Non-renormalization theorems from extension to local coupling

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Abstract

The extension of coupling constants to space time dependent fields, the local couplings, makes possible to derive the non-renormalization theorems of supersymmetry by an algebraic characterization of Lagrangian $N = 1$ supermultiplets. For super-Yang-Mills theories the construction implies non-renormalization of the coupling beyond one-loop order. However, renormalization in presence of the local gauge coupling is peculiar due to a new anomaly in one-loop order, which appears as an anomaly of supersymmetry in the Wess-Zumino gauge. As an application we derive the closed all-order expression for the gauge β function and prove the non-renormalization of general N=2 supersymmetric theories from a cancellation of the susy anomaly.

1 Introduction

The extension of coupling constants to space-time dependent external fields, i.e. to local couplings, has been an important tool in renormalized perturbation theory for a long time [1]. It has been moved in the center of interest again from a string point of view since the couplings there are dynamical fields and enter quite naturally as external fields the effective field theories derived from string theories. It has been seen that local couplings in combination with holomorphicity imply the non-renormalization theorems of supersymmetric field theories [2, 3, 4, 5].

Most of the results have been derived in the framework of Wilsonian renormalization and by using the Wilsonian effective action. We have now started a rigorous and scheme-independent construction of supersymmetric field theories with local couplings. In the Wess–Zumino model [6], in SQED [7] and in softly broken SQED [8] it was shown that local couplings allow the derivation of the non-renormalization theorems [9, 10] as well as the generalized non-renormalization theorem [11] from renormalization properties of the extended model in an algebraic context. Most interesting is the extension to super-Yang-Mills theories [12, 13]. There the classical analysis applies in the same way as for SQED, but in the procedure of renormalization one finds a new anomaly of supersymmetry, which implies finally the non-holomorphic contributions in the gauge β function of pure super-Yang-Mills theories.

2 Local coupling in super-Yang-Mills theories

We consider super-Yang-Mills theories with a simple gauge group as for example $SU(N)$ in the Wess-Zumino gauge. In the Wess–Zumino gauge the vector multiplet consists of the gauge fields $A^\mu = A^\mu_a \tau_a$, the gaugino fields $\lambda^\alpha = \lambda^\alpha_a \tau_a$ and their complex conjugate fields $\overline{\lambda}^{\dot\alpha}$ and the auxiliary fields $D = D_a \tau_a$. For constant coupling the action

$$\Gamma_{\rm SYM} = {\rm Tr} \int d^4x \,\Big(-\frac{1}{4g^2} G^{\mu\nu}(gA) G_{\mu\nu}(gA) + i\lambda^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu \overline{\lambda}^{\dot\alpha} + \frac{1}{8} D^2 \Big) \ , \tag{1}$$

with

$$\begin{aligned} G^{\mu\nu}(A) &= \partial^\mu A^\nu - \partial^\nu A^\mu + i[A^\mu, A^\nu] \ , \\ D^\mu \lambda &= \partial^\mu \lambda - ig[A^\mu, \lambda] \ , \end{aligned} \tag{2}$$

is invariant under non-Abelian gauge transformations and supersymmetry transformations.

In the Wess-Zumino gauge [14, 15] the algebra of supersymmetry transformation does not close on translations but involves an additional field dependent gauge transformation:

$$\begin{aligned} \{\delta_\alpha, \bar\delta_{\dot\alpha}\} &= 2i\sigma^\mu_{\alpha\dot\alpha} (\partial^\mu + \delta^{\rm gauge}_{A^\mu}) \ , \\ \{\delta_\alpha, \delta_\beta\} &= \{\bar\delta_{\dot\alpha}, \bar\delta_{\dot\beta}\} = 0 \ . \end{aligned} \tag{3}$$

Thus, on gauge invariant field monomials the supersymmetry algebra takes their usual form, and it is possible to classify the Lagrangians into the usual $N = 1$ multiplets.

