q^2/\Lambda^2 \gg 1$.

(ii) For large momenta of the scalar field (or, more precisely, large eigenvalues of $-D^2(\tilde{a})$) the action should take a canonical quadratic form in χ up to exponentially small corrections,

$$\mathscr{L} = \left[D_\mu(\tilde{a})\chi\right]^* D^\mu(\tilde{a})\chi + \overline{m}^2 \chi^* \chi, \tag{4.3}$$

with

$$\tilde{a}_\mu = \left[P_{\mathrm{L}} + f_\Lambda(-\partial^2)P_{\mathrm{T}}\right]a_\mu. \tag{4.4}$$

In addition, we introduce a functional measure factor similar to the one in sect. 2, e.g.

$$\mu_\Lambda[\tilde{a}] = \det\left[-D^2(\tilde{a}) + \overline{m}^2 + \Lambda^2\right]. \tag{4.5}$$

This guarantees that the high momentum fluctuations of χ decouple. On the other hand, the high momentum fluctuations of the transverse gauge bosons decouple due to the property (4.2). (This is consistent with (4.3) since the vertices of high momentum transverse gauge bosons are exponentially suppressed.) In consequence, all Green functions are ultraviolet finite. We note that this regularization may also be understood in a somewhat different language: Expressed in terms of the variable $\tilde{a}_\mu$ the vertices are unsuppressed, but the ultraviolet finiteness is now guaranteed by an effective momentum cutoff in the propagator as implied by the kinetic term

$$\mathscr{L} = \tfrac{1}{4}\tilde{f}_{\mu\nu} f_\Lambda^{-2}(-\partial^2)\tilde{f}^{\mu\nu}. \tag{4.6}$$

The measure factor (4.5) may be interpreted as an effective Pauli–Villars regularization (which can be generalized even for chiral fermions (2.24)). We emphasize that the asymtotic form (4.2), (4.3) does not exclude nontrivial interactions. For example, a quartic scalar interaction can be written in the form

$$\mathscr{L} = \tfrac{1}{2}\overline{\lambda}\left|f_\Lambda(-D^2(\tilde{a}))\chi\right|^4 \tag{4.7}$$

without disturbing the ultraviolet finiteness.

We can actually enforce the form (4.2) with (4.3) for $\Gamma_k[\varphi, A_\mu]$ by modifying the constraint (2.4), replacing $D^2(A)$ by $D^2(\tilde{A})$ in the last line and correspondingly in the measure factor μ_k ($\tilde{A}_\mu = P_{\mathrm{L}}A_\mu + f_k(-\partial^2)P_{\mathrm{T}}A_\mu$). This modification does not affect the form of Γ_k for low momentum gauge fields as discussed in this paper and in ref. [9]. It improves, however, considerably the decoupling of the high momentum transverse gauge fields compared to the discussion in the beginning of this section. All vertices in Γ_k are then exponentially suppressed if at least one momentum in the vertex obeys $q^2 \gg k^2$.

5. Wave function renormalization in the symmetric regime

The average action $\Gamma_k[\varphi, A_\mu]$ has a derivative expansion of the form

$$\Gamma_k[\varphi, A_\mu] = \int \mathrm{d}^d x\Big[U_k(|\varphi|^2) + Z_{\varphi,k}(|\varphi|^2)|D_\mu(A)\varphi|^2 + \tfrac{1}{4}Z_{F,k}(|\varphi|^2)F_{\mu\nu}F^{\mu\nu} + \dots\Big]. \tag{5.1}$$

In ref. [9] we discussed the average potential U_k and the wave function renormalization of the scalar, $Z_{\varphi,k}$. (Note that in ref. [9] we set $A_\mu = 0$ or $A_\mu = \text{const.}$. Hence there was no distinction between Γ_k and Γ_k^{un}.) Here we want to perform a one-loop calculation of $Z_{F,k}(|\varphi|^2)$ for the symmetric regime, where φ has no vacuum expectation value. Therefore the relevant quantity is $Z_{F,k} \equiv Z_{F,k}(\varphi = 0)$. This constant can be read off from $\Gamma_k(\varphi = 0, A_\mu)$ as the coefficient of the term $\sim F_{\mu\nu}^2$. The dependence of $Z_{F,k}$ on the scale k is sufficient to extract the corresponding running of the gauge coupling. Gauge invariance implies that the same coupling $\bar{e}$ appears in $D_\mu(A) = \partial_\mu + i\bar{e}A_\mu$ in (5.1) as originally in the action (2.1). Therefore $\bar{e}$ is independent of k and the renormalized coupling $e_{\text{R}}^2 = Z_F^{-1}\bar{e}^2$ depends on k only through Z_F.

Our first task is to evaluate the functional integral for Γ_k^{un} on the RHS of eq. (2.3) by the saddle point method. For $\varphi = 0$, the minimum of the action $S + S_{\text{constr}}$ has to be determined from the classical field equations

$$\left[-D^2(a) + \bar{m}^2\right]\chi + \bar{\lambda}|\chi|^2\chi + H(-D^2(A))f_k^2(-D^2(A))\chi = 0,$$

$$(-\partial^2\delta^{\mu\nu} + \partial^\mu\partial^\nu)a_\nu - \frac{f_k(-\partial^2)}{1 - f_k^2(-\partial^2)}(-\partial^2\delta^{\mu\nu} + \partial^\mu\partial^\nu)\left[A_\nu - f_k(-\partial^2)a_\nu\right]$$

$$+ \frac{1}{\alpha}\frac{1}{1 - f_k^2(-\partial^2)}\partial^\mu\partial^\nu(A_\nu - a_\nu) = 0. \tag{5.2}$$

The solution corresponding to the global minimum of the action is

$$\chi^{\min}(x) = 0,$$

$$a_\mu^{\min}(x) = \left[P_{\text{L}} + f_k(-\partial^2)P_{\text{T}}\right]A_\mu(x), \tag{5.3}$$

where P_{L} and P_{T} are the projectors on longitudinal and transversal photons, respectively. After expanding the integration variables χ and a_μ around this configuration and performing the gaussian integrals over the fluctuations, one obtains from eq. (2.3)

$$\Gamma_k^{\text{un}(1)}[0, A_\mu] = \int \mathrm{d}^d x\left(\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \Gamma_k^{\text{un}(1)}[0, A_\mu] + \text{higher loops}\right), \tag{5.4}$$

with the one-loop contribution

$$\Gamma_k^{\text{un}}[0, A] = \ln\det\left(-D^2\left(a_\mu^{\min}\right) + \bar{m}^2 + \frac{-D^2(A) + \bar{m}^2}{1 - f_k^2(-D^2(A))}f_k^2(-D^2(A))\right). \tag{5.5}$$

From eq. (2.20) with eq. (2.9) we obtain for the one-loop contribution of the complete average action

$$\begin{aligned}\Gamma_k^{(1)}[0, A_\mu] &\equiv \Gamma_k^{\mathrm{un}(1)}[0, A_\mu] - \ln(N_k[A_\mu]) + \ln(\mu_k[A_\mu]) \\ &= \ln\det\left(f_k^2(-D^2(A)) + \left[1 - f_k^2(-D^2(A))\right]\frac{-D^2(a^{\min}) + \overline{m}^2}{-D^2(A) + \overline{m}^2}\right) \\ &\quad + \ln(\mu_k[A]).\end{aligned} \tag{5.6}$$

(Note that $\ln N_k$ and $\ln \mu_k$ are of the same loop order as $\Gamma_k^{\mathrm{un}(1)}$.) The complicated determinant on the RHS of (5.6) results from combining the determinants of eqs. (5.5) and (2.9). It disappears completely if we restrict ourselves to a gauge field which is only slowly varying on the scale of k^{-1}:

$$f_k(-\partial^2)A_\mu \approx A_\mu. \tag{5.7}$$

In fact, we show in Appendix B that up to second order in A_μ (and to lowest order in ∂^2/k^2)

$$\begin{aligned}\Delta\Gamma_k^{(1)}[A_\mu] &\equiv \ln\det\left(f_k^2(-D^2(A)) + \left[1 - f_k^2(-D^2(A))\right]\frac{-D^2(a^{\min}) + \overline{m}^2}{-D^2(A) + \overline{m}^2}\right) \\ &= c\int \mathrm{d}^d x\, F_{\mu\nu}\left(\frac{-\partial^2}{k^2}\right)^{\beta-1} F^{\mu\nu} + \mathrm{O}(F_{\mu\nu}^4),\end{aligned} \tag{5.8}$$

where c is a constant. This shows that for slowly varying fields a_μ with momenta $Q^2 \ll k^2$, the determinant on the RHS of eq. (5.6) is suppressed by powers of $(Q^2/k^2)^{\beta-1}$. For $\beta > 1$ it does not enter the calculation of Z_F and will be neglected from now on. Thus we are left with

$$\Gamma_k^{(1)}[0, A_\mu] = \ln(\mu_k[A]). \tag{5.9}$$

At the one-loop level the average action is completely determined by the measure μ_k. The cancellation between $\Gamma_k^{\mathrm{un}(1)}$ and $\ln N_k$ will not persist beyond one-loop, however. If we choose the measure as in (2.23) the one-loop average action coincides for $d = 4$ with the conventional Heisenberg–Euler effective action [12] for a particle with mass $\overline{m}^2 + k^2$.

