# Non-renormalization theorems in supersymmetric field theories from extension to local coupling

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E.K., hep-th/0107239, NPB 620 (2002) 55;
E.K., hep-ph/0110323, PRD 65 (2002) 105003.

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2. Local coupling in Super-Yang-Mills theories
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4. The anomalous breaking of supersymmetry
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## 1. Introduction

Coupling constants $\longrightarrow$ H. Epstein, V. Glaser, 1975
N. Bogolubov, D. Shirkov 1969
space-time dependent external fields

String theory $\longrightarrow$ J. Louis, V. Kaplunovsky NPB 422, 57
N. Seiberg, PLB 318, 469
Non-renormalization theorems of susy from
holomorphicity of the effective action
$\longleftrightarrow$ Non-holomorphic dependence of the gauge $\beta$ function Shifman, Vainshtein NPB 277, 456

Rigorous construction of supersymmetric field theories with local coupling:

- Wess-Zumino model: R. Flume, E.K., NPB 569, 625; hep-th/9907120
  Non-renormalization theorems of chiral vertices
- SQED E.K., D. Stöckinger, NPB 626, 73; hep-th/0105028
  NRT of the gauge coupling, $l \geq 2$
- SYM E.K., NPB 620, 55; hep-th/0107239
  NRT of the gauge coupling, $l \geq 2$
  An anomalous breaking of supersymmetry

## 2. Local coupling in Super-Yang-Mills

Super-Yang-Mills theories with gauge group SU(N), supersymmetry in the Wess-Zumino-gauge with the vector multiplet $(A^\mu, \lambda^\alpha, \lambda^{\dot\alpha}, D)$ :

$$\Gamma_{\mathrm{SYM}} = \mathrm{Tr} \int d^4x \, \Big( -\frac{1}{4g^2} G^{\mu\nu}(gA) G_{\mu\nu}(gA) + i\lambda^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu \overline{\lambda}^{\dot\alpha} + \frac{1}{8} D^2 \Big) \ ,$$

Susy algebra of the Wess-Zumino gauge:

$$\{\delta_\alpha, \delta_{\dot\alpha}\} = 2i\sigma^\mu_{\alpha\dot\alpha}(\delta^T_\mu + \delta^{\mathrm{gauge}}_{gA^\mu})$$

$\Rightarrow$ Gauge invariant quantities are usual $N = 1$ super-multiplets

$L_{\mathrm{SYM}}$ is the $F$ component of a chiral multiplet,

$$\mathcal{L}_{\mathrm{SYM}} = -\frac{1}{2}\lambda\lambda + \theta^\alpha \Lambda_\alpha + \theta^2 L_{\mathrm{SYM}} \ ,$$

with

$$\begin{aligned} L_{\mathrm{SYM}} \ &= \ \mathrm{Tr}\Big( -\frac{1}{4} G^{\mu\nu}(gA) G_{\mu\nu}(gA) - \frac{i}{8} \widetilde{G}^{\mu\nu}(gA) G_{\mu\nu}(gA) \\ &\qquad + ig\lambda^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu (g\overline{\lambda}^{\dot\alpha}) + \frac{1}{8} g^2 D^2 \Big) \ . \end{aligned}$$

$\widetilde{G}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}$ is the dual fieldstrength tensor

Local gauge coupling $g(x)$

via a chiral and antichiral multiplet $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$:

$$\boldsymbol{\eta} = \eta + \chi\theta + f\theta^2 \ , \qquad \overline{\boldsymbol{\eta}} = \overline{\eta} + \overline{\chi}\overline{\theta} + \overline{f}\overline{\theta}^2 \ ,$$

and define

$$\begin{aligned} \Gamma_{\text{SYM}} &= \int dS \, \boldsymbol{\eta} \mathcal{L}_{\text{SYM}} + \int d\bar{S} \, \overline{\boldsymbol{\eta}} \bar{\mathcal{L}}_{\text{SYM}} \\ &= \int d^4x \left( \eta \, L_{\text{SYM}} - \frac{1}{2} \chi \Lambda - \frac{1}{2} g^2 f \lambda\lambda + \text{c.c.} \right) \ . \end{aligned}$$

E.K., NPB 620, hep-th/0107269
E.K., D. Stöckinger, NPB 626, hep-th/0105028

**Physical interpretation**:

- The real part of $\eta$ is related to the local coupling:

$$\eta + \overline{\eta} = \frac{1}{g^2(x)}$$

- The imaginary part of $\eta$ takes the role of a space-time dependent $\Theta$-angle:

$$\eta - \overline{\eta} = 2i\Theta(x)$$

$$\Gamma_{\mathrm{SYM}} = \int d^4x \left( -\frac{1}{4g^2} G^{\mu\nu} G_{\mu\nu} - \frac{i}{8} (\eta - \overline{\eta}) G^{\mu\nu} \widetilde{G}_{\mu\nu} + \ldots \right)$$

**Properties**:

- $g(x)$ is the loop expansion parameter:
  1PI diagrams of loop order l $\rightarrow$ powers $g^{2l}$

  The limit to constant coupling yields ordinary SYM-theories.