The Lagrangians of supersymmetric field theories are the F or D components of an $N = 1$ multiplet. In particular one finds that the Lagrangian of the super-Yang-Mills action is the highest component of a chiral and an antichiral multiplet. Using a superfield notation in the chiral representation the chiral Lagrangian multiplet $\mathcal{L}_{\rm SYM}$ is given by

$$\mathcal{L}_{\rm SYM} \ = \ -\frac{1}{2} g^2 {\rm Tr} \lambda^\alpha \lambda_\alpha + \Lambda^\alpha \theta_\alpha + \theta^2 L_{\rm SYM} \ , \tag{4}$$

with the chiral super-Yang-Mills Lagrangian $L_{\rm SYM}$

$$\begin{aligned} L_{\rm SYM} \ = \ & {\rm Tr} \big(-\frac{1}{4} G^{\mu\nu}(gA) G_{\mu\nu}(gA) + ig\lambda^\alpha \sigma^\mu_{\alpha\dot\alpha} D_\mu (g\overline{\lambda}^{\dot\alpha}) + \frac{1}{8} g^2 D^2 \\ & - \frac{i}{8} \epsilon^{\mu\nu\rho\sigma} G_{\mu\nu}(gA) G_{\rho\sigma}(gA) \big) \ . \end{aligned} \tag{5}$$

By complex conjugation one obtains the the respective antichiral multiplet $\bar{\mathcal{L}}_{\rm SYM}$.

The crucial point for the present considerations is the fact that the topological term ${\rm Tr}\, G\tilde{G}$ appears in the supersymmetric Lagrangians (5) and is related to the kinetic term $-\frac{1}{4}{\rm Tr}(GG)$ via supersymmetry. Using a local, i.e. space-time-dependent, gauge coupling one is able to include the complete Lagrangian with the topological term into the action and the renormalization of the gauge coupling will be related to the renormalization of the topological term.

For this purpose we introduce a chiral and an antichiral field multiplet with dimension zero $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$:

$$\boldsymbol{\eta} = \eta + \theta\chi + \theta^2 f \ , \overline{\boldsymbol{\eta}} = \overline{\eta} + \overline{\theta}\overline{\chi} + \overline{\theta}^2 \overline{f} \ ,$$

and couple them to the Lagrangian multiplets of the super-Yang-Mills action:

$$\begin{aligned}\Gamma_{\rm SYM} &= -\frac{1}{4}\int dS\ \boldsymbol{\eta}\mathcal{L}_{\rm SYM} - \frac{1}{4}\int d\bar{S}\ \overline{\boldsymbol{\eta}}\bar{\mathcal{L}}_{\rm SYM} \\ &= \int d^4x \left(\eta L_{\rm SYM} - \frac{1}{2}\chi^\alpha\Lambda_\alpha - \frac{1}{2}fg^2{\rm Tr}\lambda^\alpha\lambda_\alpha + \text{ c.c.}\right) . \end{aligned} \tag{6}$$

Now it is obvious that the local coupling $g(x)$ is related to the real part of the field η:

$$\eta + \overline{\eta} = \frac{1}{g^2(x)} . \tag{7}$$

The imaginary part of η couples to the topological term. Thus, it takes the role of a space time dependent Θ angle

$$\eta - \overline{\eta} = 2i\Theta . \tag{8}$$

In perturbation theory the dependence on the superfields $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$ will be governed by the following properties of the classical action (6):

- $g(x)$ is the loop expansion parameter and the 1PI Green functions satisfy in loop order l the topological formula

$$N_g\Gamma^{(l)} = (N_A + N_\lambda + N_D + 2(l-1))\Gamma^{(l)} . \tag{9}$$

- The Θ angle couples to a total derivative:

$$\Big(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\Big)\Gamma = [-\frac{i}{4}G^{\mu\nu}\tilde{G}_{\mu\nu} + i\partial(\lambda\sigma\overline{\lambda})]\cdot\Gamma , \tag{10}$$

  and the classical action satisfies the identity

$$\int d^4x \,\Big(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\Big)\Gamma = -i\int d^4x\ \frac{\delta}{\delta\Theta}\Gamma = 0 . \tag{11}$$

3 Invariant counterterms

Invariant counterterms are the local field monomials which are invariant under the classical symmetries and hold as counterterms to the non-local loop diagrams in loop order l. As such they are in one-to-one correspondence with the possible UV divergences of the theory.