The wave function renormalization can now be extracted from the term bilinear in A:

$$\ln(\mu_k[A]) = \int \mathrm{d}^d x\left[\tfrac{1}{4}(Z_{F,k} - 1)F_{\mu\nu}F^{\mu\nu} + \dots\right]. \tag{5.10}$$

6. The running gauge coupling in the one-loop approximation

In this section we evaluate the average action (5.9) for the measures (2.23) and (2.24) for terms bilinear in A_μ. We set $\varphi = 0$, so the derivative expansion (5.1) reduces to

$$\Gamma_k[0, A_\mu] = \int \mathrm{d}^d x \left(\tfrac{1}{4} Z_{F,k} F_{\mu\nu} F^{\mu\nu} + \dots\right), \tag{6.1}$$

with

$$Z_{F,k} = 1 + Z_{F,k}^{(1)} + \text{higher loops}. \tag{6.2}$$

We note that for $d \geqslant 4$ $Z_{F,k}^{(1)}$ is UV-divergent unless the theory is properly regularized. Instead of calculating the one-loop contribution $Z_{F,k}^{(1)}$, we shall evaluate the anomalous dimension

$$\begin{aligned} \eta_F &\equiv -k \frac{\mathrm{d}}{\mathrm{d}k} \ln Z_{F,k} \\ &= -k \frac{\mathrm{d}}{\mathrm{d}k} Z_{F,k}^{(1)} + \text{higher orders}. \end{aligned} \tag{6.3}$$

This is UV-finite and therefore independent of the regularization. The running of the renormalized gauge coupling e_{R} can be directly extracted from η. From eq. (5.9) we have

$$\begin{aligned} k \frac{\mathrm{d}}{\mathrm{d}k} \Gamma_k^{(1)}[0, A_\mu] &= k \frac{\mathrm{d}}{\mathrm{d}k} Z_{F,k}^{(1)} \int \mathrm{d}^d x \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathrm{O}\left(A_\mu^3\right) \\ &= k \frac{\mathrm{d}}{\mathrm{d}k} \ln(\mu_k[A]). \end{aligned} \tag{6.4}$$

Choosing the measure as in (2.23) and (2.24) one obtains explicitly

$$k \frac{\mathrm{d}}{\mathrm{d}k} \ln\left(\mu_k^{(\mathrm{I})}[A]\right) = \mathrm{Tr}\left(\frac{2k^2}{-D^2(A) + \overline{m}^2 + k^2}\right), \tag{6.5}$$

$$k \frac{\mathrm{d}}{\mathrm{d}k} \ln\left(\mu_k^{(\mathrm{II})}[A]\right) = \mathrm{Tr}\left(\frac{k\, \mathrm{d} f_k^2(-D^2(A))/\mathrm{d}k}{1 - f_k^2(-D^2(A))}\right)_{\beta=1}. \tag{6.6}$$

Since the factor $Z_{F,k}$ is defined as the coefficient of the term in Γ_k which is bilinear in A_μ, it is sufficient to compute the traces on the RHS of eqs. (6.5) and (6.6) up to terms quadratic in A_μ. This is most easily done by evaluating (6.4) for a vector potential A_μ which corresponds to a constant magnetic field B. Then $F_{\mu\nu}^2 = 2B^2$, and $k \mathrm{d}Z_{F,k}^{(1)}/\mathrm{d}k$ can be read off as the coefficient of $\frac{1}{2}B^2\Omega$ where $\Omega \equiv \int \mathrm{d}^d x$. The advantage of this field configuration is that the corresponding spectrum of $-D^2(A)$ is known explicitly. The eigenvalues are

$$\left\{ \sum_{\mu=1}^{d-2} q_\mu^2 + (2n+1)eB;\ q_\mu \in \mathbb{R},\ n = 0, 1, 2, \dots \right\}, \tag{6.7}$$

where q_μ is a $(d-2)$-dimensional momentum vector and n labels the different Landau levels. The density of states is

$$\Omega \frac{\mathrm{d}^{d-2}q}{(2\pi)^{d-2}} \frac{eB}{2\pi} \tag{6.8}$$

so that for an arbitrary function $Q(-D^2(A))$ the trace $\mathrm{Tr}[Q(-D^2)]$, provided it exists, is given by

$$\mathrm{Tr}\left[Q(-D^2(A))\right] = \Omega v_{d-2} \frac{\bar{e}B}{\pi} \sum_{n=0}^{\infty} \int_0^\infty \mathrm{d}x\, x^{(d-4)/2} Q(x + (2n+1)\bar{e}B). \tag{6.9}$$

Here we have introduced (for $d > 2$) $x \equiv \sum_{\mu=1}^{d-2} q_\mu q_\mu$ and

$$v_d \equiv \left[2^{d+1}\pi^{d/2}\Gamma\left(\tfrac{1}{2}d\right)\right]^{-1}. \tag{6.10}$$

Using the Euler–McLaurin summation formula in the form

$$\sum_{n=0}^{\infty} Q(x + (2n+1)\bar{e}B) = \frac{1}{2\bar{e}B}\int_0^\infty \mathrm{d}y\, Q(x+y) + \frac{\bar{e}B}{12}\frac{\mathrm{d}}{\mathrm{d}x}Q(x) + \mathrm{O}(B^3) \tag{6.11}$$

we can write down a systematic expansion of the RHS of eq. (6.9) in powers of B. In particular the term in Tr Q proportional to B^2, $\mathrm{Tr}(Q)_{\mathrm{quad}}$, is given by

$$\mathrm{Tr}\left[Q(-D^2(A))\right]_{\mathrm{quad}} = \Omega v_{d-2} \frac{(\bar{e}B)^2}{12\pi} \int_0^\infty \mathrm{d}x\, x^{(d-4)/2} \frac{\mathrm{d}}{\mathrm{d}x} Q(x). \tag{6.12}$$

This formula holds for all $d \geqslant 3$. For $d = 4$ it reduces to

$$\mathrm{Tr}\left[Q(-D^2(A))\right]_{\mathrm{quad}} = -\Omega \frac{(\bar{e}B)^2}{96\pi^2} Q(0). \tag{6.13}$$

For $d = 2$ the integration over $x = q^2$ is absent, and (6.12) is replaced by

$$\mathrm{Tr}\left[Q(-D^2(A))\right]_{\mathrm{quad}} = \Omega \frac{(\bar{e}B)^2}{24\pi} \frac{\mathrm{d}Q}{\mathrm{d}x}(0). \tag{6.14}$$

Returning to the traces in (6.5) and (6.6), we obtain from (6.12) for $d \geqslant 3$

$$\eta_F^{(\mathrm{I})} = v_{d-2} \frac{\bar{e}^2}{3\pi} k^2 \int_0^\infty \mathrm{d}x\, x^{(d-4)/2} (x + \overline{m}^2 + k^2)^{-2}, \tag{6.15}$$

$$\eta_F^{(\mathrm{II})} = -\tfrac{2}{3} v_{d-2} \frac{\bar{e}^2}{\pi} \frac{a}{k^2}$$

$$\times \int_0^\infty \mathrm{d}x\, x^{(d-4)/2} \frac{\mathrm{d}}{\mathrm{d}x} \left\{ x f_k(x) \left[1 - f_k^2(x) \right]^{-1} \right\}_{\beta=1}. \tag{6.16}$$

Here we made use of (6.3) with (6.4):

$$k \frac{\mathrm{d}}{\mathrm{d}k} \ln \mu_k^{(\mathrm{I,II})} = -\tfrac{1}{2} \Omega B^2 \eta_F^{(\mathrm{I,II})}. \tag{6.17}$$

In four dimensions, eq. (6.13) gives rise to

$$\eta_F^{(\mathrm{I})} = \frac{\bar{e}^2}{24\pi^2} \frac{k^2}{\overline{m}^2 + k^2}, \tag{6.18}$$

$$\eta_F^{(\mathrm{II})} = \frac{\bar{e}^2}{24\pi^2} \beta \bigg|_{\beta=1} = \frac{\bar{e}^2}{24\pi^2}. \tag{6.19}$$

We observe that for $\overline{m}^2 = 0$ both measures yield the same anomalous dimension $\bar{e}^2/24\pi^2$. This quantity is universal: It depends neither on the precise definition of the measure nor on the average scheme (the precise form of $f_k(x)$). Note also that it is only for $\beta = 1$ that $\eta_F^{(\mathrm{II})}$ produces the correct result. This is consistent with the general discussion in sect. 2.

