Counting the Entropy of Schwarzschild Black Holes

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Abstract

A novel approach is described towards a microscopic understanding of the Bekenstein-Hawking entropy of the simplest black hole – the Schwarzschild black hole. It is based at an intermediate step on the description of the black hole in terms of dual Euclidean brane/anti-brane pairs of type-IIB string-theory. By counting specific chain structures on the brane-complex one can reproduce the correct black hole entropy plus its logarithmic corrections.

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1 Introduction

We know since the early work of [1],[2],[3],[4],[5] that black holes come equipped with an entropy, known as the Bekenstein-Hawking (BH) entropy. It is determined by the area A_H of the hole's event horizon

$$\mathcal{S}_{BH} = \frac{A_H}{4G_4}\frac{k_B c^3}{\hbar} . \tag{1}$$

Since the laws of black hole thermodynamics are in a neat one-to-one correspondence with the conventional laws of thermodynamics, one expects in analogy that the BH-entropy should likewise be explainable statistical mechanically by counting the entropy of an underlying set of microstates in a microcanonical ensemble.

So far string-theory satisfied these expectations for a variety of supersymmetric black holes. Supersymmetry is an essential ingredient given that the counting of the microscopic degrees of freedom takes place in a different regime in moduli space than where the black hole actually resides. One counts the entropy of specific string excitations at weak string coupling where the string dynamics is under control and by relying on supersymmetry exports the result to the strongly coupled regime where the black hole lives but where up to now the full string-theory dynamics is not sufficiently understood [6]. In another approach which tries to avoid the supersymmetry constraint one maps the spacetime of interest by means of T- and S-Dualities and potentially also boost transformations to a spacetime whose microscopic entropy counting is under control, e.g. the D=3 BTZ black hole [7] or near-extremal branes.

Due to the lack of our knowledge of the full formulation of string-/M-theory in the strongly coupled regime, both methods have to rely on an indirect method of counting the relevant degrees of freedom. Though the resulting match in entropies is quite impressive, it is therefore difficult to actually appreciate what the degrees of freedom are that make up the black hole. In particular it is not quite clear where these are located. In this context it is interesting to note that the loop quantum gravity program which quantizes D=4 gravity directly provided a derivation of the BH-entropy for the Schwarschild case up to the fixing of an overall parameter which arises from an ambiguity of the quantization procedure [8].

Our aim here is to introduce an alternative approach which does not rely on supersymmetry and therefore might be applied to both non-supersymmetric and supersymmetric spacetimes. Moreover we do not exploit weakly coupled string technology for the entropy counting in order not to be constrained from the outset to this region in moduli space. Instead the intention of the present approach is to deal directly with the entropy counting in the strongly coupled regime where e.g. black holes are situated. Of course since we still lack a complete formulation of string-/M-theory in the strongly coupled regime, the idea is to solve the state-counting issue first, thereby extracting a suitable set of microstates in the strongly coupled regime and then in a future second step to build the dynamics on these states in order to hopefully arrive at a non-perturbative formulation of string-theory. The presentation given here which focusses on a proposition and counting of states for a D=4 spacetime with spherical event horizon is based on [9],[10],[11].

2 BH-Entropy and Dual Brane Doublets

Let us consider type II String-Theory on a D=10 Lorentzian spacetime $\mathcal{M}^{1,3} \times \mathcal{M}^{p-1} \times \mathcal{M}^{7-p}$ $(p = 1, \ldots, 5)$ with D=4 spherical symmetry

$$ds^2 = g^{(2)}_{\alpha\beta}(t,r)dx^\alpha dx^\beta + r^2 d\Omega^2 + g^{(p-1)}_{ab}(x^c)dx^a dx^b + g^{(7-p)}_{kl}(x^m)dx^k dx^l \tag{2}$$

where $x^\alpha, x^\beta = \{t, r\}$ and $d\Omega^2$ is the line-element of a unit two-sphere S^2. We consider $\mathcal{M}^{p-1}$ and $\mathcal{M}^{7-p}$ as compact such that we have a geometric background describing a compactification from D=10 down to D=4. For such a background the D=4 Newton's constant is related to the Regge slope α' and string coupling constant g_s through

$$G_4 = \frac{G_{10}}{V_{p-1}V_{7-p}} = \frac{(2\pi)^6 \alpha'^4 g_s^2}{8 V_{p-1} V_{7-p}} \; , \tag{3}$$

where $V_i = vol(\mathcal{M}^i) \equiv \int_{\mathcal{M}^i} d^i x \sqrt{g^{(i)}}$.