Constructing the invariant counterterms to $\Gamma_{\rm SYM}$ with local coupling we find from gauge invariance and supersymmetry:

$$\begin{aligned}\Gamma^{(l)}_{\rm ct,phys} &= z^{(l)}_{\rm YM}\Big(-\frac{1}{8}\int dS\ \boldsymbol{\eta}^{1-l}\mathcal{L}_{\rm SYM} - \frac{1}{8}\int d\bar{S}\ \overline{\boldsymbol{\eta}}^{1-l}\bar{\mathcal{L}}_{\rm SYM}\Big) \\ &= z^{(l)}_{\rm YM}\int d^4x\,\Big(-\frac{1}{4}(2g^2)^{l-1}G^{\mu\nu}(gA)G_{\mu\nu}(gA) \\ &\qquad - \frac{1}{8}(2g^2)^l(l-1)\Theta G^{\mu\nu}(gA)\tilde{G}_{\mu\nu}(gA) + \ldots\Big) . \end{aligned} \tag{12}$$

For local coupling the Θ angle couples to a total derivative only for $l = 0$. Hence, using the identity (11) one finds that these counterterms as well as the respective UV divergences are excluded in all loop orders except for one loop:

$$z^{(l)}_{\rm YM} = 0 \qquad \text{for} \quad l \geq 2 \ , \tag{13}$$

i.e. the Θ angle is not renormalized and determines the renormalization of the coupling via supersymmetry. The resulting counterterm action is the holomorphic effective action which has been stated already in [11]:

$$\Gamma_{\rm eff} \ = \ -\frac{1}{8}\int dS \ (\boldsymbol{\eta} + z^{(1)}_{\rm YM})\mathcal{L}_{\rm SYM} + \text{c.c.} \tag{14}$$

It is obvious from eq. (12) that the one-loop order is special. Here the renormalization of the coupling is not related to the renormalization of the Θ angle, and one has an arbitrary counterterm $z^{(1)}_{\rm YM}$ and a corresponding UV divergence. However, as we will show in the next section, the quantum corrections to $\mathrm{Tr}\, G\tilde{G}$ induce an anomaly of supersymmetry in one-loop order. It makes quantization of super-Yang-Mills theories non-trivial such that quantum results cannot be obtained from an effective action or by using multiplicative renormalization.

4 The anomalous breaking of supersymmetry

To quantize supersymmetric field theories in the Wess-Zumino gauge one includes the gauge transformations, supersymmetry transformations and translations into the nilpotent BRS operator [16, 17, 18]:

$$\mathbf{s}\phi = \delta^{\rm gauge}_c + \epsilon^\alpha \delta_\alpha + \bar{\delta}_{\dot\alpha}\epsilon^{\dot\alpha} - i\omega^\nu \delta^T_\nu \ . \tag{15}$$

The fields $c(x)$ are the usual Faddeev-Popov ghosts, ϵ^α, $\overline{\epsilon}^{\dot\alpha}$ and ω^ν are constant ghosts of supersymmetry and translations, respectively.

As for usual gauge theories BRS transformations are encoded in the Slavnov-Taylor identity:

$$\mathcal{S}(\Gamma_{\rm cl}) = 0 \tag{16}$$

with

$$\Gamma_{\rm cl} = \Gamma_{\rm SYM} + \Gamma_{g.f.} + \Gamma_{\phi\pi} + \Gamma_{ext.f.} \ . \tag{17}$$

and the Slavnov–Taylor identity expresses gauge invariance and supersymmetry of the classical action.

In addition to the Slavnov-Taylor identity the dependence on the external fields $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$ is restricted to all orders by the identity (11) [12], which determines the renormalization of the Θ angle.

