Physics Letters B 312 (1993) 97–104
North-Holland

PHYSICS LETTERS B

Integrability conditions for potential flow of the renormalisation group ⋆

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Received 13 March 1993; revised manuscript received 6 May 1993
Editor: P.V. Landshoff

The renormalisation group (RG) flow for a quantum field theory in flat D dimensional space is considered. Some conditions are derived which ensure that the β-functions are derivable from a potential on the space of couplings. This requires introducing a metric on the space of interactions and two possibilities are considered. Assuming positive definiteness of the metric the potential provides a function on the space of couplings which is non-increasing along the RG trajectories. Thus any theory which satisfies the integrability conditions also satisfies a c-theorem in D dimensions.

1. The c-theorem and renormalisation group flow

The question of the nature of the renormalisation group (RG) flow on the space of coupling constants for a quantum field theory is a recurrent one in physics. It has been shown [1] in two dimensional euclidean field theory, assuming certain positivity conditions on the Hilbert space of the theory, that there exists a function on the space of coupling constants which is non-increasing along the RG trajectories (the c-theorem). This has very important and far reaching implications for the theory because it puts constraints on the way that the RG flow can be realised, for example it can never come back to visit a point where it has already been, thus eliminating the possibility of limit cycles. The non-increasing function, c, can be interpreted as a measure of the number of degrees of freedom of the theory and its decreasing nature as the length scale, l, is increased as being due to "integrating out" the degrees of freedom on scales less than l. At fixed points (conformal field theories) it is the central charge of the theory.

The proof of the c-theorem in [1] relies heavily on the special properties of two dimensions and it is crucial that there are only four different bilinears (in terms of index structure) that can be constructed from the stress tensor, $T_{\mu\nu}$, in two dimensions. Attempts to generalise the c-theorem to higher dimensions have met with difficulties because, for $D > 2$, there is a fifth independent bilinear which cannot be eliminated [2]. Other authors have tried to generalise Zamolodchikov's result to higher dimensions [3–7], but so far without complete success.

A stronger condition than the c-theorem is that of potential flow. The possibility of potential flow was emphasised by Wallace and Zia in [8] and [9]. In the latter reference the three loop β-functions for massless φ^4 theory, with two φ^4 couplings, were shown to be derivable from a potential, and it has been conjectured that this property should hold to all orders in perturbation theory (at least in two dimensions) [10]. It can also be shown that a single scalar field coupled with Yukawa interactions to a four component fermion in four dimensions exhibits potential flow to sixth order in the couplings [11].

If the space of couplings is equipped with an invertible positive definite metric, G_{ab}, and the β functions are derivable from a potential, $V(g)$, in coupling constant space,

⋆ Work carried out with an Alexander von Humboldt research stipendium.

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$$\beta^b G_{ba} := \beta_a = -\frac{\partial}{\partial g^a} V(g), \tag{1}$$

then a c-theorem follows easily. (Here $a = 1, \ldots, n$ labels the dimensionless couplings g^a which can be thought of as real co-ordinates on the n dimensional space of interactions, denoted by $\mathcal{G}$.) The metric is necessary since the β-functions are naturally defined as vectors,

$$\beta^a = l\frac{\mathrm{d}}{\mathrm{d}l} g^a = \frac{\mathrm{d}}{\mathrm{d}t} g^a, \tag{2}$$

where $t = \ln l$, and a gradient is necessarily a co-vector. Note that the β-functions in (2) are defined by differentiation with respect to a length rather than a momentum and so have the opposite sign to the usual field theoretic definition.

The c-theorem follows from (1) by differentiation of the potential,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \beta^a \partial_a V = -G_{ab}\beta^a\beta^b \leqslant 0, \tag{3}$$

where the last inequality follows when positive definiteness of the metric is assumed. Hence the potential, $V(g)$, is itself a function on $\mathcal{G}$ which is non-increasing along the RG flow.

Of course the above statement of the c-theorem depends crucially on the choice of metric, [1,8,9,12–14]. In [9] a metric was constructed by *assumming* potential flow and using this as a criterion for defining a metric. But there are other considerations which might go into the definition of a physically reasonable metric, [12–14]. In this work both the metrics of Zamolodchikov [14] and that of O'Connor and Stephens [13] will be considered. In both cases conditions that the trace of the stress operator must satisfy, in order for the RG flow to be a potential flow, are obtained.

Before discussing the conditions for potential flow, in section 4, some relevant properties of the space of local interactions $\mathcal{G}$ and the renormalised stress operator are discussed in section 2. The two candidates for a metric function are described in section 3 and section 5 gives a summary and conclusions.