Next, let us wrap two mutually orthogonal Euclidean "electric-magnetic" dual branes, Dp and $D(6-p)$, around $S^2 \times \mathcal{M}^{p-1}$ and $\mathcal{M}^{7-p}$, resp. such that together they cover the whole internal space plus an external sphere. For such a dual brane pair it follows from the Dirac-quantization condition that the product of their tensions obeys

$$\tau_{Dp}\tau_{D(6-p)} = \frac{1}{(2\pi)^6 \alpha'^4 g_s^2} \; . \tag{4}$$

Thus we can write the inverse of the D=4 Newton's constant as

$$\frac{1}{G_4} = 8(\tau_{Dp}V_{p-1})(\tau_{D(6-p)}V_{7-p}) \; . \tag{5}$$

The right-hand-side of this expression already comes very close to the product of two Nambu-Goto actions for the Euclidean branes except for the fact that we miss the area of the external two-sphere in the first factor.

But since our goal here is to deal with the BH-entropy of D=4 spacetimes possessing spherical event horizons, let us now point out the connection between these wrapped dual branes and such D=4 spacetimes. The dual branes will act as a supergravity source and therefore will give rise to some D=10 geometry of the form

$$(\text{D=4 Spacetime}) \times (\text{Compact Internal Space}) \; . \tag{6}$$

Turning this around we want to start with a particular D=4 spacetime with spherical event horizon S^2_H whose BH-entropy we want to understand and ask whether a suitable dual brane pair exists giving rise to a D=10 geometry such that it includes the D=4 spacetime of interest in its external part. In particular in this correspondence the external sphere S^2 on which we wrap one of the branes is required to become identical with the event horizon of the D=4 spacetime

$$S^2 \equiv S^2_H \; . \tag{7}$$

We will comment below on the dual brane pair with which this can be achieved for the case of the D=4 Schwarzschild black hole and will now proceed with the general argument taking this correspondence for granted for a moment.

Given the identification (7) it is easy to see that now we can reformulate the D=4 spacetime's BH-entropy as

$$\mathcal{S}_{BH} = \frac{A_H}{4G_4} = 2S_{Dp}S_{D(6-p)} \ , \tag{8}$$

where

$$S_{Dp} = \tau_{Dp}\int_{S^2\times\mathcal{M}^{p-1}} d^{p+1}x\sqrt{\det g} \ , \qquad S_{D(6-p)} = \tau_{D(6-p)}\int_{\mathcal{M}^{7-p}} d^{7-p}x\sqrt{\det g} \tag{9}$$

are the respective Nambu-Goto actions of the involved dual branes. Actually, the factor two in front tells us to repeat the procedure once more and to employ a doublet of the Euclidean brane pairs which, however, may differ from each other. Therefore let us wrap a further dual Euclidean brane pair $Dp' - D(6-p')$, with Dp' and $D(6-p')$ again mutually orthogonal, in the same manner as before around the S^2 plus the internal space. The D=4 BH-entropy then becomes purely expressible just in terms of the respective Nambu-Goto actions

$$\mathcal{S}_{BH} = S_{Dp}S_{D(6-p)} + S_{Dp'}S_{D(6-p')} \ . \tag{10}$$

Furthermore, notice that we are free to exchange any of the appearing branes with its anti-brane and still arrive at the same expression (of course under the premise that the inclusion of the antibrane gives rise to the requested D=4 spacetime in the external part of the ensuing D=10 geometry).

It turns out that this formula works for all doublets of dual brane pairs of string- and M-Theory [9]. Thus whenever we find a doublet of dual brane pairs $(E_1, M_1), (E_2, M_2)$ which gives rise to the requested D=4 spacetime with spherical event horizon in the external part of the D=10 metric together with the identification (7), we can rewrite the BH-entropy of the D=4 spacetime as

$$\mathcal{S}_{BH} = \sum_{i=1,2} S_{E_i}S_{M_i} \ , \tag{11}$$

where the pairs (E_i, M_i) range over all possible dual pairs of type-II string-theory and M-theory

$$(E_i, M_i) \in \{(Dp_i, D(6-p_i)), (F1, NS5), (NS5, F1), (M2, M5), (M5, M2)\} \ . \tag{12}$$

Moreover, it is understood that any occuring brane or string might also be replaced by its anti-brane or anti-string partner as this exchange leaves the Nambu-Goto action unchanged.

