The SM prediction of $g-2$ of the muon

Kaoru Hagiwara[a], Alan D. Martin[b],
Daisuke Nomura[b], and Thomas Teubner[c]

[a] Theory Group, KEK, Tsukuba, Ibaraki 305-0801, Japan
[b] Department of Physics and Institute for Particle Physics Phenomenology
University of Durham, Durham DH1 3LE, U.K.
[c] Theory Division, CERN, CH-1211 Geneva 23, Switzerland

Abstract

We calculate $(g-2)/2$ of the muon, paying particular attention to the hadronic vacuum polarisation contribution and its uncertainties. The different data sets for each e^+e^- exclusive channel (as well as for the inclusive $e^+e^- \to hadrons$ channel) have been combined in order to produce the optimum estimate of the cross sections and their uncertainties. QCD sum rules are evaluated in order to resolve an apparent discrepancy between the inclusive data and the sum of the exclusive channels. We conclude $a_\mu^{\rm had,LO} = (683.1 \pm 5.9_{\rm exp} \pm 2.0_{\rm rad}) \times 10^{-10}$ which, when combined with the other contributions to $(g-2)/2$, is about 3σ below the present world average measurement.

1 Introduction

The muon anomalous magnetic moment, $a_\mu \equiv (g_\mu - 2)/2$, is one of the most precisely measured quantities in particle physics. The world average of the existing measurements is

$$a_\mu^{\rm exp} = 11659203(8) \times 10^{-10}, \tag{1}$$

which is dominated by the recent value obtained by the E821 collaboration at BNL[1]. To test the Standard Model (SM) at the quantum level and to constrain possible New Physics it is therefore important to evaluate the SM prediction of a_μ as accurately as possible. It may be written as the sum of three terms, $a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm EW} + a_\mu^{\rm had}$. The QED contribution has been calculated up to and including estimates of the 5-loop contribution, see reviews [2, 3], and reads $a_\mu^{\rm QED} = 116\ 584\ 705.6(2.9) \times 10^{-11}$. In comparison with the experimental error in eq. (1), and with the hadronic contribution error discussed later, the uncertainty in $a_\mu^{\rm QED}$ is much less important than other sources of uncertainty. The electroweak contribution is calculated through second order to be [4, 5, 6]: $a_\mu^{\rm EW} = 152(1) \times 10^{-11}$. Here again the error is negligibly small. Less accurately known is the hadronic contribution. It can be divided into three pieces, $a_\mu^{\rm had} = a_\mu^{\rm had,LO} + a_\mu^{\rm had,NLO} + a_\mu^{\rm had,l\text{-}b\text{-}l}$. The lowest-order (vacuum polarisation) hadronic contribution, $a_\mu^{\rm had,LO}$, has been calculated by a number of groups. The value $a_\mu^{\rm had,LO} = 6\ 924(62) \times 10^{-11}$ taken from Ref. [7] has been frequently used in making comparisons with the data. The next-to-leading order hadronic contribution, $a_\mu^{\rm had,NLO}$, is evaluated to be [8, 9] $a_\mu^{\rm had,NLO} = -100(6) \times 10^{-11}$. The hadronic light-by-light scattering contribution $a_\mu^{\rm had,l\text{-}b\text{-}l}$ has been recently reevaluated [10]–[15], and it is found to be $a_\mu^{\rm had,l\text{-}b\text{-}l} = 80(40) \times 10^{-11}$, where we quote the estimate of the full hadronic light-by-light contributions given in Ref. [16]. We can see that $a_\mu^{\rm had,LO}$ has the largest uncertainty, although the uncertainty in the light-by-light contribution $a_\mu^{\rm had,l\text{-}b\text{-}l}$ is also large.

In the following we discuss an update of $a_\mu^{\rm had,LO}$, which is given by the dispersion relation

$$a_\mu^{\rm had,LO} = \frac{1}{4\pi^3} \int_{s_{\rm th}}^{\infty} ds\ \sigma_{\rm had}(s)\, \frac{m_\mu^2}{3s}\, K(s)\,, \tag{2}$$

where $\sigma_{\rm had}(s)$ is the total cross section for $e^+e^- \to {\rm hadrons}\,(+\gamma)$ at centre-of-mass energy $\sqrt{s}$, and the well known kernel function K (see e.g. [17]) increases monotonically from 0.63 to 1 in the range $4m_\pi^2 < s < \infty$. To evaluate $\sigma_{\rm had}(s)$, we use experimental data up to 11.09 GeV and perturbative QCD thereafter. The most important contribution comes from the $e^+e^- \to \pi^+\pi^-$ channel; the channels $\pi^+\pi^-\pi^0$, K^+K^-, $K_S^0K_L^0$, $\pi^+\pi^-\pi^+\pi^-$, $\pi^+\pi^-\pi^0\pi^0$, etc. give subleading contributions.