In two dimensions one obtains with the help of eq. (6.14)

$$\eta_F^{(\mathrm{I})} = \frac{\bar{e}^2}{6\pi} \frac{1}{k^2} \left(\frac{k^2}{k^2 + \overline{m}^2} \right)^2, \tag{6.20}$$

$$\eta_F^{(\mathrm{II})} = \frac{\bar{e}^2}{6\pi} \frac{a}{k^2}. \tag{6.21}$$

The results differ for $a \neq 1$ even at $\overline{m} = 0$. In fact, looking at (6.15) and (6.16), it is clear that for $d \neq 4$ $\eta_F^{(\mathrm{I})}$ and $\eta_F^{(\mathrm{II})}$ will not coincide in general. In particular, any explicit dependence of $\eta_F^{(\mathrm{II})}$ on k will always involve the ratio k^2/a. We note that $\eta_F^{(\mathrm{I})}$ is well defined only for $d < 6$, whereas $\eta_F^{(\mathrm{II})}$ can be used for arbitrary d. (The measure $\mu_k^{(\mathrm{II})}$ (2.24) can easily be generalized for $\overline{m}^2 > 0$.) For completeness, we also give the result for three dimensions:

$$\eta_F^{(\mathrm{I})} = \frac{\bar{e}^2}{24\pi k}\left(\frac{k^2}{k^2+m^2}\right)^{3/2}. \tag{6.22}$$

We finally introduce the dimensionless renormalized gauge coupling

$$e^2(k) = k^{d-4} e_{\mathrm{R}}^2(k) = k^{d-4} Z_{F,k}^{-1} \bar{e}^2. \tag{6.23}$$

(The physical renormalized charge is $e_{\mathrm{R}} = Z_{F,k}^{-1/2}\bar{e}$.) Then the evolution of $e^2(k)$ is governed by the β-function

$$\beta_{e^2} \equiv k\frac{\mathrm{d}}{\mathrm{d}k} e^2(k) = (d-4+\eta_F) e^2(k). \tag{6.24}$$

We use renormalization group improvement and replace in η_F the bare coupling $\bar{e}$ by the k-dependent coupling $e_{\mathrm{R}}(k)$ and also the scalar mass term by the running mass as given by the second derivative of U_k at the origin,

$$\overline{m}^2 \to m^2(k). \tag{6.25}$$

The β-function can then be written in the generic form

$$\beta_{e^2} = (d-4)e^2 + \tfrac{4}{3} v_d l_{\mathrm{g}}^d s_{\mathrm{g}}^d (m^2/k^2) e^4. \tag{6.26}$$

Here the threshold functions s_{g}^d only depend on the dimensionless ratio m^2/k^2 and are normalized such that $s_{\mathrm{g}}^d(0) = 1$. They have a particularly simple form for the choice (2.23),

$$s_{\mathrm{g}}^{d(\mathrm{I})}\left(\frac{m^2}{k^2}\right) = \left(1 + \frac{m^2}{k^2}\right)^{(d-6)/2} \tag{6.27}$$

The constants l_{g}^d depend in general on the averaging scheme and the choice of μ_k except for $d = 4$ where

$$l_{\mathrm{g}}^4 = 1. \tag{6.28}$$

For the particular choice (2.23) one has

$$l_{\mathrm{g}}^{2(\mathrm{I})} = 1, \quad l_{\mathrm{g}}^{3(\mathrm{I})} = \tfrac{1}{4}\pi. \tag{6.29}$$

In three dimensions one has $v_3 = 1/8\pi^2$ and (6.26) coincides with (1.13).

7. The regime with spontaneous symmetry breaking

So far we have concentrated on the symmetric regime where the minimum of U_k occurs at $\varphi = 0$. For a study of spontaneous symmetry breaking we need, however, the running of the gauge coupling in the spontaneously broken regime where the potential minimum occurs for a nonvanishing scalar field $\varphi_0(k)$,

$$\rho_0(k) = |\varphi_0(k)|^2, \quad \frac{\partial}{\partial\rho} U_k(\rho_0) = 0, \quad \rho = |\varphi|^2. \tag{7.1}$$

Two changes in the formulation become necessary: First we define the constraint (2.4) with $\overline{m}^2 = 0$. In the spontaneously broken regime the mass term at the origin ($\varphi = 0$) is negative and has no direct physical relevance. The masses entering the computation should rather be given by an expansion around ρ_0. This also dictates the second change, namely the definition of η_F through the k-dependence of the wave function renormalization at $\rho_0(k)$,

$$\eta_F = -k\frac{\mathrm{d}}{\mathrm{d}k}\ln\left[Z_{F,k}(\rho_0(k))\right]. \tag{7.2}$$

Neglecting terms $\sim \partial Z/\partial\rho_0$ we can omit here the implicit k-dependence through the k-evolution of the minimum $\rho_0(k)$.

In the spontaneously broken regime the masses of the relevant physical excitations are the radial scalar mass m and the photon mass M as given by

$$m^2 = 2\bar{\lambda}\rho_0, \quad M^2 = 2\bar{e}^2\rho_0. \tag{7.3}$$

In the limit $m^2 = M^2 = 0$ the running of the gauge coupling should be the same as for the symmetric regime with $m^2 \to 0$. One concludes the general form of the evolution equation for the gauge coupling in the spontaneously broken regime,

$$\beta_{e^2} = (d-4)e^2 + \tfrac{4}{3}v_d l_{\mathrm{g}}^d \tilde{s}_{\mathrm{g}}^d(2\lambda\kappa, 2e^2\kappa)e^4. \tag{7.4}$$

The threshold functions $\tilde{s}_{\mathrm{g}}^d$ are again normalized as

$$\tilde{s}_{\mathrm{g}}^d(0, 0) = 1. \tag{7.5}$$

They depend on the dimensionless ratios of renormalized mass over k, i.e.

$$\frac{M^2(k)}{k^2} = 2e^2\kappa, \quad \frac{m^2(k)}{k^2} = 2\lambda\kappa, \tag{7.6}$$

where

$$\kappa = k^{2-d}Z_\varphi\rho_0(k), \quad \lambda = k^{d-4}Z_\varphi^{-2}\bar{\lambda}(k). \tag{7.7}$$

In three dimensions eq. (7.4) gives the evolution equation (1.13) in the regime with spontaneous symmetry breaking.