In particular, for the case of the D=4 Schwarzschild black hole one might take the self-dual $(D3, D3), (\overline{D3}, \overline{D3})$ type-IIB brane/anti-brane doublets, distributed along the ten coordinates as depicted in fig.1. An equal amount of branes and anti-branes guarantees an uncharged solution of the D=10 supergravity field equations which moreover is non-supersymmetric. Beyond this the choice of the non-dilatonic $D3$'s ensures that one finds a D=10 solution without dilaton matter present. It can then be worked out [10] that indeed the described $(D3, D3), (\overline{D3}, \overline{D3})$ configuration gives rise to a D=4 Schwarzschild black

	t	r	θ	ϕ	4	5	6	7	8	9
$D3$			•	•	•	•				
$\overline{D3}$			•	•	•	•				
$D3$							•	•	•	•
$\overline{D3}$							•	•	•	•

Figure 1: The Euclidean brane/anti-brane pairs $(D3, D3), (\overline{D3}, \overline{D3})$ suitable to describe a D=4 Schwarzschild black hole. They are oriented along the directions marked by dots. The coordinates t, r, θ, ϕ describe the external D=4 spacetime with θ, ϕ describing the S^2. The whole configuration is located at some common fixed value of $r = r_H$.

hole geometry in the external part of the metric together with the necessary identification of the spheres (7). Notice that Euclidean branes positioned somewhere in spacetime usually decay because they are localized in time. However, Euclidean branes which are positioned at an event horizon are seen by any outside observer (for whom r is bigger than the position r_H of the brane configuration) through an infinite redshift. This stretches classically any decay-time infinitely long such that from an outside observer's perspective Euclidean branes wrapping an event horizon should lead to a stationary spacetime solution [10].

In this context it is interesting to note that brane-antibrane systems cannot annihilate and decay into the closed-string vacuum classically [12]. Such a decay in which the tachyon rolls down its potential hill and thereby radiating off into the bulk the surplus of energy can occur only quantum mechanically. In weakly coupled string-theory this might be understood from the fact that in this transition open strings on the brane-antibrane system have to transform into closed strings which can enter the bulk. However this is a one-loop process from the open string point of view and thus quantum mechanical. On the other hand it is well-known that also a black hole's Hawking radiation is an intrinsic quantum mechanical phenomenon which does not exist at the classical level and likewise describes radiation sent into the bulk. It therefore seems natural to conjecture within the $(D3, D3), (\overline{D3}, \overline{D3})$ description of the D=4 Schwarzschild black hole that its *Hawking radiation might be due to the rolling of the tachyon down its potential hill* [10]. Unfortunately to understand this connection better we would have to understand first this transition better within strongly coupled string-theory.

3 Chain-States and their Entropy

To proceed further with the analysis of the BH-entropy of the Schwarzschild black hole, let us reflect upon the tension of a Euclidean brane. For Lorentzian Dp-branes their tension is usually interpreted as

$$\tau_{Dp} = \frac{\text{mass}}{\text{unit of p-volume}} \tag{13}$$

For a Euclidean brane however there is no time direction along the worldvolume which is singled out and one has to treat all worldvolume directions on an equal footing. It therefore seems more natural in this case to interpret its tension

$$\tau_{Dp} = \frac{1}{v_{Dp}} \equiv \frac{1}{l_{Dp}^{p+1}} \tag{14}$$

in terms of a smallest fundamental volume unit v_{Dp} resp. a smallest length l_{Dp}. This is a natural extension of the fact that $\sqrt{\alpha'}$ constitutes a smallest length for strings.

