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High energy asymptotics of multi-colour QCD and two-dimensional conformal field theories

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In the multi-colour limit of perturbative QCD the holomorphic factorization of wave functions of compound states of n reggeized gluons in the impact parameter space is shown. The conformally invariant Hamiltonian for each holomorphic factor has a nontrivial integral of motion. The odderon in QCD is the simplest example of the composite system with these properties.

It is known [1], that at high CMS energies $\sqrt{S}$ in the leading logarithmic approximation (LLA) ($g^2 \ln S \sim 1$), the gluon is regeizzed in QCD and the Pomeron, emerging as a shadow of multi-regge processes, is built from two reggeized gluons. The t-channel partial waves f_ω for the gluon–gluon scattering with total angular momentum $j=1+\omega$ and vacuum quantum numbers satisfy the Bethe–Salpeter equation with an integral kernel including the Regge trajectories $\omega(-\boldsymbol{P}^2_{1,2})$ for two constituent virtual gluons with momenta $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, respectively. These momenta lie in a two-dimensional plane, orthogonal to the collision axis. The gluon Regge trajectory $\omega(-\boldsymbol{P}^2)\sim -\ln(\boldsymbol{P}^2/\lambda^2)$ contains the infrared divergence for a vanishing gluon mass λ, however, in the kernel the divergence is cancelled by a term corresponding to real gluon emission [2].

To solve the problem of unitarization of the scattering amplitudes obtained in LLA it is needed to take into account the diagrams with an arbitrary number n of regeizzed gluons in the t-channels. In the case where this number is fixed, the t-channel partial waves $f^{(n)}_\omega$ satisfy a Faddeev-like equation with pair-like interactions [3]. Let us write it in the homogeneous form

$$\omega f^{(n)}_\omega = \sum_{i<k} \mathscr{K}_{ik} f^{(n)}_\omega . \qquad (1)$$

The eigenvalues of (1) yield the positions $j=1+\omega$ of the Regge singularities of f_ω in the angular momentum plane of the t-channel and its eigenfunctions are proportional to the couplings of the reggeons with external gluons.

In this paper we consider the possibility of finding the exact solution of (1) for a large number of colours N. The operators $\mathscr{K}_{ik}$ in (1) differ from the integral kernel for the Pomeron equation by a colour factor only:

$$\mathscr{K}_{ik} = \tau^a_i \tau^a_k \frac{g^2}{8\pi^2} \mathscr{H}_{ik} , \qquad (2)$$

where the τ^a_i are the generators of the colour group SU(N) acting on the indices of the gluon i. $\mathscr{H}_{ik}$ in expression (2) can be considered as a Hamiltonian for a pair of interacting particles. It looks rather simple in the coordinate (impact parameter) representation if one uses the complex coordinates ρ_k, ρ^*_k and momenta $\rho_k = \mathrm{i}\partial/\partial_{\rho k}$, $\rho^*_k = -\mathrm{i}\partial/\partial\rho^*_k$ in a two-dimensional transverse subspace (cf. refs. [4,5]):

$$\mathscr{H}_{ik} = \frac{1}{|P_i|^2 |P_k|^2} \times [P^*_i P_k \ln(|\rho_{ik}|^2) P_i P^*_k + P_i P^*_k \ln(|\rho_{ik}|^2) P^*_i P_k] + \ln(|P_i|^2 |P_k|^2) - 4\psi(1) , \qquad (3)$$

where $\rho_{ik} = \rho_i - \rho_k$ and $\psi(z) = \Gamma'(z)/\Gamma(z)$, $\psi(1) = -\gamma$ (γ denotes the Euler constant). It is obvious from (3) that $\mathscr{H}_{ik}$ is holomorphically separable [6]:

$$\mathcal{H}_{ik} = H_{ik} + H^*_{ik} , \quad (4)$$

where its holomorphic part

$$H_{ik} = P_i^{-1} \ln(\rho_{ik}) P_i + P_k^{-1} \ln(\rho_{ik}) P_k + \ln(P_i P_k) - 2\psi(1) \quad (5)$$

and its antiholomorphic part H^*_{ik} act on the coordinates ρ_i, ρ_k and ρ^*_i, ρ^*_k, respectively. Property (4) allows us to simplify (1) significantly by using the following ansatz for its solution in some cases:

$$f_\omega(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, ..., \boldsymbol{\rho}_n; \boldsymbol{\rho}_0) = \sum_r C_r f^r(\rho_1, \rho_2, ...\rho_n; \rho_0) \tilde{f}^r(\rho^*_1, \rho^*_2, ...\rho^*_n; \rho^*_0) , \quad (6)$$