We have computed in Appendix C the threshold function in the limit $e^2\kappa \to 0$. One finds

$$l_{\mathrm{g}}^{d}\tilde{s}_{\mathrm{g}}^{d}(2\lambda\kappa, 0) = k^{3-d}\frac{\partial}{\partial k}J(2\bar{\lambda}\rho_0), \tag{7.8}$$

with

$$\begin{aligned} J(w) = & -\int_0^\infty \mathrm{d}x\, x^{d/2-1}\left(\frac{1}{P+w} - \frac{1}{P}\right)\left(3\ddot{P} + \frac{9}{d}xP^{(3)} + \frac{7}{d(d+2)}x^2P^{(4)}\right) \\ & + \int_0^\infty \mathrm{d}x\, x^{d/2}\left(\frac{1}{P(P+w)} - \frac{1}{P^2}\right) \\ & \times\left(\frac{18}{d}\dot{P}\ddot{P} + \frac{32}{d(d+2)}x\dot{P}P^{(3)} + \frac{18}{d(d+2)}x\ddot{P}^2\right) \\ & - \int_0^\infty \mathrm{d}x\, x^{d/2}\left(\frac{1}{P(P+w)^2} - \frac{1}{P^3}\right)\left(\frac{6}{d}\dot{P}^3 + \frac{60}{d(d+2)}x\dot{P}^2\ddot{P}\right) \\ & + \frac{24}{d(d+2)}\int_0^\infty \mathrm{d}x\, x^{d/2+1}\left(\frac{1}{P(P+w)^3} - \frac{1}{P^4}\right)\dot{P}^4 \\ & - \frac{v_{d-2}}{8\pi v_d}\int_0^\infty \mathrm{d}x\, x^{d/2-2}(x+k^2)^{-1}. \end{aligned} \tag{7.9}$$

Here we use $x = q^2$, $P(x) = x(1-f_k^2)^{-1}$ and $\dot{P} = \partial P/\partial x$ etc.. We also observe that the k-derivative in (7.8) acts on P with

$$k\frac{\partial}{\partial k}\left(\frac{P}{x}\right) = -2x\frac{\partial}{\partial x}\left(\frac{P}{x}\right) \tag{7.10}$$

and should be applied on the integrand such that all integrals in $k\partial J/\partial k$ are finite.

One may check the limiting case

$$\lim_{w\to 0} k\frac{\partial}{\partial k}J(w) = k^{d-4}l_{\mathrm{g}}^{d}. \tag{7.11}$$

One also expects

$$\lim_{w\to\infty} k\frac{\partial}{\partial k}J(w) = 0 \tag{7.12}$$

since the running of the physical gauge coupling $e_R^2 = k^{4-d}e^2$ should stop once the scalar mass is much bigger than k. We refer for a detailed evaluation of $\tilde{s}_g^d$ to a future publication. Here we demonstrate the generic behaviour only for a "test function"

$$f_k^2(x) = k^2/(x+k^2), \quad P(x) = x + k^2. \tag{7.13}$$

We have evaluated (7.8) for this particularly simple form in Appendix C and find for $d \leqslant 4$

$$\begin{aligned} l_g^d \tilde{s}_g^d = &\frac{12}{d(d+2)}(1+2\lambda\kappa)^{(d-4)/2}\int_0^\infty \mathrm{d}y\Big\{y^{d/2}(1+y)^{-3}(1+y+2\lambda\kappa y)^{-2} \\ &\times\left[(d-2)y+d+2\right]\left[(2-\tfrac{1}{2}d)(1+y+2\lambda\kappa y)+2\lambda\kappa\right]\Big\} \\ &+(1+2\lambda\kappa)^{-1}\delta_{d4}. \end{aligned} \tag{7.14}$$

In four dimensions one obtains

$$\tilde{s}_g^4 = 2\lambda\kappa\int \mathrm{d}y \frac{y^2(y+3)}{(1+y)^3(1+y+2\lambda\kappa y)^2} + \frac{1}{1+2\lambda\kappa} \tag{7.15}$$

and verifies

$$\lim_{\lambda\kappa\to\infty} \tilde{s}_g^4 = 3/2\lambda\kappa, \quad \lim_{\lambda\kappa\to 0} \tilde{s}_g^4 = 1-\lambda\kappa \tag{7.16}$$

8. Discussion and conclusions

The main result of this paper is the running of the gauge coupling in the abelian Higgs model as a function of the average scale k, in arbitrary dimension d. It is given by eqs. (6.26) and (7.4) which correspond to the evolution equation (1.13) mentioned in the introduction. In four dimensions one recovers the well-known logarithmic running which is finally stopped once k becomes smaller than the relevant physical particle masses. Due to strong infrared effects this behaviour changes qualitatively in two and three dimensions. If the particle masses are small compared to k ($2\lambda\kappa \ll 1$, $2e^2\kappa \ll 1$ in the spontaneously broken regime, $m^2/k^2 \ll 1$ in the symmetric regime) the running of e is determined by an infrared fixpoint

$$e_*^2 = \frac{3(4-d)}{4v_d l_g^d}. \tag{8.1}$$

As long as this fixpoint dominates the evolution the physical gauge coupling runs to zero with a power of k,

$$e_{\mathrm{R}}^2 = k^{4-d} e_*^2. \tag{8.2}$$

This running is much faster than the logarithmic running in four dimensions. The running of e_{R} ceases only once k becomes smaller than the relevant particle masses.

In any model with a genuine long-distance behaviour, i.e. where the physical particle masses are much smaller than the ultraviolet cutoff (the inverse lattice distance in statistical mechanics) the evolution equation implies an upper bound on the renormalized gauge coupling. The massless evolution equation ($t = \ln(k/\Lambda)$)

$$\frac{\partial e^2}{\partial t} = (d-4)e^2 + be^4, \quad b = \tfrac{4}{3} v_d l_{\mathrm{g}}^d, \tag{8.3}$$

is easily solved:

$$e^2(k) = \frac{e^2(\Lambda)}{1 + be^2(\Lambda)\ln(\Lambda/k)} \quad \text{for} \quad d = 4,$$

$$e^2(k) = \frac{4-d}{b}\left[1 - \left(1 - \frac{4-d}{be^2(\Lambda)}\right)\left(\frac{k}{\Lambda}\right)^{4-d}\right]^{-1} \quad \text{for} \quad d < 4. \tag{8.4}$$

The upper bound obtains for $e^2(\Lambda) \to \infty$

$$e^2(k) < \frac{1}{b\ln(\Lambda/k)} \quad \text{for} \quad d = 4,$$

$$e^2(k) < \frac{4-d}{b\left[1 - (k/\Lambda)^{4-d}\right]} \quad \text{for} \quad d < 4. \tag{8.5}$$

In contrast to the slow logarithmic running for $d = 4$ the upper bound for $d < 4$ approaches rapidly the value given by the fixpoint $e_*^2 = (4-d)/b$,

$$e_{\mathrm{R}}^2(k) < e_*^2 \frac{k^{4-d}}{1 - (k/\Lambda)^{4-d}}. \tag{8.6}$$

Of course, the validity of the bound (8.5) depends on the reliability of the evolution equation for large values of e^2. It could be invalidated, however, only if β_{e^2} exhibits an (approximative) ultraviolet fixpoint for large values of e^2. We see no indication of such a fixpoint in our approach. In view of the surprisingly good description of nonperturbative phenomena like critical indices in the three-dimensional scalar theory or the Kosterlitz–Thouless phase transition in two dimensions we are actually quite confident that the bounds derived from our one-loop computation are even quantitatively correct.

The running of e_{R}^2 is finally stopped once k becomes smaller than physical particle masses as for example m. This effect is due to the threshold functions s_{g}^d or $\tilde{s}_{\mathrm{g}}^d$. One approximately obtains

$$e_{\mathrm{R}}^2(0) = e_{\mathrm{R}}^2(k_c) = e_{\mathrm{R}}^2(cm), \tag{8.7}$$

where the proportionality constant $c = k_{\mathrm{c}}/m$ depends on details of the averaging. We note that for $k \gg m$ the coupling $e_{\mathrm{R}}(k)$ is not directly related to a physical vertex. The dependence of the evolution equation ($b \sim l_{\mathrm{g}}^d$) on the average scheme is therefore not unexpected. On the other hand, $e_{\mathrm{R}}(k)$ should be in a direct relation with physical vertices for $k \ll m$. The average scheme dependence of l_{g}^d and of c should therefore cancel if our approximations are valid. If the original coupling $e^2(\Lambda)$ is not too small, the scale of the physical coupling in two and three dimensions is always set by the relevant particle mass, i.e.

$$e_{\mathrm{R}}^2(0) = \frac{3(4-d)}{4v_d l_{\mathrm{g}}^d}(cm)^{d-4}. \tag{8.8}$$

(This formula holds for $e^2(\Lambda) > e_*^2$ up to corrections $\sim cm/\Lambda$.)

