J. Phys. A: Math. Gen. **29** (1996) 365–383. Printed in the UK

Conditionally solvable path integral problems: II. Natanzon potentials

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Received 22 September 1995

Abstract. New classes of exactly solvable potentials are discussed within the path integral formalism. They are constructed from the hypergeometric and confluent Natanzon potentials, respectively. It is found that they allow incorporation of four free parameters, which give rise to fractional power behaviour, long-range and strongly anharmonic terms. We find six different classes of such potentials.

1. Introduction

In a previous paper [1] I discussed a class of potentials which are known as 'conditionally solvable potentials' [2–5]. They have the specific feature that they incorporate strongly anharmonic, fractional power behaviour, and long-range terms. They modify the usual potentials in quantum mechanics in a specific way such that they are quantum mechanically exactly solvable; however, one is not completely free to choose the parameters and the couplings of the potentials.

In this second paper I want to generalize the potentials of [1–5] to more general classes which are related to the Natanzon potentials [6], and cf [7–16]. With their six parameter structure the Natanzon potentials are designed in such a way that a wider range of shapes and potential wells is allowed in comparison to other well known potential problems in quantum mechanics. Let us mention, for example, the Morse potential, the radial harmonic oscillator, the Coulomb potential, and the class of hypergeometric potentials as contained in Pöschl–Teller and modified Pöschl–Teller potentials [17]. They have numerous applications, e.g. in the study of solvable potentials in quantum mechanics in general, in the study of molecular physics for modelling a more realistic single-particle electronic shell structure, in atomic physics for quark–antiquark forces, charge densities of nuclei, or in solid state physics. The two classes of Natanzon potentials cover all known potentials for which an analytic solution to the bound and continuous state problem can be found.

The class of the hypergeometric Natanzon potentials is defined (note the different notation used in the literature) by

$$V_{\mathrm{h}}(r) = \frac{\hbar^2}{2m}\frac{fz(z-1)+h_0(1-z)+h_1 z}{R(z)} + \frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2 - 2\frac{z'''}{z'}\right) \qquad (1)$$

where $R(z) = a_0 z^2 + b_0 z + c_0$, and $z = z(r)$ is implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. The function z varies in the interval $z \in (0, 1)$. The $\hbar^2$-term is the Schwarz derivative of z with respect to r.

0305-4470/96/020365+19$19.50

The class of the confluent Natanzon potentials can be obtained by the substitution [6] $a_0 = \sigma_2/\tau^2$, $b_0 = \sigma_1/\tau$, $f = g_2/\tau^2$, $h_1 - h_0 - f = g_1/\tau$, $z = h/\tau$, and taking into account the limit $\tau \to 0$. This yields

$$V_c(r) = \frac{\overline{h}^2}{2m}\frac{g_2h^2 + g_1h + \eta}{R(r)} + \frac{\overline{h}^2}{8m}\left(3\left(\frac{h''}{h'}\right)^2 - 2\frac{h'''}{h'}\right) \tag{2}$$

where $R(r) = \sigma_2h^2 + \sigma_1h + c_0$, and $h = h(r)$ is implicitly defined by the differential equation $h'/2h = 1/\sqrt{R(r)}$. The variable r and the function $h = h(r)$ are assumed to be positive. The $\overline{h}^2$-term is the Schwarz derivative of h with respect to r. In [18] I succeeded in calculating the path integral representations corresponding to the two potentials (1) and (2) explicitly in terms of the corresponding Green function. The energy eigenvalues (= quantization conditions) were equations of fourth degree in the energy. In spite of the fact that the bound-state energy-level conditions are rather complicated, closed form solutions in terms of the Green function were still possible. This is quite surprising because the exact analytic form of a particular Natanzon potential is only implicitly defined and may not even be known analytically.

The two path integral representations in [18] contain all former path integral solutions which are related to the radial harmonic oscillator and the (modified) Pöschl–Teller potential, respectively. In the two latter cases at most two free parameters can be freely chosen.

In particular for (2), the choices $c_0 = \eta = 0$, $g_1h \mapsto g_1h^3$, and $\sigma_1h \mapsto \sigma_1h^4$, respectively $c_0 = \eta = 0$, $g_1h \mapsto g_1h^4$, $\sigma_1h \mapsto \sigma_1h^3$ produce modifications of the two kinds of conditionally solvable potentials as discussed in [1]. They have been called a modified Coulomb potential and a radial confinement potential, respectively. The new potentials which I would like to call *conditionally solvable Natanzon potentials* have four free parameters, and seem to be entirely new. Of course, a similar modification can be done for the hypergeometric version, and gives four new classes of potentials. These modifications of the original Natanzon potentials are suitable for our purposes, and the path integral discussion of these four new classes of potentials is the main object of this paper.

Although exactly solvable, these potentials are sufficiently complicated to be worthy of serious consideration in the modelling of actual physical forces. This can be the case where one wants to study an approximation of a model, or where an exactly solvable model is used as a starting point for a comprehensive numerical investigation, cf the recent review [19], for example. By choosing a path integral approach we succeed in gaining comprehensive information about the bound-state solutions of these potentials (if they exist), and what is often more important, about the scattering states which eventually allow for the calculation of cross sections and phase shifts. In this respect, the path integral provides a convenient tool for the calculation in which the proper analytic structure of the solutions is made manifest.

In order to avoid unnecessary overlap I do not repeat the space–time transformation technique as sketched in [1], e.g [20–34], and references therein. For the actual formulation of the path integral representations of the potentials I use the canonical path integral definition as developed in [20–23, 27, 35, 36]. Again this will not be repeated here.

This paper is organized as follows. In section 2 I present the six classes of 'conditionally solvable Natanzon' potentials labelled V_1, V_2, V_3, V_4, V_5 and V_6, respectively. The well established space–time transformation technique reduces each path integral problem to one that is already known. The final result in each case includes the statement of the corresponding Green function. The poles of the Green functions yield in the first two classes implicit expressions (transcendental equations) for the energy eigenvalues, in the second two cases equations of fourth degree in the energy which admit an analytic solution,

and in the last two classes also only transcendental equations for the energy eigenvalues can be stated. The bound-state wavefunctions and the scattering states are not evaluated explicitly. Section 3 contains a summary and a short discussion.