2. The trace of the stress tensor

Consider a field theory in D dimensions which is described by an action,

$$S[g,\varphi] = \int_{\mathbb{R}^D} H(g,\varphi(x),\partial_\mu\varphi(x))\,\mathrm{d}^D x, \tag{4}$$

which depends on some set of fields, $\varphi(x)$, and their first derivatives, together with a finite set of real couplings, g^a. The couplings are all taken to be dimensionless. If there are any massive couplings in the theory these can always be made massless by multiplying by appropriate powers of the renormalisation length, l. Thus, for example, the dimensionless coupling associated with a mass, m^2, would be l^2m^2 giving rise to a β-function $\beta = (2-\delta)m^2$, where δ is the usual beta function associated with a mass. It will be assumed that the theory can be renormalised and that a quantum stress operator $\widehat{T}_{\mu\nu}$, satisfying $\partial_\mu \widehat{T}^\mu{}_\nu = 0$ and $\langle \widehat{T}^\mu{}_\nu\rangle = 0$, can be defined. Of course, quantum mechanically, the trace of the stress operator can be a non-zero operator, $\widehat{T}^\mu{}_\mu = -\widehat{\Theta} \neq 0$, even when its classical counterpart vanishes. This happens when there is a conformal anomaly.

Following ref. [1], we consider the renormalised operators

$$\widehat{\Phi}_a(x) = \frac{\partial \widehat{H}}{\partial g^a} - \left\langle \frac{\partial \widehat{H}}{\partial g^a} \right\rangle. \tag{5}$$

$\widehat{\Phi}_a$ constitute a basis for all relevant or marginal operators of the theory, i.e. any relevant or marginal operator, which is a scalar in $\mathbb{R}^D$, can be written as a linear combination of $\widehat{\Phi}_a(x)$ and the identity. These can be related to their bare counterparts, $\widehat{\Phi}_{oa}$, using the matrix of renormalisation constants, $Z_a{}^b$,

$$\widehat{\Phi}_{oa}(x) = Z_a{}^b\widehat{\Phi}_b(x),$$
$$\text{where } \widehat{\Phi}_{oa}(x) = \frac{\partial \widehat{H}(x)}{\partial g_o^a} - \left\langle \frac{\partial \widehat{H}(x)}{\partial g_o^a} \right\rangle, \tag{6}$$

g_o^a being the bare couplings. $H(x)$ here is the bare hamiltonian which can be thought of either as a function of the bare couplings or the renormalised couplings (the latter interpretation requires counterterms, of course). In this way of looking at things the bare and renormalised couplings can be considered to be two different co-ordinate sytems on the space of couplings and $Z_a{}^b = \mathrm{d}g^b/\mathrm{d}g_o^a$ plays the rôle of a co-ordinate transformation matrix, telling us how to transform tensors between the two co-ordinate systems. This co-ordinate transformation is singular when the regulator is removed but, provided the theory is renormalisable, this is not a problem.

To incorporate tensor (or spinor) operators would require a larger basis but this will not be necessary here. A consequence of the definition of $\widehat{\Phi}_a(x)$ is

$$\partial_a \widehat{\Phi}_b = \partial_b \widehat{\Phi}_a. \tag{7}$$

Note that, in terms of the "simple" fields $\varphi(x)$, which one might use in a functional integral to do perturbation theory for example, $\widehat{\Phi}_a(x)$ include composite operators and therefore must also be regularised, e.g. by introducing a cut-off $L \ll l$. We shall assume that this can be done. The $\widehat{\Phi}_a$ are defined so that $\langle \widehat{\Phi}_a(x) \rangle = 0$. This does not require that the renormalised $\partial_a \widehat{H}(x)$ have zero expectation value, though it should be finite and independent of x from translation invariance. The operators $\partial_a \widehat{H}(x)$ will appear below and they will be denoted by $\widehat{H}_a(x)$. They can be extended to include the identity operator, $\widehat{H}_0 = \mathbf{1}$, and will then be denoted by $\widehat{H}_{\widehat{a}}(x)$ where $\widehat{a} = 0, 1, \ldots, n$.

Since $\widehat{\Phi}_a(x)$ are a basis for scalar operators $\widehat{\Theta}(x)$ can be expanded as

$$\widehat{\Theta}(x) = \beta^a(g) \widehat{\Phi}_a(x) \tag{8}$$

for some set of functions β^a. It can be shown, e.g. [14], that the β^a in (8) are just the β functions of eq. (2). Note that $\langle \widehat{\Theta} \rangle = 0$ because it is defined to be, as is usual for renormalisation of the stress tensor in flat space.