We finally observe that for small $e^2(\Lambda)$ there is first a regime of slow running of e_{R} before it "feels" the attraction of the infrared fixpoint for $d < 4$. As long as

$$e^2(k) \ll e_*^2 \tag{8.9}$$

the evolution equation for $d < 4$ has the approximate solution for e_{R},

$$e_{\mathrm{R}}^2(k) = \frac{e_{\mathrm{R}}^2(\Lambda)}{1 + b e_{\mathrm{R}}^2(\Lambda) k^{d-4}/(4-d)}. \tag{8.10}$$

This regime of slow decrease of e_{R} holds for

$$k^{4-d} \gg b e_{\mathrm{R}}^2(\Lambda)/(4-d). \tag{8.11}$$

This remark is of immediate relevance for the four-dimensional abelian Higgs model at nonzero temperature T. Near the phase transition one expects particle masses much smaller than T. This implies a region of effective three-dimensional running for the couplings. Identifying the cutoff Λ with the onset of the three-dimensional running $\Lambda = T/\theta$ (with $\theta \approx 0.2$ in analogy to ref. [10]) and the three-dimensional coupling $e_R^2(\Lambda) = 4\pi T\alpha$ (with α the four-dimensional fine structure constant), one finds that the running of e_R^2 can be neglected only for

$$k \gg \tfrac{1}{6}\alpha T. \tag{8.12}$$

For masses of the order αT or smaller the three-dimensional running of the gauge coupling will be an important effect. A qualitatively similar remark applies to non-abelian gauge theories as the standard model. Here b should be replaced in (8.11) by the appropriate coefficient of the three-dimensional nonabelian β-function.

Indeed, one expects for the running of the dimensionless gauge coupling g^2 in a nonabelian SU(M) gauge theory in $d \leqslant 4$ the behaviour

$$\frac{\partial}{\partial t}g^2 = \beta_{g^2} = (d-4)g^2 - \left(\tfrac{44}{3}M - \tfrac{2}{3}\right)v_d l_{\mathrm{NA}}^d \tilde{s}_{\mathrm{NA}}^d\left(2\lambda\kappa, \tfrac{1}{2}g^2\kappa\right)g^4 \tag{8.13}$$

(and similar in the symmetric regime). Here l_{NA}^d are numerical coefficients of order one to be computed, with $l_{\mathrm{NA}}^4 = 1$ for one M-component complex scalar coupled to the gauge fields. For a vanishing expectation value $\kappa = 0$ at the origin of the scalar potential the gauge fields are massless and $\tilde{s}_{\mathrm{NA}}^d = 1$ *. For $d < 4$ one obtains the solution

$$g^2(k) = \frac{g^2(\Lambda)(\Lambda/k)^{4-d}}{1 - (44M-2)v_d l_{\mathrm{NA}}^d g^2(\Lambda)\left[(\Lambda/k)^{4-d} - 1\right]/3(4-d)}. \tag{8.14}$$

Identifying the confinement scale with the value of k where $g^2(k)$ diverges yields

$$\Lambda_{\mathrm{conf}}^{(d)} = \Lambda\left(\frac{Bg^2(\Lambda)}{4-d+Bg^2(\Lambda)}\right)^{1/(4-d)}, \tag{8.15}$$

$$B = \left(\tfrac{44}{3}M - \tfrac{2}{3}\right)v_d l_{\mathrm{NA}}^d. \tag{8.16}$$

We may apply these findings directly to the electroweak phase transition (with $M = 2$). At the origin of the scalar potential the running of $g_R^2(k) = g^2(k)k$ leads to three-dimensional confinement with

$$\Lambda_{\mathrm{conf}}^{(3)} = \frac{43 l_{\mathrm{NA}}^{(3)}}{3\pi}\alpha_{\mathrm{W}} T\left(1 + \frac{43 l_{\mathrm{NA}}^{(3)}}{3\pi}\alpha_{\mathrm{w}}\theta\right)^{-1}. \tag{8.17}$$

* If the scalar field is massive with $m^2/k^2 \geqslant 1$, the effective coefficient l_{NA}^d is somewhat enhanced by the decoupling of the scalar degree of freedom.

Here we have made again the identification $\Lambda = T/\theta$, $g_R^2(\Lambda) = 4\pi\alpha_W T$. For realistic values $\alpha_w \sim \frac{1}{30}$ the three-dimensional confinement scale is around a tenth of the temperature:

$$\Lambda_{\mathrm{conf}}^{(3)} \approx 0.1T. \tag{8.18}$$

We emphasize that "three-dimensional confinement" relates here to the behaviour of time-averaged space-like correlation functions in the thermodynamic equilibrium and not directly to time-like correlation functions. Also, g_R^2 is the gauge coupling of the transversal three-dimensional gauge bosons. The zero component A_0 of the gauge field behaves as a three-dimensional scalar field in the adjoint representation. It acquires a mass directly from the usual temperature effects for scalar fields with an additional contribution from three-dimensional confinement. For space-like momenta of the order $\Lambda_{\mathrm{conf}}^{(3)}$ (and $q_0 = 0$) this scalar is strongly interacting. For space-like correlations and for $\varphi = 0$ we conclude that the high temperature phase of the electroweak gauge theory behaves as a strong coupling theory with a three-dimensional confinement scale only moderately below the temperature.

The order of the phase transition can be characterized by the behaviour of the space-like correlations. In particular, a second-order phase transition corresponds to an infinite correlation length in the space-like direction. In consequence, the electroweak phase transition is described by the three-dimensional confinement/Higgs transition with strong gauge coupling unless a first-order transition induces a minimal value of spontaneous symmetry breaking ρ_c at $T = T_c$ such that the induced gauge boson mass $(\frac{1}{2}g_R^2\rho_c)^{1/2}$ is substantially higher than $0.1T_c$. If the transition occurs in the regime of strong coupling g_R it could actually be a smooth analytic continuation rather than a genuine phase transition [13]. If confinement operates, all bosons except for the photon will acquire a (three-dimensional) mass proportional to the confinement scale. (This is the only mechanism by which the transverse gauge bosons can acquire a mass without spontaneous symmetry breaking. Three-dimensional gauge invariance forbids the appearance of a transverse mass term which is not related to confinement.) Three-dimensional confinement therefore excludes that the electroweak phase transition is second order! A second-order phase transition necessitates at the critical temperature a massless scalar field with vanishing expectation value, in direct contradiction to confinement. We conclude that the scalar field is always massive at the transition where spontaneous symmetry breaking sets in. It remains an open question if there is a jump in some quantities (e.g. masses) indicating a first-order transition or if the behaviour is smooth such that no real phase transition occurs. The answer may depend critically on the ratio between the quartic scalar coupling and the gauge coupling, i.e. the (zero temperature) Higgs mass in units of the W-boson mass.

One last remark concerns QCD at high temperatures. Relation (8.18) is replaced for strong interactions ($M=3$) by

$$\Lambda_{\text{conf}}^{(3)} \approx 5\alpha_s(5T)T, \tag{8.19}$$

where α_s is the four-dimensional strong gauge coupling evaluated at the scale $T/\theta \propto 5T$. The scale $\Lambda_{\text{conf}}^{(3)}$ is of the same order as the temperature or even somewhat larger for the interesting range in T. In a theory without quarks the following qualitative picture for the behaviour of the space-like correlation functions emerges: As long as the temperature is small enough such that the value of the four-dimensional (zero temperature) confinement scale $\Lambda_{\text{conf}}^{(4)}$ is larger than T/θ, the three-dimensional running of the gauge coupling is ineffective. For higher temperatures, however, three-dimensional confinement replaces four-dimensional confinement and $\Lambda_{\text{conf}}^{(3)}$ sets the relevant mass scale for the theory.

Such a transition seems to indicate a smooth behaviour for the space-like correlation functions (compare ref. [14] for the behaviour of Wilson loops in space-like directions.) For very high temperatures ($T \gg \Lambda_{\text{conf}}^{(4)}$) the infrared behaviour in the space-like directions is described by three-dimensional confinement. The relevant degrees of freedom in the momentum range $q^2 < T^2$ are not weakly interacting gluons but rather the glueballs of a three-dimensional confining theory. Their mass scale is set by the temperature.

In the presence of quarks the picture of the transition is somewhat modified. As a consequence of three-dimensional confinement one expects at very high temperatures that the quark degrees of freedom are described by mesons and hadrons as far as the space-like correlations are concerned. (The picture of weakly interacting quark degrees of freedom applies here to momenta much higher than the temperature.) As a new ingredient, however, the spontaneous breaking of the chiral symmetry should disappear for sufficiently high T. According to ref. [15], this transition would be of second-order in the case of massless up and down quarks. On the other hand, the observed value of the zero temperature pion mass seems to imply a rather smooth transition.

