The longitudinal structure function $F_L(x, Q^2)$ at small x

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Abstract. The gluon contribution to the structure function $F_L(x, Q^2)$ is calculated using $k_\perp$ factorization. A generalization of this factorization is given, which allows the expression of structure functions and hard cross sections in terms of quantities that are well defined within perturbative QCD.

1. Introduction

In this paper we calculate, using $k_\perp$ factorization [1], the gluon contribution to the structure function $F_L(x, Q^2)$. With the advent of HERA the electromagnetic proton structure functions can be determined down to values of Bjorken $x \gtrsim 10^{-4}$ at $Q^2 \gtrsim 10\,\mathrm{GeV}^2$ [2] with high precision. Among the different structure functions which will be measured, $F_L(x, Q^2)$ is known to be particularly sensitive to the gluon distribution of the proton and may be used as an observable to determine this distribution [3, 4].

Usually, the evolution of the proton structure functions is calculated in fixed-order perturbation theory assuming the validity of the collinear approach of the parton model and mass factorization. For not too small values of x, $x \geqslant 10^{-2}$, this method works well, as demonstrated by various deep-inelastic scattering experiments. However, if x becomes very small, for example $x \sim 10^{-3}$–10^{-4}, these assumptions may turn out to be invalid. As discussed in [5] one should properly account for the $k_\perp$ effects of the parton entering the hard scattering process. This leads both to a modification of the parton picture *and* the factorization scheme used in comparison to the calculations done in the range of medium values of x, as discussed below. This method has been applied previously in the high-energy limit, using further appropriate approximations (see [1, 5–7] for some applications).

As an example, in this paper we calculate the longitudinal structure function $F_L(x, Q^2)$, taking the $k_\perp$ effects of the initial-state gluon into account without any approximation in x, in order to obtain a coefficient function which is valid in the full x range. This is particularly important due to the fact that the gluon density rises rapidly at small x. Since $F_L(x, Q^2)$ is obtained as a Mellin convolution in x of the gluon density and a coefficient function, the small-x part of the former samples the large-x part of the latter, and vice versa. To obtain a consistent perturbative description the $k_\perp$ factorization relation originally used in [1] cannot be applied here directly. It has to be transformed into a relation that allows us to express $F_L(x, Q^2)$ *only* in terms of quantities that are fully defined within perturbative QCD.

2. $k_\perp$ factorization

The calculation of the deep-inelastic scattering cross section presumes the factorization of the 'point-like' hard cross section of the subprocess from the parton distributions. In the case of incoming partons that are collinear with the initial-state hadrons the factorization relation is

$$H(x,\mu^2) = \int_0^1 \mathrm{d}x_1\,\mathrm{d}x_2\,\delta(x - x_1x_2)G(x_1,\mu^2)\sigma_H^{\mathrm{pt}}(x_2,\mu^2) \tag{1}$$

for an observable $H(x,\mu^2)$. Here, μ^2 denotes an appropriate factorization scale, $G(x_1,\mu^2)$ the parton distribution and $\sigma_H^{\mathrm{pt}}(x_2,\mu^2)$ the cross section of the hard subprocess. The $k_\perp$-dependent factorization structure was derived in [1] for the case that the initial-state partons are gluons. One obtains

$$H(x,\mu^2) = \int \frac{\mathrm{d}^2 k}{\pi} \int_0^1 \mathrm{d}x_1\mathrm{d}x_2\delta(x - x_1x_2)\mathcal{F}(x_1,\boldsymbol{k})\sigma_H^{\mathrm{pt}}(x_2,\boldsymbol{k},\mu^2). \tag{2}$$

Here, $\mathcal{F}(x,\boldsymbol{k},Q_0^2)$ is defined by [8]

$$G(x,\mu^2) = \int_0^{\mu^2} \mathrm{d}k^2\mathcal{F}(x,\boldsymbol{k}). \tag{3}$$

Because the gluon momentum k is expressed in the Sudakov representation $k^\mu = \xi p_1^\mu + \eta p_2^\mu + k_\perp^\mu$ it may depend on the choice of the two light-like vectors p_1 and p_2 in general, which can induce some scheme dependence when using (2). For the calculation of proton structure functions a natural choice of the light-like vectors is $q' = q + xP$ and P† where q is the 4-momentum transferred to the proton, P is the proton momentum, and x is the Bjorken variable‡.