Now the operator $\widehat{\Theta}$ should not depend on the renormalisation length, l, therefore

$$l \frac{\mathrm{d}}{\mathrm{d}l} \widehat{\Theta} = l \frac{\partial}{\partial l}\bigg|_g \widehat{\Theta} + \beta^a \partial_a \widehat{\Theta} = 0. \tag{9}$$

In terms of differential forms on $T_g^*(\mathcal{G})$ the quantities

$$\widehat{\Phi}(x) = \widehat{\Phi}_a(x) \mathrm{d}g^a \tag{10}$$

can be thought of as operator valued one-forms. These also should be independent of l giving

$$l \frac{\mathrm{d}}{\mathrm{d}l} \widehat{\Phi} = l \frac{\partial}{\partial l}\bigg|_g \widehat{\Phi} + \mathcal{L}_{\beta} \widehat{\Phi} = 0, \tag{11}$$

where $\mathcal{L}_{\boldsymbol{\beta}}$ is the Lie derivative with respect to the vector field $\boldsymbol{\beta}$ on the space of couplings $\mathcal{G}$,

$$\begin{aligned} \mathcal{L}_{\vec{\beta}} \widehat{\Phi} &= (\mathrm{d} i_{\vec{\beta}} + i_{\vec{\beta}} \mathrm{d}) \widehat{\Phi} \\ &= (\partial_a \beta^b) \widehat{\Phi}_b \mathrm{d}g^a + \beta^b (\partial_b \widehat{\Phi}_a) \mathrm{d}g^a \\ &= \mathrm{d}\widehat{\Theta} \end{aligned} \tag{12}$$

(d and $i_{\vec{\beta}}$ are respectively the exterior derivative on $T_g^*(\mathcal{G})$ and contraction with the vector $\vec{\beta}$).

These equations can then be summarised as

$$\begin{aligned} &\mathrm{d}\widehat{\Phi} = 0, \\ &\mathcal{L}_{\vec{\beta}} \widehat{\Phi} = \mathrm{d}\widehat{\Theta} = -l \frac{\partial}{\partial l}\bigg|_g \widehat{\Phi}, \\ &\mathcal{L}_{\vec{\beta}} \widehat{\Theta} = -l \frac{\partial}{\partial l}\bigg|_g \widehat{\Theta}. \end{aligned} \tag{13}$$

Note that

$$l \frac{\partial}{\partial l}\bigg|_g \widehat{\Phi}_a(x) = \partial_a \left(l \frac{\partial}{\partial l}\bigg|_g \widehat{H}(x) \right) - l \frac{\partial}{\partial l}\bigg|_g \langle \partial_a \widehat{H}(x) \rangle. \tag{14}$$

Expressions for $l(\partial/\partial l)|_g \widehat{H}(x)$ for scalar field theory are considered in [15] and [16].

3. The metric

In this section two possible candidates for a metric on $\mathcal{G}$ are briefly described. Firstly, following Zamolodchikov [1,14], a dimensionless metric on $\mathcal{G}$ can be obtained from the two point correlators $\langle \widehat{\Phi}_a(x) \widehat{\Phi}_b(y) \rangle$ by defining

$$\overline{G}_{ab}(g) = l^{2D} \langle \widehat{\Phi}_a(x) \widehat{\Phi}_b(y) \rangle \big|_{|x-y|=l} \tag{15}$$

and the assumption of positivity of the theory ensures that $\overline{G}_{ab}(g)$ is a positive definite metric on $\mathcal{G}$. Note that $\overline{G}_{ab}(g)$, being dimensionless, has no explicit l dependence and depends on the renormalisation point only implicitly, through g dependence i.e. $l(\partial/\partial l)|_g \overline{G}_{ab}(g) = 0$. This metric can be understood in terms of the operator product expansion coefficients [17],

$$\widehat{\Phi}_a(x) \widehat{\Phi}_b(y) = C_{ab}{}^0(|x-y|) \mathbf{1} + \cdots, \tag{16}$$

where the dots represent operators with zero expectation value. For small separations,

$$C_{ab}{}^{0}(|x-y|) = \sum_{s=-2D}^{\infty} C_{ab}^{(s)0}(g)|x-y|^{s}, \tag{17}$$

where $C_{ab}^{(s)0}(g)$ are independent of $|x-y|$. Thus, for small l, $\overline{G}_{ab}$ is essentially $C_{ab}^{(-2D)0}(g)$ i.e. the leading term in the operator product expansion.

An alternative metric, that of O'Connor and Stephens [13], is obtained by integrating the two point functions over all space, rather than evaluating them at a specific separation,

$$\widetilde{G}_{ab}(g) = l^{D}\int \mathrm{d}^{D}y\,\langle\widehat{\Phi}_{a}(y)\widehat{\Phi}_{b}(x)\rangle. \tag{18}$$

From translational invariance $\widetilde{G}_{ab}(g)$ is independent of the point x but it must be regularised, due to the singularity at $y\sim x$.