The special features of our analysis are (i) that it is data-driven, based on all available data, including the new data on exclusive channels from Novosibirsk, particularly $\pi^+\pi^-$ [18], and the BES data on the R ratio [19], (ii) the careful application of a clustering method, so that data of differing precision can be combined consistently, and (iii) the use of QCD sum rules to resolve an apparent discrepancy between the inclusive and exclusive determination of $\sigma_{\rm had}(s)$ for $1.4 \lesssim \sqrt{s} \lesssim 2$ GeV.

We chose not to use data on hadronic τ decays to further constrain the $e^+e^- \to 2\pi, 4\pi$ channels for $\sqrt{s} \lesssim m_\tau$, because of the possible uncertainties connected with isospin-breaking effects. A careful study of the effects of including τ data has been made recently in Ref. [20].

2 Processing the hadronic data

We applied the hadronic vacuum polarisation corrections as given by Swartz [21] and calculated the final state radiative effects for all $e^+e^- \to \pi^+\pi^-$ data, except for the new (already corrected) CMD–2 data [18], based on eq. (45) of Ref. [22]. These two corrections increase the $\pi^+\pi^-$ contribution by about 1.1×10^{-10}, to which we assign a 50% error. For data where only insufficient information is available to make reliable radiative corrections we assigned an additional $\pm 1\%$ uncertainty to their contribution to $a_\mu^{\rm had,LO}$. The net effect is an error of about $\pm 2 \times 10^{-10}$ due to radiative corrections; a further discussion will be given in [23].

We now come to the important problem of 'clustering' data from different experiments (for the same hadronic channel). To combine all data points for the same channel which fall in suitably chosen (narrow) energy bins, we determine the mean R values and their errors for all clusters by minimising the non-linear χ^2 function $\chi^2(R_m, f_k) = \sum_{k=1}^{N_{\rm exp}} \left[(1-f_k)/{\rm d}f_k\right]^2 + \sum_{m=1}^{N_{\rm clust}} \sum_{i=1}^{N_{\{k,m\}}} \left[\left(R_i^{\{k,m\}} - f_k R_m\right)/{\rm d}R_i^{\{k,m\}}\right]^2$. Here R_m and f_k are the fit parameters for the mean R value of the $m^{\rm th}$ cluster and the overall normalization factor of the $k^{\rm th}$ experiment, respectively. $R_i^{\{k,m\}}$ and ${\rm d}R_i^{\{k,m\}}$ are the R values and errors

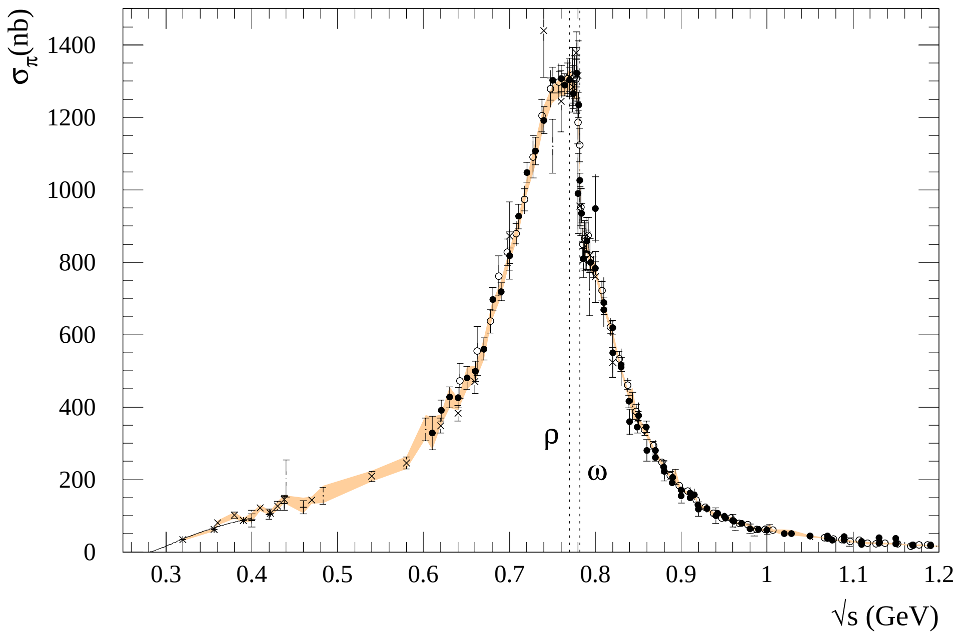

Figure 1: $e^+e^- \to \pi^+\pi^-$ data up to 1.2 GeV, where the shaded band shows the result of clustering.