2. The potentials

2.1. A generalized Coulomb potential

In the first class of potentials I consider a modification of (2) by changing the power behaviour of h in the g_1-term to a g_1h^3, in the σ_1-term to σ_1h^4, and we set $c_0 = \eta = 0$. Thus we obtain the following first *conditionally solvable confluent Natanzon potential*:

$$V_1(r) = \frac{\hbar^2}{2m}\frac{g_1h^3 + g_2h^2}{R(r)} + \frac{\hbar^2}{8m}\left(3\left(\frac{h''}{h'}\right)^2 - 2\frac{h'''}{h'}\right) \tag{3}$$

where

$$R(r) = \sigma_1h^4 + \sigma_2h^2 \qquad \frac{h'(r)}{2h(r)} = \frac{1}{\sqrt{R(r)}}\,. \tag{4}$$

The variable r and the function $h = h(r)$ are assumed to be positive. The special choice $\sigma_1 = 16$ and $\sigma_2 = 0$ gives $h = h(r) = \sqrt{r}$ and reproduces the modified Coulomb potential of [1], including, of course, the quantum potential $\Delta V = -3\hbar^2/32mr^2$, i.e.

$$V(r) = \frac{\hbar^2}{32m}\left(\frac{g_2}{r} + \frac{g_1}{\sqrt{r}} - \frac{3}{r^2}\right). \tag{5}$$

Assuming a power dependence $h = h(r) = r^\alpha$, as $r \to \infty$, we find $\alpha = \frac{1}{2}$, and the potential (5) also describes the asymptotic behaviour of $V_1(r)$. In order to calculate the path integral representation of the potential (3) we perform the transformation $r \mapsto z$ together with the time substitution $\mathrm{d}t = \mathrm{d}s/z'^2$ such that the new pseudo-time s'' can be introduced via the constraint $\int_0^{s''}\mathrm{d}s/z'^2 = T = t'' - t'$. This space–time transformation causes the emerging Schwarz derivative to cancel with the $\hbar^2$-term and gives the path integral representation in the polar coordinate h

$$K^{(V_1)}(r'', r'; T) = \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - V_1(r)\right]\mathrm{d}t\right\}$$

$$= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s''\ \exp\left(\mathrm{i}(\sigma_2E - \hbar^2g_2/2m)s''/4\hbar\right)$$

$$\times \int_{h(0)=h'}^{h(s'')=h''}\mathcal{D}h(s)\ \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{h}^2 + \frac{E\sigma_1}{4}h^2 - \frac{\hbar^2}{8m}g_1h\right)\mathrm{d}s\right] \tag{6}$$

$$= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}$$

$$\times \int_0^\infty \mathrm{d}s''\ \exp\left(\mathrm{i}(\sigma_2E - \hbar^4g_1^2/16m^2\sigma_1E - \hbar^2g_2/2m)s''/4\hbar\right)$$

$$\times \int_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s)\ \exp\left[\frac{\mathrm{i}m}{2\overline{h}}\int_0^{s''}\left(\dot{v}^2-\omega^2 v^2\right)\mathrm{d}s\right] \tag{7}$$

$$\equiv \int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\,\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\overline{h}}G^{(V_1)}(r'',r';E) \tag{8}$$

and I have performed the additional variable shift $v = h - \overline{h}^2 g_1/4m\sigma_1 E$, and set $\omega^2 = -\sigma_1 E/2m$. The path integral (6) now has exactly the form as the one in [1] for the modified Coulomb potential after the space–time transformation. There it was solved with the appropriate boundary conditions, i.e. the path integral (6) is not the path integral of a shifted harmonic oscillator over the entire $\mathbb{R}$. As pointed out in [37] the wavefunction has the wrong behaviour at the origin of a singular potential, and therefore such a 'solution' must be discarded as physically unacceptable [38]. Similarly, as in [1] it is not possible to extend the variable h to the entire $\mathbb{R}$, a feature which is in accordance with the one-dimensional Kustaanheimo–Stiefel transformation which maps $\mathbb{R}^+ \to \mathbb{R}^+$ [34, 39]. Therefore the path integral (6) is a radial path integral with $h > 0$ [40], and the path integral (7) is a radial path integral for a harmonic oscillator with $v > -\overline{h}^2/4m\sigma_1 E$, for fixed energy E. The additional linear term spoils the symmetry with respect to reflections in the variable h. In [41, 42] I developed a procedure for dealing with such problems within the path integral. We assume that we have evaluated a path integral problem with a potential $V(x)$ in, say, the entire $\mathbb{R}$. This path integral is called $K^{(V)}(T)$. The corresponding Green function is denoted by $G^{(V)}(E)$. Now we consider the path integral problem with the same potential V, but with Dirichlet boundary conditions at the location $x = a$ and we consider the half-space $x > a$. Then the Green function in the half-space $x > a$ is given [41, 42] by:

$$\frac{\mathrm{i}}{\overline{h}}\int_0^\infty \mathrm{d}T\ \mathrm{e}^{\mathrm{i}ET/\overline{h}} \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(D)}_{(x>a)}x(t)\ \exp\left\{\frac{\mathrm{i}}{\overline{h}}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2 - V(x)\right]\mathrm{d}t\right\}$$

$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}(x'',a;E)G^{(V)}(a,x';E)}{G^{(V)}(a,a;E)}\,. \tag{9}$$

The Green function corresponding to $G^{(V)}(E)$ which we need is the Green function of the harmonic oscillator, $G^{(\omega)}(E)$, and has the form

$$G^{(\omega)}(v'',v';E) = \sqrt{\frac{m}{\pi\overline{h}^3\omega}}\,\Gamma(-\nu)D_\nu\left(\sqrt{\frac{2m\omega}{\overline{h}}}\,v_>\right)D_\nu\left(-\sqrt{\frac{2m\omega}{\overline{h}}}\,v_<\right). \tag{10}$$

Here $D_\nu(z)$ is a parabolic cylinder function [43, p 1064]. Inserting (10) into (7) and complying with (9) we obtain the following solution for the Green function $G^{(V_1)}(E)$:

$$G^{(V_1)}(r'',r';E) = \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\sqrt{\frac{m}{\pi\overline{h}^3\omega}}\,\Gamma(-\nu)$$

$$\times\left\{D_\nu\left[\sqrt{\frac{2m\omega}{\overline{h}}}\left(h_>(r)-\frac{\overline{h}^2 g_1}{4m\sigma_1 E}\right)\right]D_\nu\left[-\sqrt{\frac{2m\omega}{\overline{h}}}\left(h_<(r)-\frac{\overline{h}^2 g_1}{4m\sigma_1 E}\right)\right]\right.$$

$$-D_\nu\left[\sqrt{\frac{2m\omega}{\overline{h}}}\left(h(r')-\frac{\overline{h}^2 g_1}{4m\sigma_1 E}\right)\right]D_\nu\left[\sqrt{\frac{2m\omega}{\overline{h}}}\left(h(r'')-\frac{\overline{h}^2 g_1}{4m\sigma_1 E}\right)\right]$$

$$\left.\times D_\nu\left(\sqrt{\frac{2m\omega}{\overline{h}}}\,\frac{\overline{h}^2 g_1}{4m\sigma_1 E}\right)\Big/ D_\nu\left(-\sqrt{\frac{2m\omega}{\overline{h}}}\,\frac{\overline{h}^2 g_1}{4m\sigma_1 E}\right)\right\}. \tag{11}$$

Here I have used the shorthand notation

$$\nu = -\frac{1}{2} + \frac{1}{4\omega\hbar}\left(\sigma_2 E - \frac{\hbar^2 g_2}{2m} - \frac{\hbar^4 g_1^2}{16m^2\sigma_1 E}\right). \tag{12}$$

This determines the energy spectrum by the zeros of the parabolic cylinder function, i.e.

$$D_{\nu_n}\left(-\sqrt{\frac{2m\omega_n}{\hbar}}\,\frac{\hbar^2 g_1}{4m\sigma_1 E_n}\right) = 0\,. \tag{13}$$

By $\nu_n = \nu(E_n)$ and $\omega_n = \omega(E_n)$ I have indicated the explicit dependence on E_n. The analysis in [42] showed that the poles coming from the prefactor in (11) play no role in the corresponding boundary condition problem. Equation (13) clearly generalizes the corresponding result of [1] to the case of four parameters.