Since the kinematical range of $|\boldsymbol{k}|$ is $0 \leqslant |\boldsymbol{k}| \leqslant K_{\max} = \sqrt{Q^2(1-x)/x}$, equation (2) is not an appropriate definition in general, because $\mathcal{F}(x,\boldsymbol{k})$ is not defined in perturbative QCD at values of $\boldsymbol{k}^2$ smaller than $Q_0^2 \sim 1\,\mathrm{GeV}^2$, since non-perturbative terms become significant. At such low $\boldsymbol{k}^2$ values, factorization relations are introduced to *separate* non-perturbative parts from perturbative terms. In the case of hadronic structure functions one may, fortunately, rewrite (2) in such a way that $H(x,\mu^2)$ can indeed be expressed by quantities defined perturbatively. Since the K^2-dependent coefficient function $\sigma_H^{\mathrm{pt}}(x,\boldsymbol{k},\mu^2)$ is only calculated in fixed-order perturbation theory, it is not intended to describe the observable $H(x,\mu^2)$ at arbitrary small scales μ^2. For example, in the case of hadronic structure functions, one chooses, $\mu^2 = Q^2 \equiv -q^2 \gg Q_0^2 \simeq$ a few GeV^2. A comparison with experimental data should be done only for these values of Q^2. Therefore, in the range $K^2 \equiv -k^2 \leqslant Q_0^2$ the coefficient functions $\sigma_H^{\mathrm{pt}}(x,K^2/Q^2) \equiv f_i^{\mathrm{qG}}(x,K^2/Q^2)$ approach the value $f_i^{\mathrm{qG}}(x,K^2/Q^2 \to 0)$. Thus, one may rewrite (2) as

$$F_i(x,Q^2) = \sum_q \left[\int_x^1 \frac{\mathrm{d}\eta}{\eta} f_i^{\mathrm{qG}}\left(\frac{x}{\eta}\right)\eta G(\eta,Q_0^2) + \int_x^1 \frac{\mathrm{d}\eta}{\eta}\int_{Q_0^2}^{K_{\max}^2} \mathrm{d}K^2 f_i^{\mathrm{qG}}\left(\frac{x}{\eta},\frac{K^2}{Q^2}\right)\frac{\partial \eta G(\eta,K^2)}{\partial K^2}\right] \tag{4}$$

provided f_i^{qG} contains *no* collinear singularity for $K^2 \to 0$, which is the case for $F_{\mathrm{L}}(x,Q^2)$§. This is also the reason for the well known fact that $f_{\mathrm{L}}^{\mathrm{qG}}(x,K^2/Q^2 \to 0)$ is scheme independent in $\mathcal{O}(\alpha_s)$. Here, $K_{\max}^2 = Q^2(\eta - x)/x$. Note that (4) depends on the gluon distribution $xG(x,K^2)$ only at virtualities K^2 which are large enough that it can be considered as a parton distribution. Equation (4) will be used for the calculation of the structure function $F_{\mathrm{L}}(x,Q^2)$ hereafter.

† In [1] the 4-vectors l_e and P were chosen instead.

‡ Fermion masses were neglected whenever possible.

§ The corresponding relation in the case when collinear singular terms do occur is given in [12].