Note added

Recently, the finite temperature mass of the transversal weak gauge bosons was computed [17] within resummed perturbation theory by solving gap equations. The value $M_T \approx \frac{4}{3}\alpha_W T$ is about a factor of three below the three-dimensional confinement scale $\Lambda_{\text{conf}}^{(3)}$ (8.17). The reliability of high temperature perturbation theory for processes involving transversal gauge bosons seems therefore questionable in its present form.

Appendix A

In this appendix we show that the measures $\mu_k^{(\mathrm{I})}[A]$ and $\mu_k^{(\mathrm{III})}[A]$ defined in eqs. (2.23) and (2.25), respectively, coincide for weak and slowly varying gauge fields A_μ. The calculation will be performed for a constant magnetic field B with $\bar{e}B \ll k^2$, using methods of ref. [16]. In this limit the equality of $\mu_k^{(\mathrm{I})}$ and $\mu_k^{(\mathrm{III})}$ becomes exact. (For simplicity we fix $d=4$ and omit the bars from $\bar{e}$ and $\overline{m}$.)

Using the proper time representation of the logarithm and the explicit form of the spectrum of $-D^2(A)$, eqs. (6.7) and (6.8), we obtain from (2.25)

$$\begin{aligned}\ln \mu_k^{(\mathrm{III})} &= -\int_0^\infty \frac{\mathrm{d}s}{s} \mathrm{Tr}\{\theta(-D^2-k^2)\exp[-s(-D^2+m^2)]\}\\ &= -\Omega\frac{eB}{8\pi^2}\int_0^\infty\frac{\mathrm{d}s}{s}\exp(-m^2 s)\sum_{n=0}^{\infty}\int_0^\infty \mathrm{d}x\ \theta(x+(2n+1)eB-k^2)\\ &\quad\times\exp\{-s[x+(2n+1)eB]\}. \end{aligned}\tag{A.1}$$

After shifting $x \to x-(2n+1)eB$, the integration is easily performed with the result

$$\ln \mu_k^{(\mathrm{III})} = -\Omega\frac{eB}{8\pi^2}\int_0^\infty\frac{\mathrm{d}s}{s^2}\exp(-m^2 s)\sum_{n=0}^{\infty}C_n,\tag{A.2}$$

where

$$C_n \equiv \exp[-s(2n+1)eB]\theta((2n+1)eB-k^2)+\exp(-sk^2)\theta(k^2-(2n+1)eB).\tag{A.3}$$

Because of the step functions, $\ln\mu_k$ suffers from discrete jumps whenever k^2 crosses one of the Landau levels $(2n+1)eB$. Therefore $\Gamma_k^{(1)} = \ln\mu_k$ (for any definition of the measure) is not a sensible effective action for strong fields $eB \approx k^2$. For weak fields $eB \ll k^2$ the Landau levels form a quasi-continuum (i.e. there are very many levels below k^2) so that a smooth dependence on B and k^2 results. This is seen as follows. Assume that $eB \ll k^2$ and define

$$N \equiv [k^2/2eB] \gg 1,$$

where $[x]$ denotes the largest integer contained in x. Then

$$\begin{aligned}\sum_{n=0}^{\infty}C_n &= \sum_{n=N+1}^{\infty}\exp[-s(2n+1)eB]+\sum_{n=0}^{N}\exp(-sk^2)\\ &\approx \frac{e^{-k^2 s}}{2\sinh(eBs)}+\frac{k^2}{2eB}\exp(-k^2 s).\end{aligned}\tag{A.4}$$

To arrive at the second line of eq. (A.4) we summed the geometric series and approximated N by $k^2/2eB$. This leads to the desired smooth (averaged) dependence on k^2 and B. Inserting (A.4) into (A.2) yields, up to an irrelevant constant,

$$\ln \mu_k^{(\mathrm{III})} = -\frac{\Omega}{16\pi^2}\int_0^\infty \frac{\mathrm{d}s}{s^3}\exp\left[-(m^2+k^2)s\right]\frac{eBs}{\sinh(eBs)}. \tag{A.5}$$

The RHS of (A.5) is precisely the integral representation of the Heisenberg–Euler effective action $\Gamma_{\mathrm{HE}}[A;\, m^2] \equiv \ln\det(-D^2+m^2)$ of scalar QED [16] with m^2 replaced by m^2+k^2. Thus, recalling the definition (2.23) of $\mu_k^{(\mathrm{I})}$, we arrive at the conclusion that for weak and (almost) constant fields

$$\ln\left(\mu_k^{(\mathrm{III})}[A]\right) = \Gamma_{\mathrm{HE}}[A;\, m^2+k^2] = \ln\left(\mu_k^{(\mathrm{I})}[A]\right). \tag{A.6}$$

This illustrates the points made in sect. 2 in the kinematical region where $\Gamma_k^{(1)} = \ln\mu_k$ is a good effective action, many different forms of the measure μ_k are physically equivalent.

Appendix B

Functional determinants like those appearing in eq. (5.6) cannot be computed exactly for a generic gauge field A_μ. Often it is sufficient to know the first few terms of an expansion in powers of $A_\mu(x)$. In this appendix we describe a general strategy to obtain this expansion in powers of A_μ for an arbitrary function $M(-D^2(A))$ of the covariant derivative operator

$$-D^2(A) = \Delta_1 + \Delta_2,$$

$$\Delta_1 \equiv -\partial^2, \quad \Delta_2 \equiv -2ieA^\mu\partial_\mu - ie(\partial_\mu A^\mu) + e^2A_\mu A^\mu. \tag{B.1}$$

Because Δ_1 and Δ_2 do not commute, one has to be careful in keeping track of commutator terms $\sim [\Delta_1^n, \Delta_2^m]$. We assume that the function M has a Fourier representation

$$M(x) = \int_{-\infty}^{\infty} \mathrm{d}y\, \tilde{M}(y)\, \exp(ixy) \tag{B.2}$$

so that

$$M(-D^2(A)) = \int_{-\infty}^{\infty} \mathrm{d}y\, \tilde{M}(y)\, \exp[iy(\Delta_1+\Delta_2)]. \tag{B.3}$$

For any pair of (non-commuting) operators A and B we can write down a "Dyson series"

$$e^{A+B}=e^{A}T\,\exp\left(\int_0^1 dt\; e^{-At}Be^{At}\right), \tag{B.4}$$

where T denotes the t-ordering operator. Applying this formula to (B.3), one obtains the following expansion in powers of $B\equiv iy\Delta_2$:

$$\begin{aligned}M(-D^2(A)) &= M_0+M_1+M_2+\dots\\ &\equiv\int_{-\infty}^{\infty}dy\,\tilde{M}(y)\,\exp(iy\Delta_1)\left(1+iy\int_0^1 dt\,\exp(-iy\Delta_1 t)\Delta_2\exp(iy\Delta_1 t)\right.\\ &\quad-\tfrac{1}{2}y^2\int_0^1 dt_1\int_0^1 dt_2\,T\{\exp(-iy\Delta_1 t_1)\Delta_2\exp[iy\Delta_1(t_1-t_2)]\Delta_2\\ &\quad\left.\times\exp(iy\Delta_1 t_2)\}+\dots\right).\end{aligned} \tag{B.5}$$

Here M_n contains n factors of Δ_2, i.e. the number of A_μ's contained in the various terms of M_n ranges from n to $2n$. If we are interested in an expansion up to order A_μ^2, we have to retain the terms up to M_2. They are given by

$$\begin{aligned}M_0 &= M(-\partial^2),\\ M_1 &\equiv M_1^A+M_1^{AA},\\ M_1^A &= 2e\int_{-\infty}^{\infty}dy\,\tilde{M}(y)\,y\int_0^1 dt\,\exp[-iy\partial^2(1-t)]A^\mu\partial_\mu\exp(-iy\partial^2 t),\\ M_1^{AA} &= ie^2\int_{-\infty}^{\infty}dy\,\tilde{M}(y)\,y\int_0^1 dt\,\exp[-iy\partial^2(1-t)]A^\mu A_\mu\exp(-iy\partial^2 t),\\ M_2 &= 4e^2\int_{-\infty}^{\infty}dy\,\tilde{M}(y)\,y^2\int_0^1 dt_1\int_0^{t_1}dt_2\,\exp[-iy(1-t_1)\partial^2]A^\mu\partial_\mu\\ &\quad\times\exp[-iy\partial^2(t_1-t_2)]A^\nu\partial_\nu\exp(-iy\partial^2 t_2)+O(A_\mu^3).\end{aligned} \tag{B.6}$$

Here we have taken for simplicity a transversal gauge field $\partial_\mu A^\mu=0$.