The regularisation of composite operators is not straightforward, and contains many pitfalls and subtleties. Some specific examples have been worked out in detail, e.g. in [18,19] and a general formalism is presented in [20,21]. In general $[\widehat{\Phi}_{a}(x)\widehat{\Phi}_{b}(y)]_{\mathrm{R}}$ is not the same as $[\widehat{\Phi}_{a}(x)]_{\mathrm{R}}[\widehat{\Phi}_{b}(y)]_{\mathrm{R}}$, where the subscript R denotes renormalised quantities. They differ by terms involving $\delta(x-y)$ and its derivatives with infinite co-efficients, in general. This is not a problem for the definition of the metric in eq. (15) since $l>0$ but in the next section it will be necessary to differentiate this quantity with respect to the couplings and then it will be important to use $[\widehat{\Phi}_{a}(x)\widehat{\Phi}_{b}(y)]_{\mathrm{R}}$ in the definition rather than $[\widehat{\Phi}_{a}(x)]_{\mathrm{R}}[\widehat{\Phi}_{b}(y)]_{\mathrm{R}}$. However for the other metric eq. (18) must be understood as involving $[\widehat{\Phi}_{a}(x)\widehat{\Phi}_{b}(y)]_{\mathrm{R}}$ from the start, otherwise it is infinite and not a sensible candidate for a metric.

The two point functions appearing here can be obtained from variation of a source for the operators $\widehat{\Phi}_{a}(x)$ in a generating functional for composite operators, as described in [21]. Denote the renormalised sources for the operators $\widehat{\Phi}_{a}(x)$ by $j^{a}(x)$ and let $W[g,j]=-\ln Z[g,j]$ be the generating functional for connected amplitudes. Then

$$\overline{G}_{ab}(g) = -l^{2D}\frac{\delta^{2}W}{\delta j^{a}(x)j^{b}(y)}\bigg|_{j=0;\,|x-y|=l} \tag{19}$$

and

$$\widetilde{G}_{ab}(g) = -l^{D}\int \mathrm{d}^{D}y\frac{\delta^{2}W}{\delta j^{a}(y)j^{b}(x)}\bigg|_{j=0}. \tag{20}$$

4. Integrability conditions for potential flow

Integrability conditions on the β-functions for the RG flow to be a potential flow will now be derived. Potential flow requires that the one form $\beta=\beta_{a}\mathrm{d}g^{a}$ on $T^{*}(\mathcal{G})$ is closed and hence locally exact (as stated before, questions concerning the global structure of $T^{*}_{\mathcal{G}}$ are not addressed here).

Consider first the β_{a} obtained from the metric (15). We have

$$\begin{aligned}
\partial_{a}\beta_{b} &= l^{2D}\partial_{a}\left\langle[\widehat{\Phi}_{b}(x)\widehat{\Theta}(y)]_{\mathrm{R}}\right\rangle\Big|_{|x-y|=l}\\
&= l^{2D}\partial_{a}\left\langle[\widehat{\Phi}_{b}(x)]_{\mathrm{R}}[\widehat{\Theta}(y)]_{\mathrm{R}}\right\rangle\Big|_{|x-y|=l}\\
&= l^{2D}\left\langle\partial_{a}[\widehat{\Phi}_{b}(x)]_{\mathrm{R}}[\widehat{\Theta}(y)]_{\mathrm{R}}\right\rangle\Big|_{|x-y|=l}\\
&+ l^{2D}\left\langle[\widehat{\Phi}_{b}(x)]_{\mathrm{R}}\partial_{a}[\widehat{\Theta}(y)]_{\mathrm{R}}\right\rangle\Big|_{|x-y|=l}\\
&- l^{2D}\beta^{c}\\
&\times\int \mathrm{d}^{D}z\langle[\widehat{\Phi}_{a}(z)]_{\mathrm{R}}[\widehat{\Phi}_{b}(x)]_{\mathrm{R}}[\widehat{\Phi}_{c}(y)]_{\mathrm{R}}\rangle\Big|_{|x-y|=l}.
\end{aligned} \tag{21}$$

The last term in eq. (21) can be understood by using a path integral formalism for the calculation of amplitudes and taking into consideration the variation of the couplings in the action [14]. Of course it requires regularisation, due to singularities when $z\sim x$ and $z\sim y$ but we shall see below that, when a and b are antisymmetrised, the three point function appearing in (21) is finite. Note that, even without antisymmetrising on a and b, the right hand side of (21) must be finite since the left hand side is, thus the infinities in the three terms on the right hand side must cancel (in general the operator $\partial_{a}[\widehat{\Phi}_{b}(x)]_{\mathrm{R}}$ has infinities).

It is important here that the three point correlator dies off sufficiently fast at large separations for the integral in (21) to converge. This requires that it falls off faster than $|z-x|^{-D}$.