Evidence for such a smallest volume unit on the brane's worldvolume comes from the "worldvolume uncertainty relations for D-branes". In [13] it was shown that the worldvolume $(X^0, \ldots, X^p)$ of a Dp-brane is subject to the following uncertainty relation

$$Dp \;:\quad \delta X^0 \delta X^1 \ldots \delta X^p \gtrsim g_s {\alpha'}^{\frac{p+1}{2}} \;, \tag{15}$$

where the right-hand-side was determined up to numerical factors. By employing string-dualities it was furthermore shown that similar uncertainty relations hold true for the $NS5$-brane, the fundamental string and the M-theory $M5$- and $M2$-branes

$$NS5 \;:\quad \delta X^0 \delta X^1 \ldots \delta X^5 \gtrsim g_s^2 {\alpha'}^3 \;, \qquad F1 \;:\quad \delta X^0 \delta X^1 \gtrsim \alpha' \;, \tag{16}$$

$$M5 \;:\quad \delta X^0 \delta X^1 \ldots \delta X^5 \gtrsim l_{Pl}^6 \;, \qquad M2 \;:\quad \delta X^0 \delta X^1 \delta X^2 \gtrsim l_{Pl}^3 \;, \tag{17}$$

with l_{Pl} the D=11 Planck-length. Indeed the relation for the fundamental string had been proposed earlier in [14]. By using the tensions of these objects

$$Dp:\; \tau_{Dp} = \frac{1}{(2\pi)^p g_s {\alpha'}^{(p+1)/2}} \;, \quad NS5:\; \tau_{NS5} = \frac{1}{(2\pi)^5 g_s^2 {\alpha'}^3} \;, \quad F1:\; \tau_{F1} = \frac{1}{2\pi\alpha'} \;, \tag{18}$$

$$M5:\; \tau_{M5} = \frac{1}{(2\pi)^5 l_{Pl}^6} \;, \qquad M2 \;:\; \tau_{M2} = \frac{1}{(2\pi)^2 l_{Pl}^3} \tag{19}$$

one sees that all these worldvolume uncertainty relations can be combined into the statement that the smallest volume allowed is given by the inverse of the object's tension

$$\delta X^0 \ldots \delta X^p \gtrsim \frac{1}{\tau} \;. \tag{20}$$

Thus (14) is an extrapolation of this result to the strongly coupled regime and fixes the proportionality constant.

The introduction of a smallest fundamental volume unit on the brane's worldvolume leads to a discrete structure on the brane. Namely, we can now think of it as a lattice made out of a certain number N_{Dp} of such smallest volume units which we will call cells from on. It is precisely the Nambu-Goto action which measures the number of these cells contained in a brane

$$N_{Dp} = \tau_{Dp} \int d^{p+1}x \sqrt{\det g} = S_{Dp} \;. \tag{21}$$

But this implies that the expression (11) which we have found above for the D=4 BH-entropy can be rewritten further and now becomes identical to an integer N

$$\mathcal{S}_{BH} = \sum_{i=1,2} N_{E_i} N_{M_i} =: N \tag{22}$$

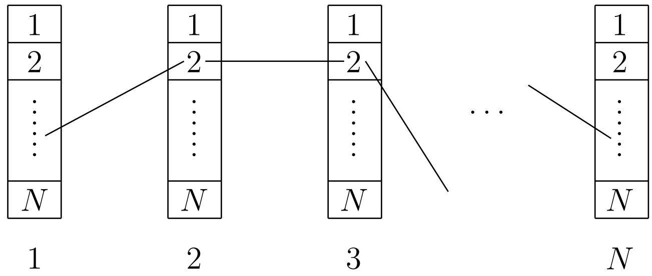

Figure 2: Constructive view of an $(N-1)$-chain where we arrange all cells of the lattice in a column and use N copies of them. We allow each link to connect any cell of a column with any cell of the succeeding column. Horizontal links correspond to loops.

with N the total number of cells contained in the joint worldvolume of the doublet of dual brane pairs.

So far we have reformulated the D=4 BH-entropy in terms of string-/M-theory notions under the inclusion of a discrete brane structure coming from a reinterpretation of the brane's tension. We will now investigate in which way this different thinking of a Schwarzschild black hole in terms of the selfdual brane/antibrane $(D3, D3), (\overline{D3}, \overline{D3})$ configuration can help us to propose a suitable set of black hole microstates on these branes. A suitable set of microstates must be able to account exactly for the BH-entropy.

To this aim, let us conceive on the combined $(D3, D3), (\overline{D3}, \overline{D3})$ worldvolume lattice an $(N-1)$-chain, which is a chain composed out of $N-1$ successive links where we allow all links to start and end democratically on any of the N cells of the lattice (see fig.2). In particular a link might start and end on the same cell thus creating a loop. Altogether the number of different chains is N^N.