- $\Theta$ couples to a total derivative:

$$\begin{aligned}
\left( \frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}} \right) \Gamma &= [-\frac{i}{4} G^{\mu\nu} \widetilde{G}_{\mu\nu} + i \partial^\mu \lambda \sigma_\mu \overline{\lambda}] \cdot \Gamma \\
&= \partial^\mu [J_\mu^{top} + j_\mu^{axial}] \cdot \Gamma
\end{aligned}$$

  and therefore

$$\int d^4x \left( \frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}} \right) \Gamma = 0$$

$g^2$ and $\Theta$ are related by supersymmetry

## 3. Invariant counterterms

Invariant counterterms $\leftrightarrow$ superficial divergences

Symmetries:

Gauge symmetry, Supersymmetry,

Loop expansion in $g^2$

$$
\begin{aligned}
\Gamma^{(l)}_{\mathrm{ct}} = \; & z^{(l)}_{g^2} \int dS \, \boldsymbol{\eta}^{-l+1} \mathcal{L}_{\mathrm{SYM}} + \int d\bar{S} \, \bar{\boldsymbol{\eta}}^{-l+1} \bar{\mathcal{L}}_{\mathrm{SYM}} \\
\sim \; & z^{(l)}_{g^2} \int d^4x \, \big( g^{2(l-1)} G^{\mu\nu} G_{\mu\nu} \\
& \qquad + (l-1)\Theta \, g^{2l} \, G^{\mu\nu} \tilde{G}_{\mu\nu} + \cdots \big)
\end{aligned}
$$

$l \geq 2$

For $g = g(x)$: $\Theta$ does not couple to a total derivative

$\Longrightarrow \quad z^{(l)}_{g^2} = 0 \quad$ for $\quad l \geq 2$

The $\Theta$ angle is not renormalized and determines the renormalization of the coupling via supersymmetry.

Holomorphicity of the effective action is governed by the non-renormalization of the $\Theta$ angle:

$$\Gamma_{\text{eff}} = \int dS\ (\boldsymbol{\eta} + z^{(1)}_{g^2})\mathcal{L}_{\text{SYM}} + \int d\bar{S}\ (\overline{\boldsymbol{\eta}} + z^{(1)}_{g^2})\bar{\mathcal{L}}_{\text{SYM}}$$

$l = 1$
$z^{(1)}_{g^2}$ is arbitrary,
i.e. one-loop renormalization of the gauge coupling

The quantum corrections to the $\Theta$ angle make

an anomalous breaking of supersymmetry

Remark

The condition $\int d^4x\ \delta_\Theta \Gamma = 0$ yields also the non-renormalization of chiral vertices:

$$\Gamma_{\text{ct,chiral}} = z^{(l)} \int dS\ \boldsymbol{\eta}^{-l}\mathcal{L}_{\text{chiral}} + \text{c.}c.$$

$\Longrightarrow z^{(l)} = 0$ for $l \geq 1$

## 4. The anomalous breaking of supersymmetry

E.K, NPB 260, hep-th/0107239

Quantization in the Wess-Zumino gauge:
BRS-transformations $\mathbf{s}$ for
gauge $c$, supersymmetry $\epsilon^\alpha, \overline{\epsilon}^{\dot\alpha}$and translations $\omega^\nu$

$$\begin{aligned} \mathbf{s}\phi &= \delta_c^{\text{gauge}} + \epsilon^\alpha \delta_\alpha + \overline{\epsilon}^{\dot\alpha} \delta_{\dot\alpha} - i\omega^\mu \delta_\mu^T \ , \\ \mathbf{s}^2 \phi &= 0 \quad \text{algebra !} \end{aligned}$$

Classical action = susy + gauge fixing + ghost

$$\Gamma_{\text{SYM}} \rightarrow \Gamma_{\text{cl}} = \Gamma_{\text{SYM}} + \Gamma_{\text{g.f.}} + \Gamma_{\text{ghost}}$$

