Local Coupling and the Supercurrent in Supersymmetric Quantum Field Theory

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In order to gain insight into the structure of quantum field theory, one often couples relevant composite operators to external fields such that functional differentiation w.r.t. these fields yields Green functions of the respective operator. A particular variant of this method consists in replacing the coupling constant by a local coupling field. The corresponding composite operator is the local, i.e. non-integrated interaction vertex whose study opens direct access to nonrenormalization theorems [1, 2]. Simultaneously, this operator is related to the supercurrent through the superconformal trace identity, i.e. the conformal anomaly can be written in terms of functional differential operators involving the local coupling. In this article, we mainly concentrate on the case of SQED.

The local coupling is a dimensionless chiral superfield η which in the limit of constant coupling corresponds to $\eta = \bar{\eta} = \frac{1}{2}g^{-2}$ [2]. We also use the abbreviation $G = (\eta + \bar{\eta})^{-1/2}$. The gauge transformations of the gauge superfield Φ and the matter fields $A_\pm$ are given by

$$\delta_\lambda \Phi = \mathrm{i}G^{-1}(\lambda - \bar{\lambda})\,, \qquad \delta_\lambda A_\pm = \mp \mathrm{i}\lambda A_\pm\,. \tag{1}$$

The classical gauge field kinetic action reads

$$I_\Phi = \int \mathrm{d}^6 z\, \eta F^\alpha F_\alpha + \int \mathrm{d}^6 \bar{z}\, \bar{\eta} \bar{F}_{\dot{\alpha}} \bar{F}^{\dot{\alpha}} \tag{2}$$

with $F_\alpha = \bar{D}^2 D_\alpha (G\Phi)$. For constant coupling, this becomes g-independent. We have the relation

$$\left(\int \mathrm{d}^6 z\, \frac{\delta}{\delta\eta} - \int \mathrm{d}^6 \bar{z}\, \frac{\delta}{\delta\bar{\eta}} \right) I_\Phi = 0\,, \tag{3}$$

therefore it makes sense to require an analogous relation for the vertex functional. For the gauge fixing we choose

$$I_{\mathrm{gf}} = \int \mathrm{d}^8 z\, G^{-2} \bar{D}^2 (G\Phi) D^2 (G\Phi)\,, \tag{4}$$

in addition we include a mass term for the gauge field in order to avoid infrared problems. The total classical action

$$\begin{aligned}\Gamma_{\mathrm{cl}} = {} & -\tfrac{1}{256} \left(\int \mathrm{d}^6 z\, \eta F^\alpha F_\alpha + \int \mathrm{d}^6 \bar{z}\, \bar{\eta} \bar{F}_{\dot{\alpha}} \bar{F}^{\dot{\alpha}} \right) + \tfrac{1}{16} \int \mathrm{d}^8 z\, \left(\bar{A}_+ \mathrm{e}^{G\Phi} A_+ + \bar{A}_- \mathrm{e}^{-G\Phi} A_- \right) \\ & + I_{\mathrm{gf}} + \tfrac{1}{16} M^2 \int \mathrm{d}^8 z\, \Phi^2 + \tfrac{1}{4} m(s-1) \left(\int \mathrm{d}^6 z\, A_+ A_- + c.c. \right)\end{aligned} \tag{5}$$

has a further symmetry,

$$W^3\Gamma_{\rm cl} = 0\,, \qquad W^3 = {\rm i}\int {\rm d}^6 z\, \left(A_+ \frac{\delta}{\delta A_+} + A_- \frac{\delta}{\delta A_-} \right) + c.c. \tag{6}$$

We use the method of algebraic renormalization together with the BPHZL subtraction procedure, in the generalization for superfields [3]. The breaking of gauge invariance is linear, such that the gauge condition

$$w^{(\lambda)}\Gamma = -\tfrac{\rm i}{128\alpha}\bar{D}^2 D^2 (G^{-2}\bar{D}^2(G\Phi)) + \frac{\rm i}{8} M^2 \hbar^n \bar{D}^2 (G^{-1}\Phi) \tag{7}$$

$$w^{(\lambda)} \equiv {\rm i}\bar{D}^2 \left(G^{-1} \frac{\delta}{\delta\Phi} \right) - {\rm i}A_+ \frac{\delta}{\delta A_+} + {\rm i}A_- \frac{\delta}{\delta A_-} \tag{8}$$

can be established also in higher $\hbar$-orders.