In the course of renormalization the Slavnov-Taylor identity (16) has to be established for the 1PI Green functions to all orders of perturbation theory. From the quantum action principle one finds that the possible breaking terms are local in one-loop order:

$$\mathcal{S}(\Gamma) = \Delta_{\rm brs} + \mathcal{O}(\hbar^2) \ . \tag{18}$$

Algebraic consistency yields the constraint:

$$\mathbf{s}_{\Gamma_{\rm cl}}\Delta_{\rm brs} = 0 \; . \tag{19}$$

Gauge invariance can be established as usually, i.e. one has

$$\mathcal{S}(\Gamma)\Big|_{\epsilon,\overline{\epsilon}=0} = \mathcal{O}(\hbar^2) \; , \tag{20}$$

and the remaining breaking terms depend on the supersymmetry ghosts ϵ and $\overline{\epsilon}$, and represent as such a breaking of supersymmetry. Having gauge invariance established, the supersymmetry algebra closes on translations. Using the supersymmetry algebra one obtains that the remaining breaking terms of supersymmetry are variations of field monomials with the quantum numbers of the action [19]:

$$\Delta_{\rm brs} = \mathbf{s}_{\Gamma_{\rm cl}}\hat{\Gamma}_{\rm ct} \; . \tag{21}$$

However, not all of the field monomials in $\hat{\Gamma}_{\rm ct}$ represent scheme-dependent counterterms of the usual form. There is one field monomial in $\hat{\Gamma}_{\rm ct}$, which depends on the logarithm of the gauge coupling, but whose BRS variation is free of logarithms:

$$\begin{aligned}
\Delta_{\rm brs}^{\rm anomaly} &= \mathbf{s}\int d^4x \;\ln g(x)(L_{\rm SYM}+\bar{L}_{\rm SYM}) \\
&= (\epsilon^\alpha\delta_\alpha + \overline{\epsilon}^{\dot\alpha}\bar{\delta}_{\dot\alpha})\int d^4x \;\ln g(x)(L_{\rm SYM}+\bar{L}_{\rm SYM}) \\
&= \int d^4x \,\Big(i\;\ln g(x)\big(\partial_\mu\Lambda^\alpha\sigma^\mu_{\alpha\dot\alpha}\overline{\epsilon}^{\dot\alpha} - \epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar{\Lambda}^{\dot\alpha}\big) \\
&\qquad - \frac{1}{2}g^2(x)(\epsilon\chi + \overline{\chi\epsilon})(L_{\rm SYM}+\bar{L}_{\rm SYM})\Big) \; .
\end{aligned} \tag{22}$$

Indeed, due to the total derivative in the first line the breaking $\Delta_{\rm brs}^{\rm anomaly}$ is free of logarithms for constant coupling and for any test with respect to the local coupling. Moreover $\Delta_{\rm brs}^{\rm anomaly}$ satisfies the topogical formula in one-loop order. Therefore $\Delta_{\rm brs}^{\rm anomaly}$ satisfies all algebraic constraints on the breakings and can appear as a breaking of the Slavnov-Taylor identity in the first order of perturbation theory.

However, being the variation of a field monomial depending on the logarithm of the coupling $\Delta_{\rm brs}^{\rm anomaly}$ cannot be induced by divergent one-loop diagrams, which are all power series in the coupling. Thus, the corresponding counterterm is not related to a naive contribution induced in the procedure of subtraction and does not represent a naive redefinition of time-ordered Green functions. Indeed, it is straightforward to prove with algebraic methods that the coefficient of the anomaly is gauge and scheme independent [12]. Therefore, $\Delta_{\rm brs}^{\rm anomaly}$ is an anomalous breaking of supersymmetry in perturbation theory and we remain with

$$\mathcal{S}(\Gamma) = r_\eta^{(1)}\Delta_{\rm brs}^{\rm anomaly} + \mathcal{O}(\hbar^2) \; . \tag{23}$$

Evaluating the Slavnov–Taylor identity one can find an expression for $r_\eta^{(1)}$ in terms of convergent loop integrals [13]. Using background gauge fields $\hat{A}^\mu$ and Feynman gauge