In order to further simplify these equations, one either expands in the number of derivatives acting on A_μ and truncates this series at a certain stage, or one assumes that $A_\mu(x)$ has some particularly simple form. We shall do the latter and assume that $A_\mu(x)$ is a transverse plane wave with momentum Q^μ:

$$A_\mu(x)=\alpha_\mu\exp(iQ_\nu x^\nu),\quad \alpha_\mu Q^\mu=0. \tag{B.7}$$

(The extension to complex A_μ poses no problem in our context.) For this form of the gauge field it is easy to commute operators like $\exp(-iy\partial^2 t)$ through $A_\mu(x)$, so that M_1^A, say, becomes

$$M_1^A = 2eA^\mu \int_{-\infty}^{\infty} \mathrm{d}y\, \tilde{M}(y)\, y \exp\left[-iy(\partial + iQ)^2\right] \int_0^1 \mathrm{d}t \exp\left\{iyt\left[(\partial + iQ)^2 - \partial^2\right]\right\}\partial_\mu . \tag{B.8}$$

Now the t-integration can be performed and the Fourier transformation can be inverted. In this way one finds to second order in A_μ (and all orders in Q^μ):

$$\begin{aligned} M(-D^2(A)) = {} & M(-\partial^2) - 2ieA^\mu\partial_\mu \check{M}_1(\partial, Q) \\ & + e^2 A_\mu A^\mu \check{M}_1(\partial, 2Q) - 4e^2 A^\mu \partial_\mu \check{M}_2(\partial, Q) A^\nu \partial_\nu + \mathrm{O}(A^3), \end{aligned} \tag{B.9}$$

with

$$\begin{aligned} \check{M}_1(\partial, Q) &= \frac{M(-\partial^2) - M\left(-(\partial + iQ)^2\right)}{(\partial + iQ)^2 - \partial^2}, \\ \check{M}_2(\partial, Q) &= \frac{1}{(\partial - iQ)^2 - \partial^2}\left(\frac{M(-\partial^2) - M\left(-(\partial + iQ)^2\right)}{(\partial + iQ)^2 - \partial^2}\right. \\ &\qquad \left. - \frac{M\left(-(\partial - iQ)^2\right) - M\left(-(\partial + iQ)^2\right)}{4iQ\partial}\right). \end{aligned} \tag{B.10}$$

In the limit $Q \to 0$ the operators Δ_1 and Δ_2 commute, so that (B.9) should reduce to the naive Taylor series. Indeed, from (B.10) it is easy to verify that

$$\lim_{Q\to 0} \check{M}_1(\partial, Q) = \dot{M}(-\partial^2), \quad \lim_{Q \to 0} \check{M}_2(\partial, Q) = \tfrac{1}{2}\ddot{M}(-\partial^2), \tag{B.11}$$

where the dot denotes a derivative with respect to $-\partial^2$.

As a first application we now use the expansion (B.9) to establish eq. (5.8) of sect. 5. Let us evaluate $\Delta\Gamma_k^{(1)}[A_\mu]$ for A_μ given by (B.7). Eqs. (B.1) and (5.3) imply that

$$\begin{aligned} & -D^2(a^{\min}) = -D^2(A) + \delta_1 + \delta_2, \\ & \qquad \delta_1 \equiv 2ie\left[1 - f_k(Q^2)\right]A^\mu\partial_\mu, \quad \delta_2 \equiv -e^2\left[1 - f_k^2(Q^2)\right]A^\mu A_\mu . \end{aligned} \tag{B.12}$$

Inserting this into $\Delta\Gamma_k^{(1)}$ as defined in eq. (5.8) we obtain

$$\Delta\Gamma_k^{(1)}[A] = \mathrm{Tr}\,\ln\left[1 + M(-D^2(A))(\delta_1 + \delta_2)\right], \tag{B.13}$$

with

$$M(x) \equiv \frac{1 - f_k^2(x)}{x + \overline{m}^2}. \tag{B.14}$$

Expanding up to second order in A_μ, one finds

$$\begin{aligned}
\Delta\Gamma_k^{(1)}[A] &= G_1 + G_2 + G_3 + \mathrm{O}\left(A_\mu^4\right),\\
G_1 &\equiv 2ie\left[1 - f_k(Q^2)\right]\mathrm{Tr}\left[M(-D^2(A))A^\mu\partial_\mu\right],\\
G_2 &\equiv -e^2\left[1 - f_k^2(Q^2)\right]\,\mathrm{Tr}\left[M(-D^2(A))A^\mu A_\mu\right],\\
G_3 &\equiv 2e^2\left[1 - f_k(Q^2)\right]\,\mathrm{Tr}\left[M(-D^2(A))A^\mu\partial_\mu M(-D^2(A))A^\nu\partial_\nu\right].
\end{aligned} \tag{B.15}$$

In G_2 and G_3 we may replace $M(-D^2(A))$ by $M(-\partial^2)$ immediately. Because $1 - f_k(Q^2) \sim (Q^2/k^2)^\beta$, we observe that these terms are of the order of $(Q^2/k^2)^\beta$. In G_1 we use (B.9), so that

$$G_1 = 4e^2\left[1 - f_k(Q^2)\right]\,\mathrm{Tr}\left[\check{M}_1(\partial, Q)\left(A^\mu\partial_\mu\right)^2\right]. \tag{B.16}$$

Eq. (B.11) shows that $\check{M}_1$ becomes independent of Q for $Q^2 \ll k^2$. Hence G_1 is of order $(Q^2/k^2)^\beta$. Invoking gauge invariance, this implies that $\Delta\Gamma_k^{(1)}$ has an expansion of the form anticipated in eq. (5.8).

Appendix C

In this appendix we calculate the one-loop contribution to the anomalous dimension η_F in the regime with spontaneously broken gauge symmetry: $-\overline{m}^2 \equiv \overline{\mu}^2 > 0$. Because the relevant renormalization constant is $Z_F^{(1)} \equiv Z_{F,k}^{(1)}(\varphi_0^2)$ where $\varphi_0^2 \equiv \rho_0 \equiv \overline{\mu}^2/\overline{\lambda}$ is the minimum of the classical potential, we will evaluate $\Gamma_k[\varphi, A_\mu]$ for $\varphi = \varphi_0 =$ const. and for a plane wave $A_\mu(x)$ with momentum $Q^2 \ll k^2$. We make the approximation of keeping only the terms of order $\bar{e}^2$. This simplifies the calculation considerably, since the relevant term in eq. (5.1), $\frac{1}{4}Z_F^{(1)}F_{\mu\nu}F^{\mu\nu}$ with $Z_F^{(1)} \sim \bar{e}^2$, contains $\bar{e}$ only in the combination $\bar{e}A_\mu$. Therefore we are allowed to drop from the integrand of the functional integral all terms containing factors of $\bar{e}$ which are not accompanied by corresponding factors of A_μ. For the field configuration at hand the minimum of $S_k \equiv S + S_{\mathrm{constr}}$ is given by $\chi^{\min}(x) = \varphi_0$, $a_\mu^{\min} = A_\mu$.