Now define

$$S_{abc}(|x-y|) := l^{2D} \int \mathrm{d}^D z \, \langle [\widehat{\Phi}_a(z)]_\mathrm{R} [\widehat{\Phi}_b(x)]_\mathrm{R} [\widehat{\Phi}_c(y)]_\mathrm{R} \rangle . \tag{22}$$

Making use of (7), eq. (21) now yields

$$\partial_{[a}\beta_{b]} = -l^{2D} \langle [\widehat{\Phi}_{[a}(x)]_\mathrm{R} \partial_{b]} [\widehat{\Theta}(y)]_\mathrm{R} \rangle |_{|x-y|=l} - S_{[ab]c}(l)\beta^c. \tag{23}$$

The vanishing of the right hand side of this equation provides integrability conditions for potential flow.

These conditions can be interpreted as follows. Since $\widehat{\Phi}_a(x)$ are a basis for scalar operators of dimension $-D$, we can expand

$$\partial_b \widehat{\Theta}(y) = \eta_b{}^c \widehat{\Phi}_c(y). \tag{24}$$

(In principle there could be a term proportional to the identity operator on the right hand side of eq. (24) but this does not affect the ensuing analysis and will be omitted.) This gives

$$\partial_{[a}\beta_{b]} = \overline{\eta}_{[ab]} - S_{[ab]c}\beta^c, \tag{25}$$

where the matrix $\overline{\eta}_{ab}$ is defined as

$$\overline{\eta}_{ab} = \eta_a{}^c \overline{G}_{cb}, \tag{26}$$

and S_{abc} is $S_{abc}(|x-y|)$ evaluated at $|x-y| = l$. Now from (13) and (24)

$$\partial_a \Theta(x) = -l \frac{\partial}{\partial l}\bigg|_g \widehat{\Phi}_a(x) = \eta_a{}^c \widehat{\Phi}_c(x). \tag{27}$$

It follows that $\eta_a{}^b$ can be thought of as the linear transformation matrix that generates the change in the basis $\widehat{\Phi}_a(x)$ under an infinitesimal change in the renormalisation point, with the renormalised couplings kept fixed. Using the metric this linear transformation can be decomposed into a rotation, a shear and a dilation. Thus the condition for potential flow relates the rotational part of $\overline{\eta}_{ab}$ to the antisymmetric part of $S_{abc}\beta^c$,

$$\overline{\eta}_{[ab]} = S_{[ab]c}\beta^c. \tag{28}$$

We can obtain some information about $\overline{\eta}_{[ab]}$ by considering $l(\partial/\partial l)|_g \widehat{\Theta} = -\beta^a \partial_a \widehat{\Theta}$, which is the full dimension (canonical plus anomalous) of the operator $\widehat{\Theta}$. The anomalous dimension of the stress operator vanishes because it consists of D conserved currents, one for each generator of translations in D euclidean dimensions, and conserved currents do not get renormalised. $\widehat{\Theta}$ has canonical dimension $-D$, hence

$$-l \frac{\partial}{\partial l}\bigg|_g \widehat{\Theta} = \beta^a \partial_a \widehat{\Theta} = D\widehat{\Theta} = D\beta^a \widehat{\Phi}_a. \tag{29}$$

This equation is derived, from general arguments, in [14] and it can be verified, to all orders in perturbation theory, for massive $\lambda\varphi^4$ in $D \leqslant 4$ using the techniques in [18], provided the improved stress tensor is used. Actually, since the stress operator is only conserved when the equations of motion are used, (29) is only true modulo the equations of motion. (Note that for the metric (15) any linear combination of $\widehat{\Phi}_a$ which is proportional to the equations of motion must be removed from our basis set anyway, since otherwise the metric is degenerate.)

If we further assume that the basis for operators $\widehat{\Phi}_a$ is linearly independent and that $\partial_a \widehat{\Theta}$ are linearly independant operators too, and therefore also constitute a basis since they are the same in number as $\widehat{\Phi}_a$, we can conclude from (29) that

$$\partial_a \widehat{\Theta} = D\widehat{\Phi}_a \quad \Leftrightarrow \quad \eta^a{}_b = D\delta^a{}_b. \tag{30}$$

It follows that

$$\overline{\eta}_{ab} = D\overline{G}_{ab} \tag{31}$$

and so

$$\overline{\eta}_{[ab]} = 0, \tag{32}$$

since $\overline{G}_{ab}$ is symmetric by definition. i.e. $\overline{\eta}_{ab}$ is irrotational. The assumption that $\partial_a \widehat{\Theta}$ are linearly independent does not seem to be a particularly strong one – for instance eq. (30) can be shown to be true for massive $\lambda\varphi^4$ theory in four dimensions. When this assumption holds the integrability conditions for the metric (15) reduce to the statement

$$\partial_{[a}\beta_{b]} = S_{[ab]c}\beta^c = 0. \tag{33}$$

The first equation here is an identity and provides the promised proof that $S_{[ab]c}\beta^c$ is finite, even though S_{abc} itself is not, the second equation gives the integrability conditions for the metric (15).