3. The structure function

The deep-inelastic scattering cross section may be written as

$$\frac{d^2\sigma}{dQ^2 dy} = 2\pi\alpha^2 \frac{Ms}{(s-M^2)^2} \frac{1}{Q^4} L_{\mu\nu} W^{\mu\nu}. \tag{5}$$

For electromagnetic interactions† the leptonic and hadronic tensor $L_{\mu\nu}$ and $W_{\mu\nu}$ are given by

$$\begin{aligned} L_{\mu\nu} &= 2(l_\mu l'_\nu + l'_\mu l_\nu - g_{\mu\nu}\, l \cdot l') \\ W_{\mu\nu} &= \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(x, Q^2) + \frac{1}{M^2}\left[\left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right)\right] W_2(x, Q^2) \end{aligned} \tag{6}$$

with l and l' the incoming and outgoing lepton 4–momenta, and M the proton mass. In the Bjorken limit the longitudinal structure function $F_L(x, Q^2)$ is obtained via the projection

$$\frac{P\cdot q}{M^2} W_2(x, Q^2) - 2x W_1(x, Q^2) \to F_L(x, Q^2) = \frac{8x^3}{Q^2} P_\mu P_\nu W^{\mu\nu}. \tag{7}$$

The coefficient function for the gluon contribution to $F_L(x, Q^2)$, which yields the dominant part, is given by

$$\begin{aligned} f_L^{qG}(K^2, x, Q^2) = \frac{\alpha_s e_q^2}{4\pi} \Bigg(&\frac{4Q^4}{K^4 x} G_{1L}(\beta, \zeta) + \frac{xQ^2}{K^2} \frac{1}{\sqrt{1-\zeta}} \log\left|\frac{1+\sqrt{1-\zeta}}{1-\sqrt{1-\zeta}}\right| G_{2L}(\beta, \zeta) \\ &+ \frac{2xQ^2}{K^2} G_{3L}(\beta, \zeta)\Bigg) \end{aligned} \tag{8}$$

where

$$\zeta = \frac{4K^2x^2}{Q^2} \qquad \cos\beta = \frac{1-\zeta/2}{\sqrt{1-x\zeta}} \tag{9}$$

and β denotes the angle between gluon and proton in the virtual-photon–virtual-gluon CMS. The functions $G_{iL}(\beta, \zeta)$ in (8) may be expressed in a polynomial form by

$$G_{iL}(\beta, \zeta) = -\sum_{j=0}^{4} g_{ji}^{L}(\beta) \left(\frac{\zeta}{W(\zeta)}\right)^j \tag{10}$$

where

$$W(\zeta) = 1 - \zeta + \sqrt{1-\zeta}. \tag{11}$$

† At $Q^2 \leqslant 500\,\mathrm{GeV}^2$ the contribution due to γ–Z interference and $|Z|^2$ terms turns out to be very small in the kinematical range accessible at HERA [9].