The quadratic action for small fluctuations around this solution can be read off from ref. [9], Appendix B. For our approximation the part which depends on $\bar{e}A_\mu$ reduces to

$$\delta^2 S_k = \int d^dx \left[\sigma\left(P_{\text{sym}} + 2\bar{\lambda}\rho_0\right)\sigma + \omega P_{\text{sym}}\omega + 2\omega P_{\text{as}}\sigma\right] + \dots, \tag{C.1}$$

where

$$P_{\text{sym}} \equiv \tfrac{1}{2}\left[P(-D^2(A)) + P(-D^2(-A))\right],$$

$$P_{\text{as}} \equiv \frac{1}{2i}\left[P(-D^2(A)) - P(-D^2(-A))\right],$$

$$P(x) = x\left[1 - f_k^2(x)\right]^{-1}. \tag{C.2}$$

Eq. (C.1) is obtained by expanding S_k around its minimum up to second order in $\delta\chi \equiv 2^{-1/2}(\sigma + i\omega)$ and δa_μ. In (C.1) the terms containing δa_μ are either of higher order in $\bar{e}$ or they decouple from A_μ. They can be omitted in our approximation. (For details we refer to ref. [9].) The one-loop approximation of Γ_k^{un} as defined in (2.3) is given by a gaussian integral over σ and ω. Using (C.1), we find

$$\Gamma_k^{\text{un}(1)} = \tfrac{1}{2}\,\text{Tr}\ln\begin{bmatrix} P_{\text{sym}} + 2\bar{\lambda}\rho_0 & -P_{\text{as}} \\ +P_{\text{as}} & P_{\text{sym}} \end{bmatrix}. \tag{C.3}$$

Because $A_\mu(x)$ is chosen to be a plane wave, the formalism of Appendix B can be used in order to expand P_{sym} and P_{as} in the number of A_μ's:

$$P_{\text{sym}} = P(-\partial^2) + \bar{e}^2 A_\mu A^\mu \check{P}_1(\partial, 2Q) - 4\bar{e}^2 A^\mu\partial_\mu \check{P}_2(\partial, Q) A^\nu\partial_\nu + \mathrm{O}\left(A_\mu^4\right),$$

$$P_{\text{as}} = -2\bar{e}A^\mu\partial_\mu \check{P}_1(\partial, Q) + \mathrm{O}\left(A_\mu^3\right). \tag{C.4}$$

Here $\check{P}_1$ and $\check{P}_2$ are defined as in (B.10) with M replaced by P. Inserting (C.4) into (C.3) and keeping only terms $\sim \bar{e}^2 A^2$ leads to

$$\begin{aligned}\Gamma_k^{\text{un}(1)} = {} & \tfrac{1}{2}\bar{e}^2\,\text{Tr}\Big(\Big\{\left[P(-\partial^2) + 2\bar{\lambda}\rho_0\right]^{-1} + P(-\partial^2)^{-1}\Big\} \\ & \times\left[A_\mu A^\mu \check{P}_1(\partial, 2Q) - 4A^\mu\partial_\mu \check{P}_2(\partial, Q) A^\nu\partial_\nu\right]\Big) \\ & + 2\bar{e}^2\,\text{Tr}\Big\{\left[P(-\partial^2) + 2\bar{\lambda}\rho_0\right]^{-1} A^\mu\partial_\mu \check{P}_1(\partial, Q) P(-\partial^2)^{-1} A^\nu\partial_\nu \check{P}_1(\partial, Q)\Big\}\end{aligned} \tag{C.5}$$

For a generic function $P(x)$ this is a rather complicated expression.

To get a first feeling for the qualitative properties of (C.5) we use the cutoff function

$$f_k(x) = k^2(x+k^2)^{-1} \tag{C.6}$$

which leads to a very simple form for $P(x)$:

$$P(x) = x + k^2. \tag{C.7}$$

In this case the infrared cutoff k^2 acts precisely like a mass term. Eq. (C.5) reduces to

$$\Gamma_k^{\mathrm{un}(1)} = \int \mathrm{d}^d x\, \mathrm{d}^d y\, A^\mu(y) A^\nu(x) \int \frac{\mathrm{d}^d Q}{(2\pi)^d} \mathrm{e}^{iQ\cdot(x-y)} J_{\mu\nu}(Q), \tag{C.8}$$

with

$$J_{\mu\nu}(Q) \equiv -\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{(Q_\nu + p_\nu) p_\mu}{\left[(Q+p)^2 + 2\bar{\lambda}\rho + k^2\right](p^2+k^2)}. \tag{C.9}$$

In order to extract the $F_{\mu\nu}F^{\mu\nu}$-term from (C.8) it is sufficient to expand (C.9) to second order in Q^2: $J_{\mu\nu}(Q) = 2\bar{e}^2 \delta_{\mu\nu} Q^2 J_0 + \dots$. In terms of the scalar quantity J_0 the renormalization constant is given by $Z_F = 4\bar{e}^2 J_0$, so that $\eta_F = -4\bar{e}^2 k \mathrm{d}J_0/\mathrm{d}k$. An elementary calculation leads to the formula

$$\begin{aligned} \eta_F = \frac{16\bar{e}^2 v_d}{d(d+2)} \Bigg(&(d-2)\frac{k^2}{M^2}\Lambda^{d-4} \\ &+ \tfrac{1}{2}(4-d)k^2 M^{d-6} \int_0^{\Lambda^2/M^2} \mathrm{d}y\, y^{d/2} \frac{(d-2)y + (d+2)}{(1+y)^3 (y + k^2/M^2)} \\ &+ 2k^2\bar{\lambda}\rho_0 M^{d-8} \int_0^{\Lambda^2/M^2} \mathrm{d}y\, y^{d/2} \frac{(d-2)y+(d+2)}{(1+y)^3 (y+k^2/M^2)^2} \Bigg), \end{aligned} \tag{C.10}$$

where Λ is an ultraviolet cutoff and $M^2 \equiv 2\bar{\lambda}\rho_0 + k^2$. (If $d \leqslant 4$ this expression is finite for $\Lambda \to \infty$.)

Let us finally turn to the evaluation of (C.5) for a generic function $P(x)$. Going to momentum space, one obtains integrals similar to (C.9) but with additional factors of $\check{P}_1(ip, Q)$ and $\check{P}_2(ip, Q)$ appearing in the numerator of the integrand. These factors have to be expanded up to second order in Q using the formulas (B.11) of the previous appendix. After combining them with the Q^2-dependent terms coming from the denominator and performing the (symmetric) integration over p^μ, one obtains again a result of the form (C.8) with $J_{\mu\nu}(Q) = 2\bar{e}^2\delta_{\mu\nu}Q^2J_0(2\bar{\lambda}\rho_0) + \dots$. The function J_0 is given by

$$J_0(w) = v_d\int_0^\infty dx\, x^{d/2-1}\left[-\frac{6}{d}\frac{x\dot{P}\ddot{P}}{P(P+w)} - \frac{2}{d(d+2)}\frac{x^2\left(\frac{16}{3}\dot{P}P^{(3)} + 3\ddot{P}^2\right)}{P(P+w)}\right.$$
$$+\frac{2}{d}\frac{x\dot{P}^3}{(P+w)^2P} + \frac{20}{d(d+2)}\frac{x^2\dot{P}^2\ddot{P}}{(P+w)^2P} - \frac{8}{d(d+2)}\frac{x^2\dot{P}^4}{(P+w)^3P}$$
$$+\ddot{P}\left(\frac{1}{P+w} + \frac{1}{P}\right) + \frac{3}{d}xP^{(3)}\left(\frac{1}{P+w} + \frac{1}{P}\right)$$
$$\left.+\frac{7}{3d(d+2)}x^2P^{(4)}\left(\frac{1}{P+w} + \frac{1}{P}\right)\right]. \tag{C.11}$$

Therefore $\Gamma_k^{\mathrm{un}(1)}$ contributes the piece

$$\eta_F^{\mathrm{un}}(w) = -4\bar{e}^2k\frac{\mathrm{d}}{\mathrm{d}k}J_0(w) \tag{C.12}$$

to the anomalous dimension. From eq. (2.20) we see that we have to add the contributions from $-\ln N_k$ and $\ln \mu_k$, respectively. Because N_k is given by (2.9) with $\bar{m} = 0$, we have $-\ln N_k = -\ln \det[P(-D^2)]$. The contribution of the latter determinant follows from (C.12) with (C.11) for $w = 0$. This is due to the fact that for $w \equiv 2\bar{\lambda}\rho_0 = 0$ the quadratic form in (C.1) may be recombined into $\chi^*P(-D^2)\chi$. Finally the term coming from $\ln \mu_k$ is exactly the anomalous dimension in the symmetric phase with $\bar{m} = 0$, see eq. (5.9). Denoting this contribution by η_F^{sym}, the complete result reads

$$\eta_F = \eta_F^{\mathrm{un}}(2\bar{\lambda}\rho_0) - \eta_F^{\mathrm{un}}(0) + \eta_F^{\mathrm{sym}}. \tag{C.13}$$

Here η_F^{sym} is given explicitly by eq. (6.15) for $m = 0$, or by eq. (6.16), respectively. Some properties of η_F are discussed in sect. 7.

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