The main motivation to consider chains as long as possible (which might cover the whole lattice) comes from an analogy with the weakly coupled string at finite temperature. We have an uncertainty in space-resolution (the cells) which leads for a relativistic object to an uncertainty in energy

$$\Delta E \simeq \Delta P \simeq \frac{1}{l_{Dp}} = (\tau_{Dp})^{\frac{1}{p+1}} = \frac{1}{\sqrt{\alpha'}(g_s(2\pi)^p)^{\frac{1}{p+1}}} \,. \tag{23}$$

Therefore at strong coupling where $g_s \simeq 1$, the temperature associated with ΔE is of order the Hagedorn-temperature. However at this temperature we know that for the weakly coupled string it is entropically favourable to allocate the energy of the system to just one single long string instead to many short ones. Moreover it is known that the string behaves at high excitation levels like a random walk [15] which in addition motivates the consideration of discrete random-walk like states also for the strongly coupled regime.

The counting of different chains has up to now been classical, meaning that all cells were regarded as distinguishable. However, in a proper quantum treatment the cells should better be regarded as indistinguishable bosonic degrees of freedom. How to account for this quantum feature is well-known from statistical mechanics – we simply have to divide by the Gibbs-correction factor $N!$. This gives the quantum-mechanically corrected number of

$$\Omega(N) = \frac{N^N}{N!} \tag{24}$$

different $N-1$ chains. To evaluate the entropy of the chain-states in the thermodynamic large N limit (as adequate to macroscopic black holes) we use Stirling's approximation, $\ln(N!) = N \ln N - N + \mathcal{O}(\ln N)$, and our reformulation of the D=4 BH-entropy (22) to obtain

$$\mathcal{S} = \ln \Omega(N) = N = \mathcal{S}_{BH} \tag{25}$$

up to corrections of $\mathcal{O}(\ln N)$. Thus we can conclude that the proposed $(N-1)$-chains constitute a reasonable set of black hole microstates in the sense that they can account for the exact black hole entropy.

4 Corrections to BH-Entropy

Having found agreement between the chain and the black hole entropy at leading large N order, one might be curious what happens at subleading order. Corrections to the BH-entropy for D=4 black holes had been determined in supersymmetric cases from String-Theory while results in non-supersymmetric cases came from Quantum Geometry [16] or the conformal field theory approach of Carlip [17].

The general result consists in a subleading logarithmic correction to the semiclassical BH-entropy

$$-k \ln \mathcal{S}_{BH} \tag{26}$$

with a positive constant $k > 0$ (a negative subleading correction is in accordance with the holographic principle). Though initially a value of $k = 3/2$ was favoured, it seems that recently this has been corrected to $k = 1/2$ [18]. The basic reason being that one obtains a correction of the form

$$-\frac{3}{2} \ln \mathcal{S}_{BH} + \ln c\ . \tag{27}$$

However, the central charge c is argued not just to be constant but instead given by $\ln c = \ln \mathcal{S}_{BH} + const$ which amounts to $k = 1/2$. Moreover, $k = 1/2$ has also been found independently by other methods exploiting the AdS/CFT duality [19].

For the proposed $(N-1)$-chains corrections to the chain entropy come simply from a more accurate approximation of $N!$ by the Stirling-series, e.g.

$$N! = \sqrt{2\pi N} N^N e^{-N} \big(1 + \frac{1}{12N} + \mathcal{O}(\frac{1}{N^2})\big)\ . \tag{28}$$

Using this corrected Stirling approximation for the evaluation of the chain entropy and once more considering the identity (22), we get the corrected chain-entropy formula

$$\mathcal{S} = \ln \Omega(N) = \mathcal{S}_{BH} - \frac{1}{2} \ln \mathcal{S}_{BH} - \ln \sqrt{2\pi} - \frac{1}{12\mathcal{S}_{BH}} + \mathcal{O}\big(\frac{1}{\mathcal{S}_{BH}^2}\big)\ . \tag{29}$$

This shows that the proposed chains can also easily account for the subleading logarithmic correction and give the precise numerical coefficient $k = 1/2$. We can therefore conclude that the chains constitute an interesting possible set of black hole microstates whose dynamics should be worth investigating in the future.

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