$\xi = 1$ the anomaly coefficient is explicitly related to insertions of the topological term and the axial current of gluinos into self energies of background fields:

$$g^2 r_\eta^{(1)} = -\frac{1}{2}\Sigma^{(1)}_{\eta-\overline{\eta}}(p_1, -p_1)\Big|_{\xi=1} , \tag{24}$$

where $\Sigma_{\eta-\overline{\eta}}$ is defined by

$$\begin{aligned}\Gamma_{\eta-\overline{\eta}\hat{A}^\mu_a \hat{A}^\nu_b}(q, p_1, p_2) &= \Big(\big[i\,\mathrm{Tr}\big(\partial(g^2\lambda\sigma\overline{\lambda}) - \frac{1}{4}G^{\mu\nu}\tilde{G}_{\mu\nu}(gA + \hat{A})\big)\big]\cdot\Gamma\Big)_{\hat{A}^\mu_a \hat{A}^\nu_b}(q, p_1, p_2) \\ &= i\epsilon^{\mu\nu\rho\sigma} p_{1\rho} p_{2\sigma}\delta_{ab}\big(-2 + \Sigma_{\eta-\overline{\eta}}(p_1, p_2)\big) . \end{aligned}\tag{25}$$

From gauge invariance with background fields and local couplings $\Sigma_{\eta-\overline{\eta}}$ is unambiguously determined in perturbation theory and not subject of renormalization. Explicit evaluation of the respective one-loop diagrams yields

$$r_\eta^{(1)} = (-1 + 2)\frac{C(G)}{8\pi^2} = \frac{C(G)}{8\pi^2} , \tag{26}$$

where the first term comes from the axial anomaly of gauginos and the second term from the insertion of the topological term [13].

We want to mention that the anomaly coefficient vanishes in SQED, since one-loop diagrams to $\Gamma_{\eta-\overline{\eta}A^\mu A^\nu}$ do not exist. The anomaly coefficient also vanishes in $N = 2$ theories. For $N = 2$ super-Yang-Mills theories the analysis of the previous sections holds in the same form, where $\mathcal{L}_{\mathrm{SYM}}$ and $\boldsymbol{\eta}$ are now $N = 2$ chiral multiplets. The algebraic analysis yields an anomaly of the same form as for $N = 1$ theories, however, in the explicit evaluation the anomaly coefficient of $N = 2$ theories vanishes since the two fermionic fields just cancel the contribution arising from the topological term $\mathrm{Tr}\, G^{\mu\nu}\tilde{G}_{\mu\nu}$ (cf. (25, 26)).

5 Renormalization and the gauge β function

In the framework of perturbation theory the anomaly of supersymmetry cannot be removed by a local counterterm. However, one is able to proceed with algebraic renormalization nevertheless by rewriting the anomalous breaking in the form of a differential operator [12]:

$$\begin{aligned}\Delta^{\mathrm{anomaly}}_{\mathrm{brs}} &= \int d^4x\,\Big(g^6 r_\eta^{(1)}(\epsilon\chi + \overline{\chi\epsilon})\frac{\delta}{\delta g^2} \\ &\qquad - i r_\eta^{(1)}\partial_\mu \ln g^2\big((\sigma^\mu\overline{\epsilon})^\alpha\frac{\delta}{\delta\chi^\alpha} + (\epsilon\sigma^\mu)^{\dot\alpha}\frac{\delta}{\delta\overline{\chi}^{\dot\alpha}}\big)\Big)\Gamma_{\mathrm{cl}}\end{aligned}\tag{27}$$

Then one has

$$\big(\mathcal{S} + r_\eta^{(1)}\delta\mathcal{S}\big)\Gamma = \mathcal{O}(\hbar^2) . \tag{28}$$

For algebraic consistency one has to require the nilpotency properties of the classical Slavnov-Taylor operator also for the extended operator. Nilpotency determines an algebraic consistent continuation of (27). This continuation is not unique but contains at the same time all redefinitions of the coupling compatible with the formal power series expansion of perturbation theory.