2.2. A radial confinement potential

The second class of *conditionally solvable confluent Natanzon potentials* we want to consider has the form

$$V_2(r) = \frac{\hbar^2}{2m}\frac{g_1 h^4 + g_2 h^2}{R(r)} + \frac{\hbar^2}{8m}\left(3\left(\frac{h''}{h'}\right)^2 - 2\frac{h'''}{h'}\right) \tag{14}$$

where

$$R(r) = \sigma_1 h^3 + \sigma_2 h^2 \qquad \frac{h'(r)}{2h(r)} = \frac{1}{\sqrt{R(r)}}\,. \tag{15}$$

The special choice $\sigma_1 = 9$ and $\sigma_2 = 0$ gives $h = h(r) = r^{2/3}$ and reproduces the radial confinement potential of [1], including the quantum potential $\Delta V = -5\hbar^2/72mr^2$,

$$V(r) = \frac{\hbar^2}{18m}\left(g_1 r^{2/3} + g_2 r^{-2/3} - \frac{5}{4r^2}\right) \tag{16}$$

and the radial confinement potential (16) is also the asymptotic solution of V_2, for $h = h(r)$ powerlike as $r \to \infty$. The necessary space–time transformation has the form $r \mapsto h$ accompanied by the time substitution $\mathrm{d}t = R(h)\mathrm{d}s/4h^2$, as before. This gives the following path integral representation in the polar coordinate h:

$$\begin{aligned}
K^{(V_2)}(r'', r'; T) &= \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - V_2(r)\right]\mathrm{d}t\right\} \\
&= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s''\ \exp\left(\mathrm{i}(\sigma_2 E - \hbar^2 g_2/2m)s''/4\hbar\right) \\
&\quad\times \int_{h(0)=h'}^{h(s'')=h''} \mathcal{D}h(s)\ \exp\left[\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left(\frac{m}{2}\dot{h}^2 - \frac{\hbar^2}{8m}g_1 h^2 + \frac{E\sigma_1}{4}h\right)\mathrm{d}s\right] \qquad (17) \\
&= \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar} \\
&\quad\times \int_0^\infty \mathrm{d}s''\ \exp\left(\mathrm{i}(\sigma_2 E + m\sigma_1^2 E^2/2\hbar^2 g_1 - \hbar^2 g_2/2m)s''/4\hbar\right)
\end{aligned}$$

$$\times \int_{v(0)=v'}^{v(s'')=v''} \mathcal{D}v(s)\ \exp\left[\frac{\mathrm{i}m}{2\overline{h}}\int_0^{s''}\left(\dot{v}^2-\omega^2 v^2\right)\mathrm{d}s\right] \tag{18}$$

$$\equiv \int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\,\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\overline{h}}G^{(V_2)}(r'',r';E) \tag{19}$$

and I have performed the additional variable shift $v = h - m\sigma_1 E/\overline{h}^2 g_1$, and set $\omega = \overline{h}\sqrt{g_1}/2m$. Compared with (6) the roles of the quadratic and the linear term are interchanged. The path integral (17) now has exactly the form as in [1] for the radial confinement potential. Taking the corresponding result of [1] we obtain the following solution for the Green function $G^{(V_2)}(E)$:

$$\begin{aligned}
G^{(V_2)}(r'',r';E) = {} & \left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\sqrt{\frac{m}{2\pi\overline{h}^3\omega}}\,\Gamma(-\nu) \\
& \times\left\{D_\nu\left[\sqrt{\frac{m\omega}{\overline{h}}}\left(h_>(r)-\frac{m\sigma_1}{\overline{h}^2 g_1}E\right)\right]D_\nu\left[-\sqrt{\frac{m\omega}{\overline{h}}}\left(h_<(r)-\frac{m\sigma_1}{\overline{h}^2 g_1}E\right)\right]\right. \\
& -D_\nu\left[\sqrt{\frac{m\omega}{\overline{h}}}\left(h(r')-\frac{m\sigma_1}{\overline{h}^2 g_1}E\right)\right]D_\nu\left[\sqrt{\frac{m\omega}{\overline{h}}}\left(h(r'')-\frac{m\sigma_1}{\overline{h}^2 g_1}E\right)\right] \\
& \left.\times D_\nu\left(\sqrt{\frac{m\omega}{\overline{h}}}\,\frac{m\sigma_1}{\overline{h}^2 g_1}E\right)\Big/ D_\nu\left(-\sqrt{\frac{m\omega}{\overline{h}}}\,\frac{m\sigma_1}{\overline{h}^2 g_1}E\right)\right\}.
\end{aligned} \tag{20}$$

Here I have used the shorthand

$$\nu = -\frac{1}{2}+\frac{1}{4\overline{h}\omega}\left(\sigma_2 E+\frac{m\sigma_1^2E^2}{2\overline{h}^2 g_1}-\frac{\overline{h}^2 g_2}{2m}\right). \tag{21}$$

This determines the energy levels E_n by the zeros of the parabolic cylinder function, i.e.

$$D_{\nu_n}\left(-\sqrt{\frac{m\omega}{\overline{h}}}\,\frac{m\sigma_1}{\overline{h}^2 g_1}E_n\right)=0\,. \tag{22}$$

I have indicated by $\nu_n = \nu(E_n)$ the explicit dependence on E_n. This result again generalizes the corresponding case of [1].

In the special case of $g_1 = 0$ we must consider the path integral solution of the linear potential. In this case for the Green function $G^{(g_1=0)}(E)$ (note $E' = \sigma_2 E - \overline{h}^2 g_2/2m$) we obtain

$$\begin{aligned}
G^{(V_2)}_{g_1=0}(r'',r';E) = {} & \frac{4}{3}\frac{m}{\overline{h}^2}\left(\frac{\sqrt{R(r')R(r'')}}{4h(r')h(r'')}\right)^{1/2}\left[\left(h(r')+\frac{4E'}{\sigma_1 E}\right)\left(h(r'')+\frac{4E'}{\sigma_1 E}\right)\right]^{1/2} \\
& \times\left\{K_{\frac{1}{3}}\left[\frac{\sqrt{-2m\sigma_1 E}}{3\overline{h}}\left(h_>(r)+\frac{4E'}{\sigma_1 E}\right)^{\frac{3}{2}}\right]\right. \\
& \times I_{\frac{1}{3}}\left[\frac{\sqrt{-2m\sigma_1 E}}{3\overline{h}}\left(h_<(r)+\frac{4E'}{\sigma_1 E}\right)^{\frac{3}{2}}\right] \\
& \left.-K_{\frac{1}{3}}\left[\frac{\sqrt{-2m\sigma_1 E}}{3\overline{h}}\left(h(r')+\frac{4E'}{\sigma_1 E}\right)^{\frac{3}{2}}\right]\right.
\end{aligned}$$

$$\times K_{\frac{1}{3}}\left[\frac{\sqrt{-2m\sigma_1 E}}{3\hbar}\left(h(r'')+\frac{4E'}{\sigma_1 E}\right)^{\frac{3}{2}}\right]$$

$$\times\ I_{\frac{1}{3}}\left[\frac{\sqrt{-2m}}{3\sigma_1\hbar E}\left(\sigma_2 E-\frac{\hbar^2 g_2}{2m}\right)^{\frac{3}{2}}\right]\Bigg/ K_{\frac{1}{3}}\left[\frac{\sqrt{-2m}}{3\sigma_1\hbar E}\left(\sigma_2 E-\frac{\hbar^2 g_2}{2m}\right)^{\frac{3}{2}}\right]\Bigg\}\,. \tag{23}$$