from experiment k contributing to cluster m. For $\mathrm{d}R_i^{\{k,m\}}$ the statistical and point-to-point systematic errors are added in quadrature, whereas $\mathrm{d}f_k$ is the overall systematic error of the k^{th} experiment. Minimization of χ^2 with respect to the $(N_{\mathrm{exp}} + N_{\mathrm{clust}})$ parameters, f_k and R_m, gives our best estimates for these parameters together with their error correlations.

Our definition of χ^2 implies piecewise constant R values but imposes no further constraints on the form of the hadronic cross section ('minimum bias'). Due to use of the overall normalization factors it also results in an adjustment of the different sets within their systematic uncertainties. This means, for example, that sparse but precise data will dominate the normalization, but that the information on the shape of R from sets with larger systematic uncertainties is preserved.

The error estimate for each hadronic channel is then done using the complete covariance matrix returned by our χ^2 minimization. Therefore statistical and systematic (point-to-point as well as overall) errors from the different sets are taken into account including correlations between different energies (clusters). The dispersion integral (2) is performed integrating (using the trapezoidal rule) over the clustered data directly for all hadronic channels, including the ω and ϕ resonances. Thus we avoid possible problems due to missing or double-counting of non-resonant backgrounds, and interference effects are taken into account automatically. As an example we display in Fig. 1 the most important $\pi^+\pi^-$ channel.

In the region between 1.43 and ~ 2 GeV we have the choice between summing up the exclusive channels or relying on the inclusive measurements. Surprisingly, the sum of the exclusive channels overshoots the inclusive data, even after having corrected the latter for missing two-body and (some) purely neutral modes. The discrepancy is shown in Fig. 2,

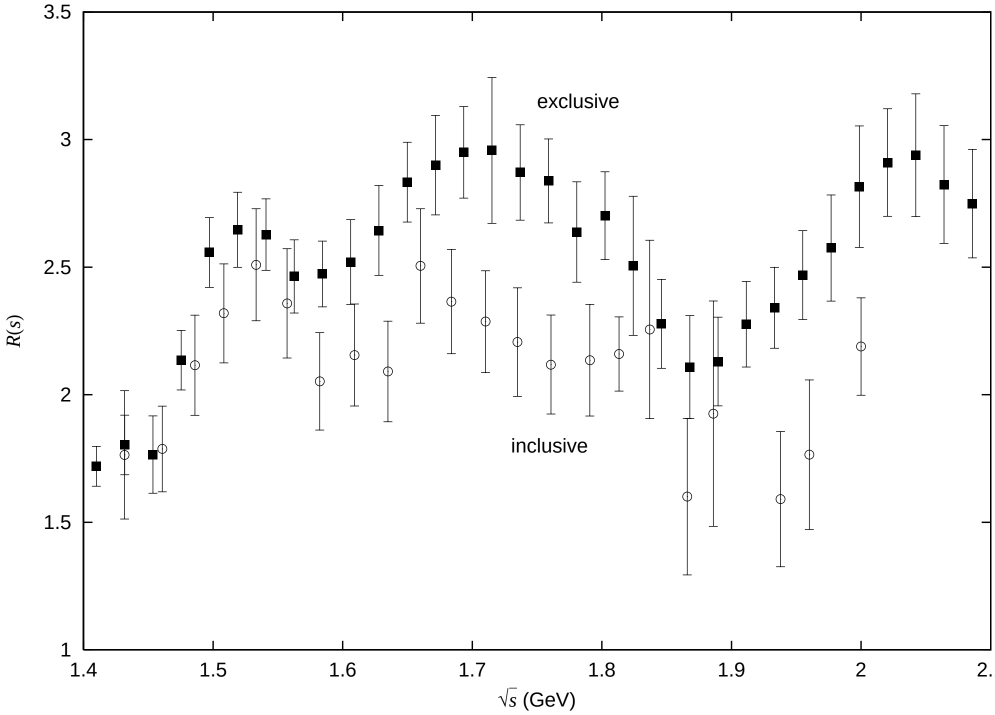

Figure 2: The inclusive and the sum of exclusive channel values of R, *after* the data have been clustered.

where we display data points with errors after application of our clustering algorithm.