As a result one finds an algebraic consistent Slavnov-Taylor identity in presence of the anomaly:

$$\mathcal{S}^{r_\eta}(\Gamma) = 0 \quad \text{and} \quad \int d^4x \left(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\right)\Gamma = 0 \ , \tag{29}$$

where the anomalous part is of the form:

$$\begin{aligned}\mathcal{S}^{r_\eta}(\Gamma) = \mathcal{S}(\Gamma) \quad - \quad & \int d^4x \, \Big(g^4 \delta F(g^2)(\epsilon\chi + \overline{\chi\epsilon})\frac{\delta}{\delta g^2} \\ & + i\frac{\delta F}{1+\delta F}\partial_\mu g^{-2}\big((\sigma^\mu\overline{\epsilon})^\alpha\frac{\delta}{\delta\chi^\alpha} + (\epsilon\sigma^\mu)^{\dot\alpha}\frac{\delta}{\delta\overline{\chi}^{\dot\alpha}}\big)\Big)\Gamma \ ,\end{aligned} \tag{30}$$

with

$$\delta F(g^2) = r_\eta^{(1)} g^2 + \mathcal{O}(g^4) \ . \tag{31}$$

The lowest order term is uniquely fixed by the anomaly, whereas the higher orders in $\delta F(g^2)$ correspond to the scheme-dependent finite redefinitions of the coupling.

The simplest choice for δF is given by

$$\delta F = r_\eta^{(1)} g^2 \ , \tag{32}$$

and another choice is provided by

$$\frac{\delta F}{1+\delta F} = r_\eta^{(1)} g^2 \ . \tag{33}$$

As seen below, the latter choice gives the NSZV expression of the gauge β function [20, 11].

Algebraic renormalization with the anomalous Slavnov-Taylor operator (29) is performed in the conventional way. In particular one can derive the β functions from an algebraic construction of the renormalization group equation in presence of the local coupling. Starting from the classical expression of the RG equation

$$\kappa\partial_\kappa \Gamma_{\rm cl} = 0 \tag{34}$$

we construct the higher orders of the RG equation by constructing the general basis of symmetric differential operator with the quantum numbers of the action:

$$\mathcal{R} = \kappa\partial_\kappa + \mathcal{O}(\hbar) \ , \tag{35}$$

with

$$\mathbf{s}_\Gamma^{r_\eta}\mathcal{R}\Gamma - \mathcal{R}\mathcal{S}^{r_\eta}(\Gamma) = 0 \ , \tag{36}$$

$$\Big[\int d^4x \left(\frac{\delta}{\delta\eta} - \frac{\delta}{\delta\overline{\eta}}\right), \mathcal{R}\Big] = 0 \ . \tag{37}$$

The general basis for the symmetric differential operators consists of the differential operator of the supercoupling $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$ and several field redefinition operators. The differential operator of the coupling determines the β function of the coupling, whereas the field redefinition operators correspond to the anomalous dimensions of fields.

For the present paper we focus on the operator of the β function and neglect the anomalous dimensions. For proceeding we first construct the RG operator $\mathcal{R}_{\rm cl}$, which is symmetric with respect to the classical Slavnov-Taylor operator, i.e. we set $\delta F = 0$. In a second step we extend it to a symmetric operator with respect to the full anomalous Slavnov-Taylor operator.

Using a superspace notation we find:

$$\mathcal{R}_{\rm cl} = -\sum_{l\geq 1} \hat{\beta}_g^{(l)} \Big(\int dS\, \boldsymbol{\eta}^{-l+1} \frac{\delta}{\delta \boldsymbol{\eta}} + \int d\bar{S}\, \overline{\boldsymbol{\eta}}^{-l+1} \frac{\delta}{\delta \overline{\boldsymbol{\eta}}} \Big) + \dots \ , \tag{38}$$

and

$$\mathbf{s}_\Gamma \mathcal{R}_{\rm cl} \Gamma - \mathcal{R}_{\rm cl} \mathcal{S}(\Gamma) = 0 \ . \tag{39}$$

Evaluating then the consistency equation (37) we obtain

$$\hat{\beta}_g^{(l)} = 0 \quad \text{for} \quad l \geq 2 \ . \tag{40}$$

These restrictions are the same restrictions as we have found for the invariant counterterms of the super-Yang-Mills action (see (12) with (13)). Hence the only independent coefficient is the one-loop β-function.