Now consider a similar analysis applied to the other metric under consideration, (18). Then we have

$$\partial_a\beta_b = l^D\int \mathrm{d}^Dx\,\partial_a\left\langle[\widehat{\Phi}_b(x)\widehat{\Theta}(y)]_{\mathrm{R}}\right\rangle$$
$$= l^D\int \mathrm{d}^Dx\left\langle\partial_a[\widehat{\Phi}_b(x)\widehat{\Theta}(y)]_{\mathrm{R}}\right\rangle$$
$$- l^D\beta^c\int \mathrm{d}^Dz\int \mathrm{d}^Dx$$
$$\times\left\langle[\widehat{\Phi}_a(z)]_{\mathrm{R}}[\widehat{\Phi}_b(x)\widehat{\Phi}_c(y)]_{\mathrm{R}}\right\rangle. \quad (34)$$

We must be careful however, because

$$\partial_a[\widehat{\Phi}_b(x)\widehat{\Theta}(y)]_{\mathrm{R}}$$
$$\neq \partial_a[\widehat{\Phi}_b(x)]_{\mathrm{R}}[\widehat{\Theta}(y)]_{\mathrm{R}} + [\widehat{\Phi}_b(x)]_{\mathrm{R}}\partial_a[\widehat{\Theta}(y)]_{\mathrm{R}}, \quad (35)$$

when $x = y$. The difference is dictated by the short distance behaviour of the theory and it is shown in [21] that

$$[\widehat{H}_b(x)\widehat{H}_c(y)]_{\mathrm{R}} = [\widehat{H}_b(x)]_{\mathrm{R}}[\widehat{H}_c(y)]_{\mathrm{R}}$$
$$+ L_{bc}^{(p,q)\widehat{d}}\partial_x^p\partial_y^q\left\{\widehat{H}_{\widehat{d}}(x)\delta(x-y)\right\}, \quad (36)$$

or

$$[\widehat{\Phi}_b(x)\widehat{\Phi}_c(y)]_{\mathrm{R}} = [\widehat{\Phi}_b(x)]_{\mathrm{R}}[\widehat{\Phi}_c(y)]_{\mathrm{R}}$$
$$+ L_{bc}^{(p,q)\widehat{d}}\partial_x^p\partial_y^q\left\{\widehat{H}_{\widehat{d}}(x)\delta(x-y)\right\}. \quad (37)$$

Here $\widehat{H}_a = \partial_a\widehat{H}$ and $\widehat{d} = 0,\ldots,n$ so as to include the identity operator. $L_{bc}^{(p,q)\widehat{d}}$ are independent of position, but depend on the renormalised couplings and on the regularisation parameter. For example in dimensional regularisation, $L_{bc}^{(p,q)\widehat{d}}$ would consist of a series of poles in ϵ. The notation on the right hand side of eq. (37), involving p and q, is a condensed notation for the sum over all possible derivatives with respect to x and y, with all possible contractions that are consistent with the fact that the left hand side is a scalar. That the δ-function terms involve only $\widehat{\Phi}_a$ and the identity, rather than a larger set of operators, is a consequence of dimensional analysis and is explained in [21]. Using eq. (37) in eq. (35) gives

$$\partial_a[\widehat{\Phi}_b(x)\widehat{\Theta}(y)]_{\mathrm{R}}$$
$$= \partial_a[\widehat{\Phi}_b(x)]_{\mathrm{R}}[\widehat{\Theta}(y)]_{\mathrm{R}} + [\widehat{\Phi}_b(x)]_{\mathrm{R}}\partial_a[\widehat{\Theta}(y)]_{\mathrm{R}}$$
$$+ \partial_x^p\partial_y^q\{\partial_a(\beta^c L_{bc}^{(p,q)\widehat{d}}\widehat{H}_{\widehat{d}}\delta(x-y))\}. \quad (38)$$

Taking the expectation value, antisymmetrising a and b and integrating over x gives, with (30) and (7),

$$l^D\int \mathrm{d}^Dx\langle\partial_{[a}[\widehat{\Phi}_{b]}(x)\widehat{\Theta}(y)]_{\mathrm{R}}\rangle$$
$$= l^D\int \mathrm{d}^Dx\,\partial_x^p\partial_y^q\langle\partial_{[a}(\beta^c L_{b]c}^{(p,q)\widehat{d}}\widehat{H}_{\widehat{d}}(x)\delta(x-y))\rangle. \quad (39)$$