$I_\nu(z)$ and $K_\nu(z)$ are modified Bessel functions [43, p 958]. The possible bound states are determined by

$$K_{1/3}\left[\frac{\sqrt{-2m}}{3\sigma_1\hbar E_n}\left(\sigma_2 E_n-\frac{\hbar^2 g_2}{2m}\right)^{3/2}\right]=0\,. \tag{24}$$

In terms of the Airy function $\mathrm{Ai}(z)$ this gives

$$\mathrm{Ai}\left[\left(\frac{\sqrt{-2m}}{2\sigma_1\hbar E_n}\right)^{2/3}\left(\sigma_2 E_n-\frac{\hbar^2 g_2}{2m}\right)\right]=0 \tag{25}$$

which gives a cubic equation in E_n and has the form ($\alpha_n>0, n\in\mathbb{N}_0$, are the zeros of the Airy function, i.e. $\mathrm{Ai}(-\alpha_n)=0$ [44, p 166])

$$\left(\sigma_2 E_n-\frac{\hbar^2 g_2}{2m}\right)^3-\frac{2\sigma_1^2\hbar^2\alpha_n^3}{m}E_n^2=0 \tag{26}$$

which can be cast into the canonical form (the case $\sigma_2=0$ is equivalent with the case discussed in [1] and must be treated separately)

$$E_n^3-\frac{\hbar^2}{2m\sigma_2^3}\left(3g_2\sigma_2^2+4\sigma_1^2\alpha_n^3\right)E_n^2+3\left(\frac{\hbar^2 g_2}{2m\sigma_2}\right)^2E_n-\left(\frac{\hbar^2 g_2}{2m\sigma_2}\right)^3=0\,. \tag{27}$$

One *real* solution of this cubic equation is given by [50]

$$E_n=\sqrt[3]{\sqrt{D}-\frac{Q}{2}}-\sqrt[3]{\sqrt{D}+\frac{Q}{2}}-\frac{R}{3} \tag{28}$$

$$D=\left(\frac{P}{3}\right)^3+\left(\frac{Q}{2}\right)^2\qquad P=\frac{3S-R^2}{3}\qquad Q=\frac{2R^3}{27}-\frac{RS}{3}+T \tag{29}$$

$$R=-\frac{\hbar^2}{2m\sigma_2^3}\left(3g_2\sigma_2^2+4\sigma_1^2\alpha_n^3\right)\qquad S=3\left(\frac{\hbar^2 g_2}{2m\sigma_2}\right)^2\qquad T=-\left(\frac{\hbar^2 g_2}{2m\sigma_2}\right)^3. \tag{30}$$

An asymptotic analysis of the cubic equations shows that for $\alpha_n\to\infty$, i.e. $n\to\infty$, we have a behaviour of the bound-state energy levels according to $E_n\propto-|g_2|\alpha_n^{-3/2}$, $g_2<0$, and the accumulation point is $E_\infty=0$.

2.3. A modified Rosen–Morse potential: I

In the third class of potentials I want to investigate, I consider the hypergeometric Natanzon potential with the following modifications

$$V_3(r)=\frac{\hbar^2}{2m}\frac{\frac{3}{4}z(1-z)+h_0(1-z)+h_1z^{1/2}}{R(z)}+\frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2-2\frac{z'''}{z'}\right) \tag{31}$$

where $R(z) = b_0 z + c_0$, and $z = z(r)$ is implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. The variable z varies in the interval $z \in (0,1)$. For $R(z) = 1$, we find $z = \frac{1}{2}(1 + \tanh x)$, $x \in \mathbb{R}$, and the emerging potential has the form

$$V(r) = \frac{\hbar^2}{2m}\left(h_0 + 1 - \frac{h_0 - \frac{3}{4}}{1+\mathrm{e}^{-2r}} + \frac{h_1}{\sqrt{1+\mathrm{e}^{-2r}}} - \frac{3}{4(1+\mathrm{e}^{-2r})^2}\right) \tag{32}$$

which looks exactly like the first modified Rosen–Morse potential in [1]. In the original Natanzon potential V_{h} this choice yields the usual Rosen–Morse potential, which justifies our notion.

In order to calculate the path integral representation corresponding to the potential (31) we perform the transformation $r \mapsto z$ together with the time substitution $\mathrm{d}t = \mathrm{d}s/z'^2$. This space–time transformation causes the emerging Schwarz derivative to cancel with the $\hbar^2$-term and gives the path integral representation

$$K^{(V_3)}(r'', r'; T) = \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - V_3(r)\right]\mathrm{d}t\right\}$$

$$= \left(\frac{\sqrt{R(r')R(r'')}}{4z(r')z(r'')(1-z(r'))(1-z(r''))}\right)^{1/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\ \mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s''$$

$$\times \int_{z(0)=z'}^{z(s'')=z''} \mathcal{D}z(s)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{z}^2 + \frac{E\,R(z)}{4z^2(1-z)^2}\right.\right.$$

$$\left.\left. - \frac{\hbar^2}{8m}\left(\frac{h_0}{z^2(1-z)} + \frac{h_1}{z^{3/2}(1-z)^2} - \frac{3}{4z(z-1)}\right)\right]\mathrm{d}s\right\}. \tag{33}$$

We perform a further space–time transformation $z = z(x) = \tanh^2 x$, $x > 0$, together with the time substitution $\mathrm{d}t = 4\tanh^2 x\,\mathrm{d}s/\cosh^4 x$. The quantum potential emerging from the Schwarz derivative of z with respect to x is given by

$$\Delta V = \frac{\hbar^2}{8m}\left(4 + \frac{3}{\sinh^2 x} - \frac{3}{\cosh^2 x}\right) \tag{34}$$

and we obtain the path integral representation for the emerging Manning–Rosen [45] potential with the solution according to [23, 25, 46–49]

$$K^{(V_3)}(r'', r'; T) = \left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\ \mathrm{e}^{-\mathrm{i}ET/\hbar}$$

$$\times \int_0^\infty \mathrm{d}s''\ \exp\left(\mathrm{i}[(b_0+c_0)E - \hbar^2/2m]s''/\hbar\right)$$

$$\times \int_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac{1}{4}}{\sinh^2 x} + h_1\coth x\right)\right]\mathrm{d}s\right\} \tag{35}$$

$$= \int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\,\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G^{(V_3)}(r'', r'; E) \tag{36}$$

with the Green function $G^{(V_3)}(E)$ of the third conditionally solvable Natanzon potential given by