where f^r and $\tilde{f}^r$ are analytic and antianalytic functions with the only possible singularities at the coinciding points $\rho_{ij} = 0$. In (6) $\boldsymbol{\rho}_0$ is the coordinate of the compound state of n gluons (f_ω depends on the differences $\boldsymbol{\rho}_{i0}$ only). This ansatz is applied to the Odderon, being a compound state of three reggeized gluons [6,7], and to the case of the arbitrary number of gluons in multi-colour QCD, with $N \to \infty$. In both cases the Feynman diagrams have cylinder topology in colour space, if one substitutes each gluon line by quark and antiquark lines. Only the nearest gluons on the circle of this cylinder interact. We enumerate them in corresponding order considering the gluons $i = n+1$ and $i = 1$ as coinciding ones and simplify the colour structure at large N in (2) substituting

$$-\tau^a_i \tau^a_k \to \delta_{k,i+1} \frac{N}{2} . \quad (7)$$

This procedure corresponds to irreducible one-cylinder diagrams from which multi-cylinder contributions can be derived easily. Thus, at large N the holomorphic function $f^r(\rho_1, \rho_2 ... \rho_0)$ in (6) satisfies the equation

$$\varepsilon f^r = \mathcal{H} f^r, \quad \mathcal{H} = \sum_{i=1}^{n} H_{i,i+1} , \quad (8)$$

and the antiholomorphic function $\tilde{f}^r$ satisfies the analogous equation obtained by the substitution $H \to H^*$, $\varepsilon \to \tilde{\varepsilon}$. The position ω of the Regge singularities of the partial waves f_ω is proportional to the sum of the "energies" ε and $\tilde{\varepsilon}$

$$\omega = -\frac{g^2}{16\pi^2} N(\varepsilon + \tilde{\varepsilon}) . \quad (9)$$

Now we pass to the central point of our discussion: it turns out that the holomorphic Hamiltonian $\mathcal{H}$ (8) has a nontrivial integral of motion: there is a conformally invariant differential operator

$$\mathcal{A} = \rho_{12}\rho_{23} ... \rho_{n1} P_1 P_2 ... P_n , \quad (10)$$

which commutes with the Hamiltonian:

$$\mathcal{A}\mathcal{H} = \mathcal{H}\mathcal{A} . \quad (11)$$

To verify (11) one should use (5) for $H_{i,i+1}$ on its left-hand side and the equivalent expression

$$H_{ik} = 2 \ln(\rho_{ik}) + \rho_{ik} \ln(P_i P_k) \rho_{ik}^{-1} - 2\psi(1) \quad (12)$$

on its right-hand side.

The existence of the operator $\mathcal{A}$ (10) with the property (11) means that the differential equation

$$\mathcal{A} f = \lambda f \quad (13)$$

and the integral equation (8) have common eigenfunctions f. Therefore, the solution of (8) is reduced to the finding of all solutions of the comparatively simple conformally-invariant equation (13).

Note, that another representation of the pair Hamiltonians H_{ik} is

$$H_{ik} = \ln(\rho^2_{ik} P_i) + \ln(\rho^2_{ik} P_k) - 2 \ln(\rho_{ik}) - 2\psi(1) , \quad (14)$$

from which the conformal invariance of $\mathcal{H}$ is also evident.

Let us consider the Odderon case ($n = 3$) now, where (6) is valid for arbitrary N [6,7]. Using the conformally covariant ansatz [6]

$$f(\rho_1, \rho_2, \rho_3; \rho_0) = \left(\frac{\rho_{12}\rho_{13}\rho_{23}}{\rho^2_{10}\rho^2_{20}\rho^2_{30}} \right)^{m/3} \psi^m(x) , \quad (15)$$

where $m = \frac{1}{2} + i\nu + \frac{1}{2} r$ is the conformal weight of the Odderon and ψ^m depends only on the anharmonic ratio $x = \rho_{12}\rho_{30}/\rho_{10}\rho_{32}$, one can rewrite (7) with H_{ik} given by (14) in the following form:

$$\varepsilon\psi^m(x)=\Big[\ln\Big(\frac{1-x}{x}\,\mathrm{i}\nabla_1\Big)+\ln\Big(\frac{x}{1-x}\,\mathrm{i}\nabla_1\Big)+\ln\Big(\frac{x}{1-x}\,\mathrm{i}\nabla_2\Big)+\ln\Big(\frac{1}{x(1-x)}\,\mathrm{i}\nabla_2\Big)+\ln\Big(\frac{1-x}{x}\,\mathrm{i}\nabla_3\Big)+\ln\Big(\frac{1}{x(1-x)}\,\mathrm{i}\nabla_3\Big)-6\psi(1)\Big]\psi^m(x)\,, \qquad (16)$$

where

$$\nabla_1=\tfrac{1}{3}m(1-2x)+x(1-x)\partial\,,$$
$$\nabla_2=\tfrac{1}{3}m(1+x)+x(1-x)\partial\,,$$
$$\nabla_3=-\tfrac{1}{3}m(2-x)+x(1-x)\partial\,, \qquad (17)$$

and $\partial\equiv\partial_x$.

Due to (11) ψ^m satisfies also (13), where the operator $\mathscr{A}$ (10) in the x-representation can be written in terms of products of ∇_i:

$$\mathrm{i}\mathscr{A}(x)=\nabla_1\frac{1}{x(1-x)}\nabla_2\nabla_3=\nabla_1\frac{1}{x(1-x)}\nabla_3\nabla_2=\ldots=\frac{1}{x}\nabla_3\frac{x}{1-x}\nabla_2\frac{1}{x}\nabla_1\,. \qquad (18)$$

Therefore, $\psi^m(x)$ satisfies the third order ordinary differential equation (13). Analogous differential equations where solved by Dotsenko and Fateev for the Green functions of two-dimensional conformal field theories in terms of generalized hypergeometric functions [8]. One can expect that (13) is also exactly solvable. In this case, the eigenvalue ε would be calculated exactly by putting the known solution on the right-hand side of (16). Furthermore, the operator $\mathscr{H}$ in (16) is a function of $\mathscr{A}$:

$$\mathscr{H}=\mathscr{H}(\mathscr{A})\,. \qquad (19)$$

This function can be calculated in the singular points $x\to0$, 1, ∞. For example, for $x\to0$ one obtains

$$\mathscr{H}(\mathscr{A})=\ln(\mathscr{A})=-\ln(x)+\tfrac{1}{2}\psi(\tfrac{1}{3}m+x\partial)+\tfrac{1}{2}\psi(1+\tfrac{1}{3}m+x\partial)+\tfrac{1}{2}\psi(1-2\cdot\tfrac{1}{3}m+x\partial)+\tfrac{1}{2}\psi(1-\tfrac{1}{3}m-x\partial)+\tfrac{1}{2}\psi(-\tfrac{1}{3}m-x\partial)+\tfrac{1}{2}\psi(2\cdot\tfrac{1}{3}m-x\partial)\,. \qquad (20)$$

Due to the group representation theory the wave function f_ω (6) is expanded in the series over the states with all admissible conformal weights $M_{12}=\frac{1}{2}+\mathrm{i}\sigma+\frac{1}{2}r$, $\tilde{M}_{12}=\frac{1}{2}+\mathrm{i}\sigma-\frac{1}{2}r$ in the subchannel (1,2), for example. Taking into account the definition (15) we have for $\psi^{m,\tilde{m}}$:

$$\psi^{m,\tilde{m}}(x,x^*)=\sum_{r=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}\sigma\, x^{M_{12}-m/3}x^{*\tilde{M}_{12}-m/3}\varphi_{M_{12},\tilde{M}_{12}}\,. \qquad (21)$$

The Fourier coefficients $\rho_{M_{12},\tilde{M}_{12}}$ can be calculated in explicit form from (16) for the simplified Hamiltonian (20). We obtain for $\tilde{m}=m$

$$\psi^{m,\tilde{m}}(x,x^*)\,|_{x\to0}=|x|^{4m/3}+|x|^{2-2m/3}[a_1+a_2\ln(|x|)]\,. \qquad (22)$$

This asymptotics of $\psi^{m,\tilde{m}}$ was used for finding the appropriate trial function in the variational approach to the Odderon problem in QCD in ref. [7].

Note, that the small x behaviour (22) of $\psi^{m,\tilde{m}}(x)$ can be obtained from (13) and from an analogous equation for $\tilde{f}$ in (6). But (13) has solutions with stronger singularities $\sim x^{-m/3}$, $(1-x)^{-m/3}$ and $x^{2/3}$ at the points $x=0$, 1 and ∞, respectively. Presumably, one should select the eigenvalue λ in (13) from the condition that such singularities incompatible with the representation (21) would disappear.

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