Performing the x integration, with y finite, the only term that can survive is that with $p = 0$, i.e. no x derivatives. The resulting expectation value is translationally invariant and thus independent of y, hence $p = q = 0$ and thus

$$l^D\int \mathrm{d}^Dx\left\langle\partial_{[a}[\widehat{\Phi}_{b]}(x)\widehat{\Theta}(y)]_{\mathrm{R}}\right\rangle$$
$$= l^D\left\langle\partial_{[a}(\beta^c L_{b]c}{}^{\widehat{d}}\widehat{H}_{\widehat{d}}\right\rangle$$
$$= l^D\partial_{[a}\left(\beta^c L_{b]c}{}^{\widehat{d}}\left\langle\widehat{H}_{\widehat{d}}\right\rangle\right)$$
$$+ l^D\beta^c L_{c[b}{}^{\mathrm{d}}\int \mathrm{d}^Dx\left\langle[\widehat{\Phi}_{a]}(x)]_{\mathrm{R}}[\widehat{\Phi}_{\mathrm{d}}(y)]_{\mathrm{R}}\right\rangle, \quad (40)$$

where $L_{bc}{}^{\widehat{d}} = L_{bc}^{(0,0)\widehat{d}}$ (we have used the fact that $L_{bc}^{(0,0)\widehat{d}} = L_{cb}^{(0,0)\widehat{d}}$ is symmetric in b and c).

Now consider the second term in eq. (34)

$$l^D\beta^c\int \mathrm{d}^Dz\int \mathrm{d}^Dx\left\langle[\widehat{\Phi}_a(z)]_{\mathrm{R}}[\widehat{\Phi}_b(x)\widehat{\Phi}_c(y)]_{\mathrm{R}}\right\rangle$$
$$= l^D\beta^c\int \mathrm{d}^Dz\int \mathrm{d}^Dx$$
$$\times\{\langle[\widehat{\Phi}_a(z)]_{\mathrm{R}}[\widehat{\Phi}_b(x)]_{\mathrm{R}}[\widehat{\Phi}_c(y)]_{\mathrm{R}}\rangle$$
$$+ L_{bc}^{(p,q)\mathrm{d}}\partial_x^p\partial_y^q\left\langle[\widehat{\Phi}_a(z)]_{\mathrm{R}}[\widehat{\Phi}_{\mathrm{d}}(x)]_{\mathrm{R}}\delta(x-y)\right\rangle\}. \quad (41)$$

The first term on the right hand side of this equation is manifestly symmetric under interchange of a and b, the second term simplifies as before, giving

$$
l^D \beta^c \int d^D z \int d^D x \left\langle [\widehat{\Phi}_{[a}(z)]_R [\widehat{\Phi}_{b]}(x)\widehat{\Phi}_c(y)]_R \right\rangle
= l^D \beta^c L_{c[b}{}^{d} \int d^D z \left\langle [\widehat{\Phi}_{a]}(z)]_R [\widehat{\Phi}_d(x)]_R \right\rangle . \tag{42}
$$

When inserted into (34) this cancels the last term in eq. (40) giving

$$
\partial_{[a}\beta_{b]} = l^D \partial_{[a}(\beta^c L_{b]c}{}^{\widehat{d}} \langle \widehat{H}_{\widehat{d}} \rangle) = \partial_{[a}(\beta^c L_{b]c}), \tag{43}
$$

where we have defined the dimensionless quantity

$$
L_{bc} := l^D L_{bc}{}^{\widehat{d}} \langle \widehat{H}_{\widehat{d}} \rangle . \tag{44}
$$

Eq. (43) is of the same form as would have been obtained had L_{ab} itself been used as a metric. Being singular, however, L_{ab} would not make a sensible metric. Eq. (44) is telling us that only the short distance behaviour of the metric (18) can prevent potential flow.

It may be worth noting that $\partial_a \mathbf{1} = 0$, and, from eq. (20),

$$
\partial_a \left\langle [\widehat{H}_d(y)]_R \right\rangle = - \int d^D x \left\langle [\widehat{\Phi}_a(x) \widehat{H}_d(y)]_R \right\rangle
= - \int d^D x \left\langle [\widehat{\Phi}_a(x) \widehat{\Phi}_d(y)]_R \right\rangle
= -l^{-D} \widetilde{G}_{ad}, \tag{45}
$$