$$G^{(V_3)}(r'', r'; E) = \frac{m}{\hbar^2}\left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4} \frac{\Gamma(m_1 - L_E)\Gamma(L_E + m_1 + 1)}{\Gamma(m_1 + m_2 + 1)\Gamma(m_1 - m_2 + 1)}$$
$$\times\left(\frac{2\sqrt{z(r')}}{1+\sqrt{z(r')}}\frac{2\sqrt{z(r'')}}{1+\sqrt{z(r'')}}\right)^{\frac{1}{2}(m_1+m_2+1)}$$
$$\times\left(\frac{1-\sqrt{z(r')}}{1+\sqrt{z(r')}}\frac{1-\sqrt{z(r'')}}{1+\sqrt{z(r'')}}\right)^{\frac{1}{2}(m_1-m_2)}$$
$$\times {}_2F_1\left(-L_E + m_1, L_E + m_1 + 1; m_1 - m_2 + 1; \frac{1-\sqrt{z_>(r)}}{1+\sqrt{z_>(r)}}\right)$$
$$\times {}_2F_1\left(-L_E + m_1, L_E + m_1 + 1; m_1 + m_2 + 1; \frac{2\sqrt{z_<(r)}}{1+\sqrt{z_<(r)}}\right) \tag{37}$$

where

$$L_E = \tfrac{1}{2}\left(\sqrt{1 - h_1 - 2m(b_0 + c_0)E/\hbar^2} - 1\right) \tag{38}$$

$$m_{1,2} = \sqrt{h_0 + 1 - 2mc_0E/\hbar^2} \pm \tfrac{1}{2}\sqrt{h_1 + 1 - 2m(b_0 + c_0)E/\hbar^2} \tag{39}$$

and where ${}_2F_1(a, b; c; z)$ is the hypergeometric function [43, p 1039]. Furthermore I have used the shorthand

$$\eta^2 = h_0 + 1 - \frac{2mc_0E}{\hbar^2}\,. \tag{40}$$

Note that the number η is a square root and the specific sign it takes may vary in different examples. From the poles of the Green function and from the spectral expansion of the Manning–Rosen potential [49], respectively, we derive the quantization condition for the bound-state wavefunctions ($n \in \mathbb{N}$):

$$\sqrt{h_0 + 1 - 2mc_0E_n/\hbar^2} + \frac{\hbar}{\sqrt{2m}}\left(n + \frac{1}{2}\right)$$
$$= \tfrac{1}{2}\left(\sqrt{1 + h_1 - 2m(b_0 + c_0)E_n/\hbar^2} - \sqrt{1 - h_1 - 2m(b_0 + c_0)E_n/\hbar^2}\right). \tag{41}$$

This equation, which is actually an equation of fourth degree in the variable E_n, is with its four parameters an obvious generalization of the corresponding case in [1]. Introducing the abbreviations $B_1 = \frac{\hbar^2}{2m}(h_1-1)$, $B_2 = \frac{\hbar^2}{2m}(h_1+1)$, $C_1 = 4\frac{\hbar^2}{2m}(h_0+1)$, $\alpha_1 = 4c_0$, $\alpha_2 = b_0+c_0$, $\tilde{n} = \hbar(n + \frac{1}{2})/\sqrt{2m}$ it can be rewritten as

$$\sqrt{C_1 - \alpha_1 E_n} + 2\tilde{n} = \sqrt{B_1 - \alpha_2 E_n} - \sqrt{B_2 - \alpha_2 E_n} \tag{42}$$

and can be cast in the canonical form $\left(C_2 = C_1 + 4\tilde{n}^2 - \hbar/m, b_2 = (2b_0 - c_0)\right)$

$$E_n^4 + bE_n^3 + cE_n^2 + dE_n + e = 0 \tag{43}$$

$$a = (b_2^2 - 4\alpha_2^2)^2$$

$$b = \frac{4}{a}\Big[(b_2^2 - 4\alpha_2^2)(2\alpha_2(B_1 + B_2) - b_2C_2 + 8\alpha_1\tilde{n}^2) + 64\alpha_1\alpha_2^2\tilde{n}^2\Big]$$

$$\begin{aligned} c = \frac{8}{a}\Big[&2(4\alpha_1\tilde{n}^2 - \alpha_2B_1)^2 + \tfrac{1}{2}(2\alpha_2B_2 - b_2C_2)^2 + \tfrac{1}{4}b_2^2C_2^2 \\ &+ B_1B_2(8\alpha_2^2 - b_2^2) - \alpha_2C_2(\alpha_2C_2 + 2b_2B_1) \\ &- 4\tilde{n}^2(2\alpha_1b_2C_2 + b_2^2C_1 + 4\alpha_1\alpha_2B_2 + 4\alpha_2^2C_1)\Big] \end{aligned} \tag{44}$$

$$\begin{aligned} d = \frac{8}{a}\Big[&(B_2\alpha_2 - \tfrac{1}{2}b_2C_2 - 4\alpha_1\tilde{n}^2 + \alpha_2B_1)(C_2^2 - 4B_1B_2 + 16C_1\tilde{n}^2) \\ &+ 8C_2\tilde{n}^2(\alpha_1C_2 + 2b_2C_1)\Big] \end{aligned}$$

$$e = \frac{1}{a}\Big[(C_2^2 - 4B_1B_2 + 16C_1\tilde{n}^2)^2 - 64C_1C_2^2\tilde{n}^2\Big].$$

Equation (43) can be solved [50] by considering the solutions of the quadratic equation

$$E_n^2 + (b + A)\frac{E_n}{2} + y + \frac{by - d}{A} = 0 \tag{45}$$

where $A = \pm\sqrt{8y + b^2 - 4c}$, i.e. with the four solutions

$$E_{n_{1,2,3,4}} = -(b + A) \pm \sqrt{(b + A)^2 - 16\left(y + \frac{by - d}{A}\right)} \tag{46}$$

and y is any of the real roots of the cubic equation

$$8y^3 - 4cy^2 + (2bd - 8e)y + e(4c - b^2) - d^2 = 0\,. \tag{47}$$

2.4. A modified Rosen–Morse potential: II

In the fourth class of potentials we want to investigate, we modify the hypergeometric Natanzon potential according to

$$V_4(r) = \frac{\hbar^2}{2m}\frac{\frac{3}{4}z(1-z) + h_0(1-z) + h_1(1-z)^{1/2}}{R(z)} + \frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2 - 2\frac{z'''}{z'}\right) \tag{48}$$

where $R(z) = b_0z + c_0$, and $z = z(r)$ is again implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. Repeating the considerations of (32) gives an additional factor e^{-x} in the h_1-term, therefore reproducing the modified Rosen–Morse potential of [1]. We perform the same space–time transformation as before and obtain (η^2 as in (40))

$$K^{(V_4)}(r'', r'; T) = \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - V_4(r)\right]\mathrm{d}t\right\}$$

$$= \left(\frac{\sqrt{R(r')R(r'')}}{4z(r')z(r'')(1 - z(r'))(1 - z(r''))}\right)^{1/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s''$$

$$\times \int_{z(0)=z'}^{z(s'')=z''} \mathcal{D}z(s)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{z}^2+\frac{ER(z)}{4z^2(1-z)^2}\right.\right.$$

$$\left.\left.-\frac{\hbar^2}{8m}\left(\frac{h_0}{z^2(1-z)}+\frac{h_1}{z^2(1-z)^{3/2}}-\frac{3}{4z(z-1)}\right)\right]\mathrm{d}s\right\}$$

$$=\left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}$$

$$\times\int_0^\infty \mathrm{d}s''\ \exp\left(\mathrm{i}[b_0E+\hbar^2(h_0-\tfrac{1}{4})/2m]s''/\hbar\right)$$

$$\times \int_{u(0)=u'}^{u(s'')=u''} \mathcal{D}u(s)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{u}^2-\frac{\hbar^2}{2m}\left(\frac{\eta^2-\frac{1}{4}}{\tanh^2 u}+h_1\frac{\coth u}{\sinh u}\right)\right]\mathrm{d}s\right\} \tag{49}$$

$$=\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\,\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G^{(V_4)}(r'',r';E)\,. \tag{50}$$