For constant coupling, the Callan-Symanzik equation expresses renormalizability of the theory,

$$\left(\mu\partial_\mu + \beta \frac{\partial}{\partial g} - \gamma_\Phi N_\Phi - \gamma_A N_A \right) \Gamma = {\rm soft}\,, \tag{9}$$

where μ runs over all mass parameters and N_Φ, N_A are counting operators for the gauge resp. matter fields. When we pass over to a local coupling, the obvious generalization of the anomalous dimension term for the matter fields is

$$\sum_k \gamma_A^{(k)} g^{2k} \hbar^k \int {\rm d}^6 z \left(A_+ \frac{\delta}{\delta A_+} + A_- \frac{\delta}{\delta A_-} \right) \longrightarrow \sum_k \int {\rm d}^6 z\, \gamma_A^{(k)} G^{2k} \hbar^k \left(A_+ \frac{\delta}{\delta A_+} + A_- \frac{\delta}{\delta A_-} \right), \tag{10}$$

where g is replaced by G. But the right-hand side of (10) is *not supersymmetric*, simply because G is a real superfield and the integral extends only over chiral superspace. So we run into the problem that there is no obvious possibility to formulate a Callan-Symanzik equation for the model with local coupling due to the presence of chiral matter fields. A possible solution to this problem has been proposed in [4]. It consists of introducing an additional external real field L such that the W^3-symmetry is gauged. However, since this contains a γ_5-transformation of the electron field, there is a one-loop anomaly which we express in terms of the operator

$$\mathscr{B}_\Phi^{\rm loc} \equiv \frac{\delta}{\delta\eta} + \tfrac{1}{2}\bar{D}^2 \left(G^2\, \Phi \frac{\delta}{\delta\Phi} \right) . \tag{11}$$

Thus our additional gauge Ward identity reads

$$\tilde{w}^3\Gamma \equiv \left(A_+ \frac{\delta}{\delta A_+} + A_- \frac{\delta}{\delta A_-} - \bar{D}^2 \frac{\delta}{\delta L} + 2\beta^{(1)}\hbar\mathscr{B}_\Phi^{\rm loc} \right) \Gamma = {\rm soft}\,, \tag{12}$$

where "soft" includes soft contributions from the mass and gauge fixing terms. In order to prove this Ward identity, one has to use Zimmermann identities for the mass and gauge fixing terms and use the commutation relation $[\tilde{w}^3(z), \bar{\tilde{w}}^3(z')] = 0$.

From the WI (12) we derive another chiral WI [6]:

$$w^K\Gamma + 2\beta^{(1)}\hbar\left(D^2\mathscr{B}_\Phi^{\rm loc} - \bar{D}^2\bar{\mathscr{B}}_\Phi^{\rm loc}\right)\Gamma + {\rm i}\partial^a V_a^K\cdot\Gamma = {\rm soft} \tag{13}$$

with

$$w^K = D^2\left(A_+\frac{\delta}{\delta A_+} + A_-\frac{\delta}{\delta A_-}\right) - \bar{D}^2\left(\bar{A}_+\frac{\delta}{\delta\bar{A}_+} + \bar{A}_-\frac{\delta}{\delta\bar{A}_-}\right) \tag{14}$$

$$V_a^K = 2{\rm i}\sigma_a^{\alpha\dot\alpha}[D_\alpha, \bar{D}_{\dot\alpha}]\frac{\delta\Gamma_{\rm eff}}{\delta L}\,. \tag{15}$$

(13) describes a chiral transformation of the matter fields which is broken by a 1-loop anomaly.

With the help of the real external field L it is possible to establish a Callan-Symanzik equation also in the case of local coupling,

$$\left(\mu\partial_\mu - \beta^{(1)}\hbar\mathscr{B}_\Phi + \mathscr{B}_A\right)\Gamma = {\rm soft} \tag{16}$$

with

$$\mathscr{B}_\Phi = \int {\rm d}^6z\,\mathscr{B}_\Phi^{\rm loc} + \int {\rm d}^6\bar{z}\,\bar{\mathscr{B}}_\Phi^{\rm loc}\,, \tag{17}$$

$$\mathscr{B}_A = \int {\rm d}^6z\,\mathscr{B}_A^{\rm loc} + \int {\rm d}^6\bar{z}\,\bar{\mathscr{B}}_A^{\rm loc}\,, \qquad \mathscr{B}_A^{\rm loc} = \tfrac{1}{2}\sum_{r,s}\gamma_A^{(r,s)}\hbar^r\bar{D}^2\left(G^{2r}L^s\frac{\delta}{\delta L}\right)\,. \tag{18}$$

The operator $\mathscr{B}_A$ represents here all dilatational anomalies arising from the presence of chiral fermions. Since $\mathscr{B}_A^{\rm loc}$ should commute with $\tilde{w}^3$, the coefficients $\gamma_A^{(r,s)}$ obey certain restrictions.