3 Results

In Table 1 we list the contributions to $a_\mu^{\mathrm{had,LO}}$ from different energy regimes: From the two-pion threshold up to 0.32 GeV, chiral perturbation theory is applied and for the high energy tail above 11.09 GeV, R is calculated using perturbative QCD. The J/ψ, $\psi(2S)$ and $\Upsilon(1S-6S)$ resonance contributions are evaluated in narrow-width-approximation. Apart from those contributions we use the direct integration of clustered data as described above. For the controversial region from 1.43 to 2 GeV we present two results: if we use the lower lying inclusive data the corresponding contribution is considerably smaller than the one resulting from the use of the sum over the exclusive channels.

4 Resolution of the ambiguity: QCD sum rules

To resolve the ambiguity between the exclusive and inclusive data, we evaluate QCD sum rules of the form $\int_{s_{\mathrm{th}}}^{s_0} \mathrm{d}s\; R(s) f(s) = \int_C \mathrm{d}s\; D(s) g(s)$. $\sqrt{s_0} = 3.7$ GeV is chosen to be just below the open charm threshold and C is a circular contour of radius s_0. $D(s)$ is the Adler D function, $D(s) \equiv -12\pi^2 s\, \mathrm{d}\left(\Pi(s)/s\right)/\mathrm{d}s$, where $R(s) = 12\pi\, \mathrm{Im}\, \Pi(s)/s$. Experimental data for $R(s)$ are used to evaluate the left-hand-side, while QCD is used to determine $D(s)$. We take $f(s)$ to be of the form $(1-s/s_0)^n (s/s_0)^m$, with $n+m = 0, 1$ or 2. Once

Table 1: Contributions to $a_\mu^{\rm had,LO}$ from different energy intervals. The second line and the total also include a small 0.13×10^{-10} contribution from the $\pi^0\gamma$ channel near its threshold.

energy range (GeV)	comments	$a_\mu^{\rm had,LO} \times 10^{10}$
$2m_\pi \ldots 0.32$	chiral PT	2.30 ± 0.05
$0.32 \ldots 1.43$	excl. only	596.73 ± 5.18
$1.43 \ldots 2.00$	excl. only	38.14 ± 1.72
	incl. only	32.43 ± 2.46
$2.00 \ldots 11.09$	incl. only	42.09 ± 1.25
J/ψ and $\psi(2S)$	nar. width	7.31 ± 0.43
$\Upsilon(1S-6S)$	nar. width	0.10 ± 0.00
$11.09 \ldots \infty$	pQCD	2.14 ± 0.01
$\sum$ of all	ex-ex-in	688.81 ± 6.17
	ex-in-in	683.11 ± 5.89

$f(s)$ is chosen, the functional form of $g(s)$ is readily evaluated. The sum rules with $n = 1$ or 2 and $m = 0$ are found to maximize the fractional contribution of the left-hand-side coming from the relevant $1.43 < \sqrt{s} < 2$ GeV interval. The evaluations of these two sum rules are shown in Table 2. Consistency clearly selects the inclusive, as opposed to the exclusive, determination of $R(s)$.

Table 2: Sum rule results as discussed in the text. The main QCD error comes from $\alpha_S(M_Z^2) = 0.117 \pm 0.002$.

sum rule	l.h.s. (data)	r.h.s. (QCD)
$n = 1, m = 0$	15.34 ± 0.39 (incl) 15.99 ± 0.35 (excl)	15.34 ± 0.08
$n = 2, m = 0$	10.40 ± 0.25 (incl) 10.90 ± 0.22 (excl)	10.30 ± 0.06

5 Conclusions

We have undertaken a data-driven determination of the hadronic vacuum polarisation contribution to $a_\mu^{\rm had,LO}$, using all available e^+e^- data and a non-linear χ^2 approach to cluster data for the same channel in narrow bins. We found that there was a discrepancy between the inclusive value for $\sigma(e^+e^- \to \text{hadrons})$ and the sum of the exclusive channels in the region $1.4 \lesssim \sqrt{s} \lesssim 2$ GeV, which gave an uncertainty of about 6×10^{-10} in $a_\mu^{\rm had,LO}$. We used a QCD sum rule analysis to resolve the discrepancy in favour of the inclusive data. Thus finally we find that the SM predicts

$$a_\mu^{\mathrm{had,LO}} = (683.1 \pm 5.9_{\mathrm{exp}} \pm 2.0_{\mathrm{rad}}) \times 10^{-10}. \quad (3)$$