Here the path integral for the hyperbolic Scarf-like potential [48, 51] has been used. The Green function $G^{(V_4)}(E)$ of the fourth conditionally solvable Natanzon potential is thus given by

$$G^{(V_4)}(x'',x';E)=\frac{2m}{\hbar^2}\left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4}\frac{\Gamma(m_1-L_\eta)\Gamma(L_\eta+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}$$

$$\times\left[\frac{1}{4}\left(1+\frac{1}{\sqrt{1-z(r')}}\right)\left(1+\frac{1}{\sqrt{1-z(r'')}}\right)\right]^{-(m_1-m_2)/2}$$

$$\times\left(\frac{\sqrt{z(r')}}{1+\sqrt{1-z(r')}}\frac{\sqrt{z(r'')}}{1+\sqrt{1-z(r'')}}\right)^{m_1+m_2+1/2}$$

$$\times\,{}_2F_1\left(-L_\eta+m_1,L_\eta+m_1+1;m_1-m_2+1;2\left(1+\frac{1}{\sqrt{1-z_<(r)}}\right)^{-1}\right)$$

$$\times\,{}_2F_1\left(-L_\eta+m_1,L_\eta+m_1+1;m_1+m_2+1;\frac{z_>(r)}{\left(1+\sqrt{1-z_>(r)}\right)^2}\right). \tag{51}$$

Here denote

$$m_{1,2}=\frac{1}{2}\sqrt{h_0+h_1+1-\frac{2m}{\hbar^2}c_0E}\pm\sqrt{1-\frac{2m}{\hbar^2}b_0E} \tag{52}$$

$$L_\eta=\frac{1}{2}\left(\sqrt{h_0-h_1+1-\frac{2m}{\hbar^2}c_0E}-1\right). \tag{53}$$

With the abbreviations $B_1 = \frac{\hbar^2}{2m}(h_0+1-h_1)$, $B_2 = \frac{\hbar^2}{2m}(h_0+1+h_1)$, $C_1 = 2\hbar^2/m$, $\alpha_1 = 4b_0$ we get the following quantization condition ($n \in \mathbb{N}$):

$$\sqrt{C_1 - \alpha_1 E_n} + 2\tilde{n} = \sqrt{B_1 - c_0 E_n} - \sqrt{B_2 - c_0 E_n}\,. \tag{54}$$

This equation is again an equation of fourth degree in E_n. Equation (54) is an obvious generalization of the corresponding case of [1], where the eigenvalue equation was a cubic equation, and can be solved in a similar way as the previous case.

2.5. A modified Manning–Rosen potential: I

In the fifth class of potentials we want to investigate, we modify the hypergeometric Natanzon potential according to

$$V_5(r) = \frac{\hbar^2}{2m}\frac{fz(z-1)+\frac{3}{4}(1-z)+h_1 z^{3/2}}{R(z)} + \frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2 - 2\frac{z'''}{z'}\right) \tag{55}$$

where $R(z) = a_0 z^2 + z b_0$, and $z = z(r)$ is again implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. If we make the ansatz $R = z^2$, we get $z = 1 - \mathrm{e}^{-2r}$, $r > 0$. In the original Natanzon potential we obtain the Manning–Rosen potential. In the present case we get

$$V(r) = \frac{\hbar^2}{2m}\left(f + 1 - \frac{f-\frac{3}{4}}{1-\mathrm{e}^{-2r}} + \frac{h_1}{\sqrt{1-\mathrm{e}^{-2r}}} - \frac{3}{4(1-\mathrm{e}^{-2r})^2}\right). \tag{56}$$

Due to its singular structure this potential may be called a modified Manning–Rosen potential. Such a 'conditionally solvable potential' seems to be new. We perform the same space–time transformations as in the previous two cases and obtain

$$\begin{aligned} K^{(V_5)}(r'', r'; T) &= \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - V_5(r)\right]\mathrm{d}t\right\} \\ &= \left(\frac{\sqrt{R(r')R(r'')}}{4z(r')z(r'')(1-z(r'))(1-z(r''))}\right)^{1/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s'' \\ &\quad\times\int_{z(0)=z'}^{z(s'')=z''}\mathcal{D}z(s)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{z}^2 + \frac{ER(z)}{4z^2(1-z)^2}\right.\right. \\ &\quad\left.\left. -\frac{\hbar^2}{8m}\left(\frac{f}{z(1-z)} + \frac{h_1}{z^{1/2}(1-z)^2} + \frac{3}{4z^2(1-z)}\right)\right]\mathrm{d}s\right\} \\ &= \left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar} \\ &\quad\times\int_0^\infty \mathrm{d}s''\ \exp\left(\mathrm{i}[(a_0+b_0)E - \hbar^2/2m)s''/\hbar\right) \end{aligned} \tag{57}$$

$$\times \int_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{x}^2+\frac{\hbar^2}{2m}\left(\frac{\nu^2+\frac{1}{4}}{\cosh^2 x}-h_1\tanh x\right)\right]\mathrm{d}s\right\} \tag{58}$$

$$= \int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G^{(V_5)}(r'',r';E)\,. \tag{59}$$

Equation (58) corresponds to the path integral of the Rosen–Morse potential as discussed in [26, 47]. However, we encounter in the present case the same difficulty as in the discussion of the two confluent conditionally solvable Natanzon potentials. The transformation $z=z(x)$ maps $(0,1)\to\mathbb{R}^+$, and (58) is therefore a radial path integral. But the Rosen–Morse potential is defined in the entire $\mathbb{R}$ and due to the odd $\tanh x$-term we cannot continue to the entire real line. Hence, we must apply the same technique as in the first two examples and therefore the Green function $G^{(V_5)}(E)$ of the fifth conditionally solvable Natanzon potential is given by

$$G^{(V_5)}(E)(r'',r';E)=G(r'',r';E)-\frac{G\big(r'',z(0);E\big)G\big(z(0),r';E\big)}{G\big(z(0),z(0);E\big)} \tag{60}$$

with the Green function $G(E)$ given by

$$\begin{aligned}G(r'',r';E)=\frac{m}{\hbar^2}&\left(\frac{\sqrt{R(r')R(r'')}}{z(r')z(r'')}\right)^{1/2}\frac{\Gamma(m_1-L_B)\Gamma(L_B+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)}\\
&\times\left(\frac{1-\sqrt{z(r')}}{2}\frac{1-\sqrt{z(r'')}}{2}\right)^{(m_1-m_2)/2}\\
&\times\left(\frac{1+\sqrt{z(r')}}{2}\frac{1+\sqrt{z(r'')}}{2}\right)^{(m_1+m_2)/2}\\
&\times{}_2F_1\left(-L_B+M_1,L_B+m_1+1;m_1+m_2+1;\frac{1+\sqrt{z_>(r)}}{2}\right)\\
&\times{}_2F_1\left(-L_B+M_1,L_B+m_1+1;m_1-m_2+1;\frac{1-\sqrt{z_<(r)}}{2}\right).\end{aligned} \tag{61}$$