In the limit of constant coupling, $\eta \to \frac{1}{2}g^{-2}$, and for $L = 0$, the contribution of $\mathscr{B}_A$ may be rewritten using the $\tilde{w}^3$ Ward identity,

$$\mathscr{B}_A\Gamma \to -\gamma_A\left(N_A + 2\beta^{(1)}\hbar\mathscr{B}_\Phi\right)\Gamma + {\rm soft} \tag{19}$$

with

$$\gamma_A = -\tfrac{1}{2}\sum_r \gamma_A^{(r,0)}\hbar^r g^{2r}\,, \tag{20}$$

such that the Callan-Symanzik equation for constant coupling reads

$$\left(\mu\partial_\mu + \beta^{(1)}(1+2\gamma_A)\hbar g^2\left(g\partial_g - N_\Phi\right) - \gamma_A N_A\right)\Gamma = {\rm soft}\,. \tag{21}$$

Thus the full β function is identified as

$$\beta = \beta^{(1)}\hbar g^3(1+2\gamma_A)\,. \tag{22}$$

The superconformal Ward identity (see [3, 5]) reads

$$-16w_\alpha\Gamma = \bar{D}^{\dot\alpha}V_{\alpha\dot\alpha}\cdot\Gamma + \tfrac{4}{3}\beta^{(1)}\hbar D_\alpha\mathscr{B}_\Phi^{\rm loc}\Gamma - \tfrac{4}{3}D_\alpha\mathscr{B}_A^{\rm loc}\Gamma + {\rm soft}\,. \tag{23}$$

$\frac{\delta\Gamma_{\text{eff}}}{\delta\eta}$ may be interpreted as the g-derivative of an effective Lagrangian density, such that in the limit $L=0$, $G=g=const$, (23) reads

$$-16w_\alpha\Gamma = \bar{D}^{\dot\alpha}V_{\alpha\dot\alpha}\cdot\Gamma - \tfrac{4}{3}\beta^{(1)}\hbar g^3(1+2\gamma_A)\partial_g L_{\text{eff}}\cdot\Gamma + \text{soft}$$
$$+\tfrac{2}{3}\beta^{(1)}\hbar g^2(1+2\gamma_A)D_\alpha\bar{D}^2\left(\Phi\frac{\delta\Gamma}{\delta\Phi}\right) + \tfrac{4}{3}\gamma_A D_\alpha\left(A_+\frac{\delta}{\delta A_+} + A_-\frac{\delta}{\delta A_-}\right)\Gamma\,. \quad (24)$$

The second line contains anomalous dimensions for Φ, A_+, A_-. Here, the $\tilde{w}^3$-WI (12) has been used to evaluate the $\frac{\delta}{\delta L}$-term in $\mathscr{B}_A^{\text{loc}}$. Instead, it is also possible to rewrite the $\frac{\delta}{\delta L}$-term as a current contribution and a B-breaking. Of the resulting supercurrent V'_a only the R part is relevant,

$$-16w\Gamma = 2\mathrm{i}\partial^a V'_a\cdot\Gamma + \tfrac{4}{3}\beta^{(1)}\hbar(D^2\Delta_{F^2} - \bar{D}^2\bar{\Delta}_{F^2})\cdot\Gamma + \text{soft} \quad (25)$$
$$V'_a = V_a + \tfrac{16}{3}\gamma_A D^\alpha\sigma_{a\alpha\dot\alpha}\bar{D}^{\dot\alpha}\frac{\delta\Gamma_{\text{eff}}}{\delta L} - \tfrac{32}{3}\mathrm{i}\gamma_A\partial_a\frac{\delta\Gamma_{\text{eff}}}{\delta L} \quad (26)$$
$$\Delta_{F^2} = \left\{\frac{\delta\Gamma_{\text{eff}}}{\delta\eta} + \tfrac{1}{2}g^2\bar{D}^2\left(\Phi\frac{\delta\Gamma_{\text{eff}}}{\delta\Phi}\right)\right\}_{\eta=\frac{1}{2}g^{-2}}\,. \quad (27)$$

This shows that conformal R-symmetry is broken by a one-loop anomaly if one uses the modified current V'_a. The breaking is identical to that of (13), therefore one can combine both WIs to a conserved WI,

$$(-24w + w^K)\Gamma = \mathrm{i}\partial^a(3V'_a - V^K_a)\cdot\Gamma + \text{soft}\,, \quad (28)$$

which corresponds to an R transformation with non-conformal weights for A_+, A_-: we have $n(A_+) = n(A_-) = -1$. With these weights the matter field mass term is invariant, therefore there is no $F^\alpha F_\alpha$ anomaly (The Zimmermann identity for the gauge field mass term does not produce generically chiral breaking terms).