The symmetries are combined in

$$\mathcal{S}(\Gamma_{\text{cl}}) = 0 \quad \text{(Super-)Slavnov-Taylor identity}$$

P.L. White, Class. Qu. Grav. 9 (1992) 1663
N. Maggiore, O. Piguet, S. Wolf, NPB 458 (1996) 365

**Renormalization**:

Establish the Slavnov-Taylor-identity to all orders.

$$\mathcal{S}(\Gamma) = \Delta_{\text{brs}} + \mathcal{O}(\hbar^2)$$

Anomalies: $\Delta_{\text{brs}}$ cannot be absorbed into counter-terms to the classical action

The anomalous breaking is the variation of a counterterm depending on the logarithm of the coupling

$$
\begin{aligned}
\Delta^{\text{anom}}_{\text{susy}} &= \mathrm{s}\int d^4x \; \ln g(x)(L_{\text{SYM}} + \bar{L}_{\text{SYM}}) \\
&= (\epsilon^\alpha \delta_\alpha + \bar{\epsilon}^{\dot\alpha}\bar{\delta}_{\dot\alpha}) \int d^4x \; \ln g(x)(L_{\text{SYM}} + \bar{L}_{\text{SYM}}) \\
&= \int d^4x \, \Big( i \; \ln g(x) \partial_\mu \big( \Lambda^\alpha \sigma^\mu_{\alpha\dot\alpha} \bar{\epsilon}^{\dot\alpha} - \epsilon^\alpha \sigma^\mu_{\alpha\dot\alpha} \bar{\Lambda}^{\dot\alpha} \big) \\
&\qquad - \frac{1}{2} g^2(x)(\epsilon\chi + \overline{\chi\epsilon})(L_{\text{SYM}} + \bar{L}_{\text{SYM}}) \Big) \; .
\end{aligned}
$$

- is independent of $\ln g$ for constant coupling and any test with respect to the coupling ,
- satisfies the topological formula in $l = 1$ .

Loop diagrams are power series in the coupling:
$\Longrightarrow$ $\Delta^{\text{anom}}_{\text{susy}}$ is an anomaly, i.e.,

- Its coefficient is scheme- and gauge-independent.
- It is not introduced in the procedure of UV subtractions, but
- is determined by convergent one-loop integrals.

Supersymmetry is anomalous for local gauge coupling:

$$\mathcal{S}(\Gamma) = r_\eta^{(1)} \Delta^{\text{anom}}_{\text{susy}} + \mathcal{O}(\hbar^2)$$

Explicitly one derives E.K., PRD 65, hep-ph/0110323

$$g^2 r_\eta^{(1)} = -\frac{1}{2}\Sigma^{(1)}_{\eta-\overline{\eta}}(p_1, -p_1)\Big|_{\xi=1} \ ,$$

where

$$\begin{aligned}&\left(\left[i \ \mathrm{Tr}\Big(\partial(g^2\lambda\sigma\overline{\lambda}) - \frac{1}{4}G^{\mu\nu}\widetilde{G}_{\mu\nu}\Big)\right]\cdot\Gamma\right)_{\widehat{A}^\mu_a \widehat{A}^\nu_b}(q, p_1, p_2) \\ &= \ i\epsilon^{\mu\nu\rho\sigma} p_{1\rho} p_{2\sigma}\delta_{ab}\Big(-2 + \Sigma_{\eta-\overline{\eta}}(p_1, p_2)\Big) \ .\end{aligned}$$

The quantum corrections to $\Sigma_{\eta-\overline{\eta}}$ are uniquely defined by extension to local coupling.

$$r_\eta^{(1)} = (-1 + 2)\frac{1}{8\pi^2}C(G) = \frac{1}{8\pi^2}C(G)$$

$\widehat{A}$ is the background gauge field.

It is possible to add counterterms

– independent of $\ln g$ –

in such a way that supersymmetry is unbroken:

$$\mathcal{S}(\Gamma) = 0 \ .$$

But then one has a redefinition of the $\Theta$ angle by finite counterterms:

$$\int d^4x \left(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\right)\Gamma = \frac{1}{2} r_\eta^{(1)} \Delta^{\mathrm{anom}}_{\eta-\overline{\eta}}$$

with

$$\Delta^{\mathrm{anom}}_{\eta-\overline{\eta}} = \int d^4x \left(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\right)\left(\int dS \ \ln \boldsymbol{\eta}\, \mathcal{L}_{\mathrm{SYM}} + \mathrm{c.c.}\right)$$

Supersymmetric gauge!