Here I have used the shorthand

$$L_B=\sqrt{f+1-\frac{2m}{\hbar^2}a_0E}-\frac{1}{2} \tag{62}$$

$$m_{1,2}=\frac{1}{2}\sqrt{-(h_1+1)-\frac{2m}{\hbar^2}(a_0+b_0)E}\pm\frac{1}{2}\sqrt{h_1-1-\frac{2m}{\hbar^2}(a_0+b_0)E} \tag{63}$$

and have further set

$$\nu^2=f+1-\frac{2ma_0E}{\hbar^2}\,. \tag{64}$$

Note that the number ν is a square root and the specific sign it takes may vary in different examples. We obtain consequently the transcendental quantization condition for the bound-state energy levels E_n

$${}_2F_1\Big(-L_B(E_n)+m_1(E_n),L_B(E_n)+m_1(E_n)+1;m_1(E_n)+m_2(E_n)+1;\tfrac{1}{2}\Big)\,. \tag{65}$$

In the case of $R = z^2$, i.e. for the potential

$$V(r) = \frac{\hbar^2}{2m}\left(\frac{B}{1-\mathrm{e}^{-2r}} - \frac{A}{\sqrt{1-\mathrm{e}^{-2r}}} - \frac{3}{4(1-\mathrm{e}^{-2r})^2}\right) \tag{66}$$

we obtain the quantization condition

$${}_2F_1\Big(-\tilde{L}_B(E_n)+\tilde{m}_1(E_n), \tilde{L}_B(E_n)+\tilde{m}_1(E_n)+1; \tilde{m}_1(E_n)+\tilde{m}_2(E_n)+1; \tfrac{1}{2}\Big) \tag{67}$$

where we have set

$$\tilde{L}_B(E) = \sqrt{\frac{1}{4} - \frac{2m}{\hbar^2}E} - \frac{1}{2} \tag{68}$$

$$\tilde{m}_{1,2}(E) = \frac{1}{2}\sqrt{B + A - \frac{3}{4} - \frac{2m}{\hbar^2}E} \pm \frac{1}{2}\sqrt{B - A - \frac{3}{4} - \frac{2m}{\hbar^2}E}\,. \tag{69}$$

2.6. A modified Manning–Rosen potential: II

In the sixth class of potentials we want to investigate, we modify the hypergeometric Natanzon potential according to

$$V_6(r) = \frac{\hbar^2}{2m}\frac{fz(z-1) + \frac{3}{4}(1-z) + h_1 z^{3/2}\sqrt{1-z}}{R(z)} + \frac{\hbar^2}{8m}\left(3\left(\frac{z''}{z'}\right)^2 - 2\frac{z'''}{z'}\right) \tag{70}$$

where $R(z) = a_0 z^2 + z b_0$, and $z = z(r)$ is again implicitly defined by the differential equation $z' = 2z(1-z)/\sqrt{R(z)}$. Repeating the analysis as in V_5 we obtain an additional factor e^{-r} in the h_1-term, therefore producing another modified Manning–Rosen potential, i.e.

$$V(r) = \frac{\hbar^2}{2m}\left(f + 1 - \frac{f - \frac{3}{4}}{1-\mathrm{e}^{-2r}} + \frac{h_1 \mathrm{e}^{-r}}{\sqrt{1-\mathrm{e}^{-2r}}} - \frac{3}{4(1-\mathrm{e}^{-2r})^2}\right). \tag{71}$$

We perform the same space–time transformations as in the previous cases and obtain with ν as in (64)

$$\begin{aligned} K^{(V_6)}(r'', r'; T) &= \int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{r}^2 - V_6(r)\right]\mathrm{d}t\right\} \\ &= \left(\frac{\sqrt{R(r')R(r'')}}{4z(r')z(r'')(1-z(r'))(1-z(r''))}\right)^{1/2}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar}\int_0^\infty \mathrm{d}s'' \\ &\quad\times\int_{z(0)=z'}^{z(s'')=z''}\mathcal{D}z(s)\ \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{z}^2 + \frac{ER(z)}{4z^2(1-z)^2}\right.\right. \\ &\quad\left.\left. - \frac{\hbar^2}{8m}\left(\frac{f}{z(1-z)} + \frac{h_1}{z^{1/2}(1-z)^2} + \frac{3}{4z^2(1-z)}\right)\right]\mathrm{d}s\right\} \\ &= \left(\frac{R(r')R(r'')}{z(r')z(r'')}\right)^{1/4}\int_{\mathbb{R}}\frac{\mathrm{d}E}{2\pi\hbar}\,\mathrm{e}^{-\mathrm{i}ET/\hbar} \end{aligned} \tag{72}$$

$$\times \int_0^\infty \mathrm{d}s''\ \exp\left(\mathrm{i}[b_0E + \hbar^2(f-\tfrac{1}{4})/2m)s''/\hbar\right) \int_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s)$$

$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left((\nu^2-\tfrac{1}{4})\tanh^2 x + h_1\frac{\tanh x}{\cosh x}\right)\right]\mathrm{d}s\right\} \tag{73}$$

$$= \int_{\mathbb{R}} \frac{\mathrm{d}E}{2\pi\,\mathrm{i}}\mathrm{e}^{-\mathrm{i}ET/\hbar}G^{(V_6)}(r'', r'; E)\,. \tag{74}$$

The path integral (73) is the path integral for the hyperbolic barrier potential as discussed in [48], however in the half-space $\mathbb{R}^+$. Applying the same method as for $G^{(V_5)}(E)$, the Green function $G^{(V_6)}(E)$ of the sixth conditionally solvable Natanzon potential is therefore given by

$$G^{(V_6)}(E)(r'', r'; E) = G(r'', r'; E) - \frac{G\big(r'', z(0); E\big)G\big(z(0), r'; E\big)}{G\big(z(0), z(0); E\big)} \tag{75}$$

with the Green function $G(E)$ given by

$$\begin{aligned} G(r'', r'; E) = {} & \frac{m}{\hbar^2}\left(\frac{\sqrt{R(r')R(r'')}}{z(r')z(r'')}\right)^{1/2}\frac{\Gamma(m_1-L_\nu)\Gamma(L_\nu+m_1+1)}{\Gamma(m_1+m_2+1)\Gamma(m_1-m_2+1)} \\ & \times\left(\frac{1-\sqrt{z(r')}}{2}\frac{1-\sqrt{z(r'')}}{2}\right)^{(m_1-m_2)/2} \\ & \times\left(\frac{1+\sqrt{z(r')}}{2}\frac{1+\sqrt{z(r'')}}{2}\right)^{(m_1+m_2+1/2)/2} \\ & \times {}_2F_1\left(-L_\nu+M_1, L_\nu+m_1+1; m_1+m_2+1; \frac{1+\sqrt{z_>(r)}}{2}\right) \\ & \times {}_2F_1\left(-L_\nu+M_1, L_\nu+m_1+1; m_1-m_2+1; \frac{1-\sqrt{z_<(r)}}{2}\right). \end{aligned} \tag{76}$$

Here I have used the abbreviations

$$L_\nu = \frac{1}{2}\left(\sqrt{f+1+\mathrm{i}h_1 - \frac{2m}{\hbar^2}a_0E} - 1\right) \tag{77}$$

$$m_{1,2} = -\frac{1}{2}\sqrt{f+1-\mathrm{i}h_1 - \frac{2m}{\hbar^2}a_0E} \pm \sqrt{\frac{1}{4} - f - \frac{2m}{\hbar}b_0E}\,. \tag{78}$$

Note that the minus sign in the first term in $m_{1,2}$ is due to the reality condition of the problem, cf [48]. Bound states with energy E_n are determined by the equation

$${}_2F_1\Big(-L_\nu(E_n)+m_1(E_n), L_\nu(E_n)+m_1(E_n)+1; m_1(E_n)+m_2(E_n)+1; \tfrac{1}{2}\Big)\,. \tag{79}$$

A more detailed investigation, and investigation of the special case $R = z^2$, is left to the reader.