The operator Δ_{F^2} defined in (27) represents a renormalized version of the term $F^\alpha F_\alpha$,

$$[\Delta_{F^2}]\cdot\Gamma = \left[-\tfrac{1}{256}F^\alpha F_\alpha + O(\hbar)\right]\cdot\Gamma + \text{soft mass terms}\,. \quad (29)$$

By applying $\mathscr{B}_\Phi^{\text{loc}}$ to the Callan-Symanzik equation (16), we find

$$W^D\left([\Delta_{F^2}(z)]\cdot\Gamma\right) = [\delta^D\Delta_{F^2}]\cdot\Gamma + (\beta\partial_g - \gamma_\Phi N_\Phi - \gamma_A N_A)\,[\Delta_{F^2}]\cdot\Gamma + \text{soft}\,, \quad (30)$$
$$w^{(\lambda)}(z')\left([\Delta_{F^2}(z)]\cdot\Gamma\right) = \tfrac{\mathrm{i}}{8\alpha}(\Box + \alpha M^2)(\delta^6(z-z')\bar{D}^2(G\Phi))\,. \quad (31)$$

(30), (31) mean that Δ_{F^2} is a *gauge invariant, finite operator.*

To conclude this article, we briefly demonstrate how to derive a nonrenormalization theorem for the two point function $\langle\mathbf{T}\,\bar{A}_+\bar{A}_-\rangle$, closely following [1]. By acting twice with the supersymmetry Ward operator W^Q_α on $\int\mathrm{d}^4x\ \mathscr{B}_\Phi^{\text{loc}}\,\Gamma$, we obtain in the limit $G\to g$

$$W^{Q\alpha}W^Q_\alpha\left[\int\mathrm{d}^4x\ \Delta_{F^2}\right]_3\cdot\Gamma = \int\mathrm{d}^6z\,\mathscr{B}_\Phi^{\text{loc}}\Gamma\,. \quad (32)$$

Using the chain rule and (3) this may be rewritten as

$$W^{Q\alpha}W^Q_\alpha \left[\int \mathrm{d}^4x\ \Delta_{F^2}\right]_3 \cdot \Gamma = -\tfrac{1}{2}g^2\left(g\partial_g - \mathscr{N}_\Phi\right)\Gamma\,. \tag{33}$$

We test this equation w.r.t. $\bar{A}_+(z_1)$, $\bar{A}_-(z_2)$ and take into account that the eigenvalue of $g\partial_g$ is just twice the loop order l:

$$-lg^2\left\langle \mathbf{T}\,\bar{A}_+\bar{A}_-\right\rangle = 4(\bar\theta_1\bar\theta_2\eta^{\mu\nu} + \mathrm{i}\bar\theta_1\bar\sigma_{\mu\nu}\bar\theta_2)\partial^1_\mu\partial^2_\nu\left\langle \mathbf{T}\,\bar{A}_+(z_1)\bar{A}_-(z_2)\int \mathrm{d}^4x\ \Delta_{F^2}\right\rangle\,. \tag{34}$$

The superficial degree of divergence of a supergraph (lowest θ component) is given by [3]

$$d_{min} = 4 - 2N + N_c + N_{\bar c} - \sum_{\text{ext legs}} \dim(\text{ext leg}) + \sum_{\text{vertices}} (\dim(\text{vertex}) - 4)\,, \tag{35}$$

where N is the total number of external legs and N_c, $N_{\bar c}$ are the numbers of chiral resp. antichiral external legs. For the higher θ components one has to add $\frac{1}{2}$ for each θ or $\bar\theta$,

$$d = d_{min} + \tfrac{1}{2}\omega\,. \tag{36}$$

For the left and right hand sides of (34) we find the degrees

$$d_{\text{left,min}} = 0\,, \qquad d_{\text{right,min}} = -1\,. \tag{37}$$

Since the r.h.s. of (34) contains two explicit θs, we have $\omega_{\text{left}} = \omega_{\text{right}} + 2$ and therefore

$$d_{\text{left}} = \tfrac{1}{2}\omega_{\text{left}}\,, \qquad d_{\text{right}} = \tfrac{1}{2}\omega_{\text{left}} - 2\,, \tag{38}$$

i.e. the degree of divergence is improved by two.

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