Eq. (43) can be written in an alternative manner by observing that $\partial \beta^b / \partial g^a$ is related to the matrix of anomalous dimensions for the theory. This is shown in [21], but a slight difference here is that, since our couplings are defined to be dimensionless, the β-functions include the canonical dimensions of the couplings as explained at the beginning of section 2. This means that $\partial \beta^b / \partial g^a$ include the canonical dimensions as well as the anomalous dimensions. We therefore define the matrix of dimensions $\Gamma_a{}^b$ (canonical plus anomalous) as

$$
\Gamma_a{}^b := \frac{\partial \beta^b}{\partial g^a}. \tag{46}
$$

The integrability conditions for the metric (18) can now be expressed as

$$
\beta^c \partial_{[a}(L_{b]c}) + \Gamma_{[a}{}^c L_{b]c} = 0. \tag{47}
$$

Alternatively, one can avoid the introduction of the infinite quantities $L_{ab}{}^{\widehat{c}}$ by writing (34) as

$$
\begin{aligned}
\partial_a \beta_b &= l^D \int d^D x \partial_a \left\langle [\widehat{\Phi}_b(x) \widehat{\Theta}(y)]_R \right\rangle \\
&= l^D \beta^c \int d^D x \partial_a \left\langle [\widehat{\Phi}_b(x) \widehat{\Phi}_c(y)]_R \right\rangle \\
&+ l^D \int d^D x [\left\langle \widehat{\Phi}_b(x) \widehat{\Phi}_c(y)]_R \right\rangle \partial_a \beta^c \\
&= -l^D \beta^c \int d^D z d^D x \left\langle [\widehat{\Phi}_a(z) \widehat{\Phi}_b(x) \widehat{\Phi}_c(y)]_R \right\rangle \\
&+ l^D \int d^D x \left\langle [\widehat{\Phi}_b(x) \widehat{\Phi}_c(y)]_R \right\rangle \partial_a \beta^c.
\end{aligned} \tag{48}
$$

This follows by noting that, due to the definition of the basis operators $\widehat{\Phi}_a$, a variation of the couplings in the action can equally well be obtained by a variation of the sources, as in [21]. Thus

$$
\begin{aligned}
&\int d^D z \left\langle [\widehat{\Phi}_a(z) \widehat{\Phi}_b(x) \widehat{\Phi}_c(y)]_R \right\rangle \\
&= \int d^D z \frac{\delta^3 W}{j^a(z) j^b(x) j^c(y)} \bigg|_{j=0} \\
&= -\partial_a \left\langle [\widehat{\Phi}_b(x) \widehat{\Phi}_c(y)]_R \right\rangle .
\end{aligned} \tag{49}
$$

The first term on the right hand side of eq. (48) is manifestly symmetric under interchange of a and b.

The integrability conditions for the metric (18) can now be expressed by the statement that the matrix

$$
M_{ab} = \Gamma_a{}^c \widetilde{G}_{cb} \tag{50}
$$

must be symmetric.

5. Conclusions

For a local renormalisable quantum field theory in flat D dimensional space integrability conditions on the β functions for the RG flow to be a potential flow have been derived. This requires the introduction of a metric on the space of local interactions and two possibilities,

$$
\overline{G}_{ab}(g) = l^{2D} \left\langle [\widehat{\Phi}_a(x) \widehat{\Phi}_b(y)]_R \right\rangle \Big|_{|x-y|=l} \tag{51}
$$

and

$$
\widetilde{G}_{ab}(g) = l^D \int d^D y \left\langle [\widehat{\Phi}_a(y) \widehat{\Phi}_b(x)]_R \right\rangle , \tag{52}
$$

have been considered. The integrability conditions for the first metric are then

$$S_{[ab]c}\beta^c = 0, \tag{53}$$

where $S_{[ab]c}$ is the three point function (22) evaluated at the renormalisation length $|x - y| = l$. For the second metric they are as in eq. (47), where L_{ab} are defined in eq. (44) and are determined from the expansion (37) and $\Gamma_a{}^b$ is the matrix of dimensions. An alternative statement for the metric (18) is that

$$M_{[ab]} = 0, \tag{54}$$

where the the matrix M_{ab} is defined in eq. (50).

Whether or not the integrability conditions will be useful in practical situations remains to be seen, but it is hoped that the analysis presented here goes some way towards exposing the structure of the two and three point functions of composite operators which seem to be an essential ingredient of any such discussion.

The nature of the potential itself, should one exist, has not been addressed here but this is clearly of prime importance since, if it could be constructed, it would be related to a higher dimensional analogue of the two dimensional concept of the central charge of the theory. It has been suggested that if the RG flow is a potential flow then the effective action should provide the required potential [10]. A great deal more work must be done before this hypothesis can be either confirmed or refuted.

Acknowledgement

It is a pleasure to thank N. Dragon, S. Ketov, O. Lechtenfeld, L. O'Raifeartaigh and especially D. O'Connor for useful conversations concerning the renormalisation group. I would also like to thank the referee for pointing out an error in an earlier version of this work and making very helpful comments.

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