Summing up all SM contributions to a_μ^{SM} as given in section 1, but with $a_\mu^{\mathrm{had,LO}}$ from (3), we conclude that

$$a_\mu^{\mathrm{SM}} = (11659166.9 \pm 7.4) \times 10^{-10}, \quad (4)$$

which is 36.1×10^{-10} (3.3σ) below the world average experimental measurement. If, on the other hand, we were to take the value of $a_\mu^{\mathrm{had,LO}}$ obtained using the sum of the exclusive data in the interval $1.43 < \sqrt{s} < 2$ GeV then we would find $a_\mu^{\mathrm{SM}} = (11659172.6 \pm 7.7) \times 10^{-10}$, which is 30.4×10^{-10} (2.7σ) below a_μ^{exp}.

Our result (3) agrees also fairly well with a recent reevaluation of the leading hadronic contribution from F. Jegerlehner, who also used the recent CMD-2 data [18] and obtained $(688.9 \pm 5.8) \times 10^{-10}$, see [24]. Another independent SM prediction has very recently been made [20]. Their total final e^+e^--based result, $(684.7 \pm 6.0_{\mathrm{exp}} \pm 3.6_{\mathrm{rad}}) \times 10^{-10}$, is very similar to ours. A detailed comparison with these two predictions will be presented elsewhere [23].

For the future, we can expect further improvement in the accuracy of the experimental $(g-2)/2$ measurement. As far as the SM prediction is concerned, we anticipate new low energy data for a variety of e^+e^- channels, partly from *radiative return* at the ϕ- and B-factories.

Acknowledgements: We thank Simon Eidelman for numerous helpful discussions concerning the data.

References

[1] E821 Collaboration, G.W. Bennett et al., Phys. Rev. Lett. **89** (2002) 101804.

[2] V.W. Hughes and T. Kinoshita, Rev. Mod. Phys. **71** (1999) S133.

[3] A. Czarnecki and W.J. Marciano, Nucl. Phys. Proc. Suppl. **76** (1999) 245.

[4] A. Czarnecki, B. Krause, and W.J. Marciano, Phys. Rev. **D52** (1995) 2619; Phys. Rev. Lett. **76** (1996) 3267.

[5] S. Peris, M. Perrottet, and E. de Rafael, Phys. Lett. **B355** (1995) 523.

[6] M. Knecht, S. Peris, M. Perrottet, and E. de Rafael, `hep-ph/0205102`.

[7] M. Davier and A. Höcker, Phys. Lett. **B435** (1998) 427.

[8] B. Krause, Phys. Lett. **B390** (1997) 392.

[9] R. Alemany, M. Davier, and A. Höcker, Eur. Phys. J. **C2** (1998) 123.

[10] M. Knecht and A. Nyffeler, Phys. Rev. **D65** (2002) 073034.

[11] M. Knecht, A. Nyffeler, M. Perrottet, and E. de Rafael, Phys. Rev. Lett. **88** (2002) 071802.

[12] M. Hayakawa and T. Kinoshita, `hep-ph/0112102` and erratum Phys. Rev. **D66** (2002) 019902 (E).

[13] J. Bijnens, E. Pallante, and J. Prades, Nucl. Phys. **B626** (2002) 410.

[14] I. Blokland, A. Czarnecki, and K. Melnikov, Phys. Rev. Lett. **88** (2002) 071803.

[15] M. Ramsey-Musolf and M.B. Wise, Phys. Rev. Lett. **89** (2002) 041601.

[16] A. Nyffeler, `hep-ph/0203243`.

[17] K. Hagiwara, A.D. Martin, Daisuke Nomura, and T. Teubner, `hep-ph/0209187`.

[18] R.R. Akhmetshin et al., Phys. Lett. **B527** (2002) 161.

[19] BES Collaboration, J.Z. Bai et al., Phys. Rev. Lett. **88** (2002) 101802.

[20] M. Davier, S. Eidelman, A. Höcker, and Z. Zhang, `hep-ph/0208177`.

[21] M.L. Swartz, Phys. Rev. **D53** (1996) 5268.

[22] A. Hoefer, J. Gluza and F. Jegerlehner, Eur. Phys. J. **C24** (2002) 51.

[23] K. Hagiwara, A.D. Martin, D. Nomura, and T. Teubner, in preparation.

[24] F. Jegerlehner, talk at the workshop "Hadronic Contributions to the Anomalous Magnetic Moment of the Muon", Centre de Physique Théorique, Marseille, France, 14 – 16 March 2002.