Finally, the coefficients $g_{ji}^{(L)}$ in (10) are

$$
\begin{aligned}
g_{01}^{(L)}(\beta) &= -\tfrac{1}{8} + \tfrac{1}{4}\cos\beta - \tfrac{1}{4}\cos^3\beta + \tfrac{1}{8}\cos^4\beta \\
g_{02}^{(L)}(\beta) &= -\tfrac{1}{4} + 2\cos\beta - \cos^2\beta - 3\cos^3\beta + \tfrac{9}{4}\cos^4\beta \\
g_{03}^{(L)}(\beta) &= -\tfrac{1}{4} + 6\cos\beta - \tfrac{9}{2}\cos^2\beta - 10\cos^3\beta + \tfrac{35}{4}\cos^4\beta \\
g_{11}^{(L)}(\beta) &= \cos\beta - \tfrac{3}{4}\cos^2\beta - \tfrac{3}{2}\cos^3\beta + \tfrac{5}{4}\cos^4\beta \\
g_{12}^{(L)}(\beta) &= \tfrac{1}{4} + \tfrac{13}{2}\cos\beta - \tfrac{15}{2}\cos^2\beta - \tfrac{21}{2}\cos^3\beta + \tfrac{45}{4}\cos^4\beta \\
g_{13}^{(L)}(\beta) &= 1 + 18\cos\beta - 24\cos^2\beta - 30\cos^3\beta + 35\cos^4\beta \\
g_{21}^{(L)}(\beta) &= \tfrac{3}{16} + \tfrac{9}{8}\cos\beta - \tfrac{9}{4}\cos^2\beta - \tfrac{15}{8}\cos^3\beta + \tfrac{45}{16}\cos^4\beta \\
g_{22}^{(L)}(\beta) &= \tfrac{5}{4} + \tfrac{27}{4}\cos\beta - 15\cos^2\beta - \tfrac{45}{4}\cos^3\beta + \tfrac{75}{4}\cos^4\beta \\
g_{23}^{(L)}(\beta) &= \tfrac{7}{2} + 18\cos\beta - 42\cos^2\beta - 30\cos^3\beta + \tfrac{105}{2}\cos^4\beta \\
g_{31}^{(L)}(\beta) &= \tfrac{3}{16} + \tfrac{6}{16}\cos\beta - \tfrac{15}{8}\cos^2\beta - \tfrac{5}{2}\cos^3\beta + \tfrac{35}{4}\cos^4\beta \\
g_{32}^{(L)}(\beta) &= \tfrac{9}{8} + \tfrac{9}{4}\cos\beta - \tfrac{45}{4}\cos^2\beta - \tfrac{15}{4}\cos^3\beta + \tfrac{105}{8}\cos^4\beta \\
g_{33}^{(L)}(\beta) &= 3 + 6\cos\beta - 30\cos^2\beta - 10\cos^3\beta + 35\cos^4\beta \\
g_{41}^{(L)}(\beta) &= \tfrac{3}{64} - \tfrac{15}{32}\cos^2\beta + \tfrac{35}{64}\cos^4\beta \\
g_{42}^{(L)}(\beta) &= \tfrac{9}{32} - \tfrac{45}{16}\cos^2\beta + \tfrac{105}{32}\cos^4\beta \\
g_{43}^{(L)}(\beta) &= \tfrac{3}{4} - \tfrac{15}{2}\cos^2\beta + \tfrac{35}{4}\cos^4\beta.
\end{aligned}
\tag{12}
$$

The structure function $F_L(x, Q^2)$ is then given by (4) for $i = L$.

In the limit $K^2 \to 0$ the coefficient function $f_L^{qG}(x, Q^2)$ simplifies considerably and one obtains the well known result [10]

$$
f_L^{qG(0)}(x, Q^2) = \frac{2}{\pi} e_q^2 \alpha_s x^2 (1 - x). \tag{13}
$$

The gluonic part of $F_L(x, Q^2)$ is then represented by

$$
F_{L(0)}(x, Q^2) = \sum_q \int_x^1 \frac{dz}{z} f_{2,L}^{qG}\left(\frac{x}{z}\right) zG(z, Q^2) \tag{14}
$$

since the integral over K^2 in (4) can be carried out analytically. Equation (4) is the generalization of (14) for the case of $k_\perp$ factorization. Note that in the limit $K^2 \to 0$ the complete coefficient function as derived in the Altarelli–Parisi approach is obtained, which is due to the fact that (4) was derived without further approximations with respect to the x behaviour. Furthermore, (4) is an expression for $F_L(x, Q^2)$ depending *only* on quantities accessible in perturbative QCD. We have not been specific in expressing $\partial xG(x, K^2)/\partial K^2$, since at small x various dynamical effects may influence the scaling violations of the single-particle gluon distribution. Among them are effects due to the Lipatov Pomeron [8] and gluon-recombination effects [11]. A detailed theoretical investigation of these terms within perturbative QCD requires further work. The value of expressions like (4) is that they allow us to extract the gluon distribution from measured structure functions, and thus to compare theoretical predictions on the evolution of $xG(x, Q^2)$ directly with data.

Results of a numerical comparison between (4) and (14), and the corresponding behaviour of $F_2(x, Q^2)$, are given in [12].

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