The one-loop operator $\mathcal{R}_{\rm cl}$ can be extended to a symmetric operator with respect to the anomalous ST identity (36):

$$\begin{aligned} \mathcal{R} &= \hat{\beta}_g^{(1)} \int d^4x\, g^3 (1 + \delta F(g^2)) \frac{\delta}{\delta g} + \dots \\ &= \hat{\beta}_g^{(1)} \int d^4x\, g^3 \big(1 + r_\eta^{(1)} g^2 + \mathcal{O}(\hbar^2)\big) \frac{\delta}{\delta g} + \dots \ . \end{aligned} \tag{41}$$

For constant coupling we find from (41) the closed expression of the gauge β function

$$\beta_g = \hat{\beta}_g^{(1)} g^3 (1 + \delta F(g^2)) = \hat{\beta}_g^{(1)} g^3 \big(1 + r_\eta^{(1)} g^2 + \mathcal{O}(\hbar^2)\big) \ . \tag{42}$$

Thus, the two-loop order is uniquely determined by the anomaly and the one-loop coefficient, whereas higher orders depend on the specific form one has chosen for the function $\delta F(g^2)$. In particular one has for the minimal choice (32) a pure two-loop β-function and for the NSZV-choice (33) the NSZV expression [20] of the gauge β function.

For $N = 2$ super-Yang-Mills the anomaly vanishes and the classical Slavnov–Taylor identity (16) can be extended to all orders. Then the classical renormalization group operator (38) is a symmetric operator to all orders and one gets from (37) a pure one-loop contribution to the gauge β function, i.e.,

$$\mathcal{S}(\Gamma^{N=2}) = 0 \quad \Longrightarrow \quad \beta_g^{N=2} = \hat{\beta}_g^{(1)} g^3 \ . \tag{43}$$

6 Conclusions

The extension of the coupling constants to an external fields is a crucial step for deriving the non-renormalization theorems of supersymmetry in a scheme-independent way and independent from the usage of superspace methods. For super-Yang-Mills theories the extended model yields the non-renormalization of the coupling beyond one-loop order due to the supersymmetry induced relation of the coupling with the Θ angle.

The non-renormalization of the Θ-angle is the real new result gained by the construction with local coupling. Using gauge invariance and the property that the Θ angle couples to a total derivative in the classical action higher order corrections a uniquely determined by convergent expressions. In this way, local coupling in addition gives a simple proof of the non-renormalization of the Adler–Bardeen anomaly [21, 22] also for non-supersymmetric theories.

For supersymmetric theories the non-renormalization of the Θ angle induces not only the non-renormalization theorem of the gauge coupling but also a supersymmetry anomaly in one-loop order. The supersymmetry anomaly is the variation of a gauge-invariant field monomial which depends on the logarithm of the local coupling. As such it cannot be induced in the procedure of regularization, since loop diagrams are power series in the coupling. Hence the anomaly found in super-Yang-Mills with local coupling has the same properties as the Adler-Bardeen anomaly: It is determined by convergent one-loop diagrams and its coefficient is gauge- and scheme-independent.

As an application we have constructed the renormalization group equation and the gauge β function. It was shown, that the non-renormalization of the Θ angle first yields vanishing coefficients for the β function in $l \geq 2$. The supersymmetry anomaly induces the 2-loop term in terms of the one-loop coefficient and the anomaly coefficient. Higher order terms are scheme dependent and are determined by finite redefinitions of the coupling. Hence, the non-holomorphic contributions in the β function of pure super-Yang-Mills theories are generated by the supersymmetry anomaly. Since $N = 2$ super-Yang-Mills theories are not anomalous, one can impose the classical Slavnov-Taylor identity and in this case a pure one-loop gauge β function is found.

The construction can be extended also to the matter part [7, 12], and to softly broken gauge theories [8, 23]. Since the soft breakings are the lowest components of Lagrangian multiplets, they are already included in the supersymmetric model with local coupling and thus softly broken supersymmetry appears as a natural extension of supersymmetric theories with local coupling.

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