3. Summary and discussion

In this paper the path integral treatments of six classes of 'conditionally solvable Natanzon' potentials have been presented. For the analysis of these problems I have used the path integral approach. In spite of the fact that the potentials are defined only implicitly and may not even be known analytically, it has been possible to calculate the Green function in terms of the variable r, and the functions $R(r)$ and $h(r)$ and $z(r)$, respectively. The poles of the Green functions have given the bound-state energy levels, i.e. we have obtained transcendental equations in terms of parabolic cylinder functions, hypergeometric functions, and equations of fourth degree, respectively, and the cuts have provided the scattering states.

The two 'conditionally solvable confluent Natanzon potentials' (5) and (16) are but two simple solutions of the potential classes V_1 and V_2, respectively, where $\sigma_2 = 0$. The general structure of the potential V_1 modifies the Coulomb interaction by adding a long-range effect. V_2 instead has a confinement character. The special case σ_1 gives trivial results, and there do not seem to be any other simple potentials.

It remains for us to consider not only the real solutions of the eigenvalue equations, but also the other possible complex solutions as well. For instance, in (27) we have found an asymptotic behaviour of the bound-state levels which is in accordance with the result of [1]. There only real solutions were allowed because the zeros of the Airy function are known to be real. By a suitable choice of parameters all solutions of (27) need not to be real. The existence of complex solutions, i.e. resonances, depends on how strong a potential barrier is above the energy of the lowest lying scattering states. It is known that (3) can have such resonance states [37], and this feature of locating resonances in scattering processes was one of the reasons to study potentials like (3). Therefore it should be worthwhile to carry out a numerical investigation and a detailed Green function analysis, respectively, along these lines. This would fix the requirements as to which of the three solutions of the cubic equation (27) or of the four solutions of (43), (54) actually contribute to the spectrum, and which are unphysical and must be discarded. In particular, resonance states would add to the variety of potential forces in nuclear physics and elementary particle physics for states which are subject to decay.

On the other hand, the potential (14) has a confinement character, and its power-dependence describes phenomenological a $q\bar{q}$ interaction, as supported by lattice QCD, cf [52, 53] and references therein. Therefore both potentials can serve as a proper refinement of important phenomenological models due to their four-parameter structure.

In [54] de Souza Dutra and Girlich have presented two-dimensional extensions of the potentials (5) and (16) according to $(\omega, \lambda > 0)$

$$V_i(x, y) = \frac{m}{2}\omega^2(x^2 + y^2) + \frac{\hbar^2}{2m}\frac{1}{x^2 + y^2}\left[\lambda^2 + V_i\Big(\arctan(y/x)\Big) - \frac{1}{4}\right] \tag{80}$$

$i = 1, 2$ and $V_1(r)$ the potential (5) and $V_2(r)$ the potential (16). With the coordinate choice $x = u\cos\gamma$, $y = u\sin\gamma$ the problem is separable. However, the same line of reasoning as sketched in subsection 2.1 concerning the proper boundary conditions applies as well. Therefore the *transcendental* equations for the eigenvalues $E_{i,n}^{(\gamma)}$ (13), (22), now for the variable $\gamma > 0$, determine the separation constant which enters into (80). This eventually yields a radial harmonic oscillator path integral with

$$V_i(u) = \frac{m}{2}\omega^2 u^2 + \frac{\hbar^2}{2m}\frac{\lambda^2 + E_{i,n}^{(\gamma)} - \frac{1}{4}}{u^2} \tag{81}$$

which is easy to solve, e.g. [21, 23, 27, 40].

In the second set of *hypergeometric conditionally solvable Natanzon potentials* the bound-state solutions are determined by an equation of fourth degree which complicates the expressions analytically. In both cases the bound-state energy levels with the wavefunctions and the scattering solutions can be obtained in principle. However, due to the complicated structure of the equations, and the fact that a more simple case has already been discussed in an earlier paper [1], an explicit evaluation is omitted. The six path integral representations (8), (19), (35), (49), (58), (73) together with the explicit form of the corresponding Green functions (11), (20), (37), (51), (60), (75) therefore contain the *former* path integral solutions as special cases, and at the same time generalize them.

In the set of potentials (31) and (48) the choice $R(z) \propto z^2$ is explicitly excluded, so one can either set $b_0 = 0$ or $c_0 = 0$. Choosing $b_0 = 0$ yields $z = \frac{1}{2}(1 + \tanh x)$, $x \in \mathbb{R}$, leading to the two modified Rosen–Morse potentials of [1]. The structure of the two potentials is such that they can describe a complicated scattering process by wells or troughs on the real line. The alternative $c_0 = 0$ gives $z = \coth^2 r$, $r > 0$, and leads to the Rosen–Morse and hyperbolic barrier potentials of [48, 51], respectively, and therefore does not give rise to any new features.

In order to include in the hypergeometric case $R \propto z^2$, where $z = 1 - \mathrm{e}^{-2r}$, $r > 0$, which gives in the original Natanzon potential the Manning–Rosen potential, I have modified the structure of V_{h} once more. I have obtained two new potentials which I have called due to their singular behaviour 'modified Manning–Rosen' potentials. The effect of the potentials can be interpreted as modifying a Coulomb interaction in a space of constant curvature, e.g. [55, 56] and references therein. Alternatively, they combine the effect of screening a Coulomb potential with a long-range behaviour according to (5). In both cases the quantization conditions are transcendental equations involving the hypergeometric function.

I have therefore found four classes of 'conditionally solvable hypergeometric Natanzon potentials' with either complicated scattering properties on the real line, or modified screening features for singular potentials. In particular, the potentials (56), (66), (71) provide explicit solutions in terms of the radial variable r which appear to be new.

The particular features of the 'conditionally solvable Natanzon potential' also clarify the origin of their solubility. The term which is proportional to $\hbar^2$ guarantees this very fact. Whereas in [2–5] this term seems to come in 'by hand' and has to be chosen in a suitable way, its structure is fixed because it *is* a Schwarzian derivative. The incorporation of this term in the potential therefore arises in a natural way, in a manner similar to the way the Schwarzian derivative appears in the space–time transformation of path integrals, and the quantum potential ΔV in the definition of path integrals on curved manifolds.

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