N=2 supersymmetric theories E.K., in preparation

The same construction can be used with

$$\left.\begin{array}{l}\mathcal{L}_{\mathsf{SYM}}(x,\theta^1,\theta^2)\\ \boldsymbol{\eta}(x,\theta^1,\theta^2)\end{array}\right\} \qquad \text{chiral } N=2 \text{ multiplets}$$

Algebra: The anomaly appears in the same form,

but due to the second fermion one gets

$$\left(\left[i\,\mathrm{Tr}\Big(\partial(g^2\lambda^i\sigma\overline{\lambda}_i) - \frac{1}{4}G^{\mu\nu}\widetilde{G}_{\mu\nu}\Big)\right]\cdot\Gamma\right)_{\widehat{A}^\mu_a\widehat{A}^\nu_b}(q,p_1,p_2)$$
$$= i\epsilon^{\mu\nu\rho\sigma}p_{1\rho}p_{2\sigma}\delta_{ab}\Big(-2+\Sigma_{\eta-\overline{\eta}}(p_1,p_2)\Big)\ .$$

and

$$r_\eta^{(1)} = (-2\cdot 1 + 2)\frac{1}{8\pi^2}C(G) = 0$$

$N=2$ Super-Yang- Mills is not anomalous with local coupling.

## 5. Renormalization and the $\beta$-functions

The anomaly can be absorbed into a differential operator

$$\mathcal{S}(\Gamma) + r_\eta^{(1)} \delta S \Gamma = 0$$

with $\delta S$ including anomalous susy trafos of the coupling $g(x)$:

$$\begin{aligned} \delta_\alpha g^{-2} &= \delta_\alpha(\eta + \overline{\eta}) = \chi_\alpha (1 + r_\eta^{(1)} g^2 + \mathcal{O}(g^4)) \\ 2i\delta_\alpha \Theta &= \delta_\alpha(\eta - \overline{\eta}) = \chi_\alpha \end{aligned}$$

$\mathcal{O}(g^4)$ are finite scheme dependent redefinitions of the coupling.

Naive (multiplicative) renormalization is not possible.
A $\Gamma_{\text{eff}}$ solving the anomalous ST-identity does not exist
$\Longrightarrow$ algebraic renormalization

Renormalization group equation and $\beta$-functions by algebraic construction

$$\left[\mathcal{R}, \mathcal{S}_\Gamma + r_\eta^{(1)} \delta S\right] = 0 \qquad \left[\mathcal{R}, \int d^4x \left(\frac{\delta}{\delta \Theta}\right)\right] = 0$$

Closed expression of the gauge $\beta$ function:

$$\widehat{\beta}_{g^2}^{(1)} \int d^4x \, g^4 (1 + r_\eta^{(1)} g^2 + \mathcal{O}(g^4)) \frac{\delta}{\delta g^2}$$

is the only symmetric differential operator of the local coupling and its superpartners.

The anomaly coefficient $r_\eta^{(1)} \longrightarrow$ two-loop $\beta$ function
$l$-loop ST-identity $\longrightarrow$ order $l+1$ of $\beta_{g^2}$

$$\beta_{g^2} = \widehat{\beta}_{g^2}^{(1)} g^4 (1 + r_\eta^{(1)} g^2 + \mathcal{O}(g^4))$$

With an appropriate choice for $\mathcal{O}(g^4)$

$$\beta_{g^2} = \widehat{\beta}_{g^2}^{(1)} \frac{g^4}{1 - r_\eta^{(1)} g^2}$$

NSZV-expression V. Novikov et al., NPB 229(1983)381

## 6. Conclusions

Local gauge coupling:

Non-renormalization of the $\Theta$ angle

Supersymmetric gauge theories:

The non-renormalization of the $\Theta$ angle

- determines in the non-renormalization of the coupling in loop orders $l \geq 2$,

  $\longrightarrow$ "holomorphicity of the effective action",

- gives rise to a supersymmetry anomaly in loop order $l = 1$.

  $\longrightarrow$ "Non-holomorphic gauge $\beta$-function"

Extensions

Matter part with axial current and the Adler–Bardeen anomaly.

A natural description of soft breakings and their non-renormalization theorems.

E.K., D. Stöckinger, PRD 64, 115012,hep-ph/0107061
and PRD 65,105014, hep-ph/0201247