Geometric Transitions and Strong Coupling Effects in Supersymmetric Field Theories

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Abstract

Geometrical transitions are powerful tools designed to study strong coupling effect in a general class of $\mathcal{N}=1$ theories realized on type IIB D5 branes wrapping 2-cycles of local Calabi-Yau threefolds or as effective field theories on D4 branes in type IIA brane configurations. In this notes we discuss the issue of the classical moduli space for $\mathcal{N}=2, \prod_{i=1}^{n} U(N_i)$ theories deformed by a general superpotential for the adjoint and bifundamental fields. We investigate the geometric transitions in the ten dimensional theories as well as in M-theory. Strong coupling effects in field theory are analyzed in the deformed geometry with fluxes. The connection between the geometrical transition and strong coupling results constitutes a background for the methods of the Dijkgraaf-Vafa approach, where nonperturbative results in field theories can be studied by using perturbative results in matrix models.

1 Geometric Transitions

One of the most exciting directions developed in string theory and field theory in of the last few years was to considerer the duality between open strings (field theory) and the closed string (supergravity). The by-now classical example of such dualities is the Maldacena conjecture [1] between closed strings propagating on AdS spaces and conformal field theories (see [2] for some more recent developments).

The same idea appeared in the context of topological theories and was pursued by Gopakumar-Vafa in studying the duality between Chern-Simons theories on S^3 cycles and closed topological strings on the resolved conifold [3]. The topological transition was lifted to a type IIA transition by Vafa, in the seminal paper [4] where it becomes a geometric transition between D6 branes on the S^3 of the deformed conifold and type IIA strings on the resolved conifold, with fluxes. After a geometric transition of the non-singular Calabi-Yau threefold to another non-singular Calabi-Yau threefold, the D-branes disappear and they are replaced by RR and NS fluxes through the dual cycles, the field theory being described in the large 't Hooft parameter region. It is difficult and cumbersome to generalize the type IIA picture, this is because not much it is known about the Chern-Simons theory for more complicated geometries (nevertheless see [5] for a worldsheet proof of the transition, which is based on the type IIA picture or [11, 12, 13] for developments of the type IIA picture).

The type IIB picture can be obtained as a mirror symmetric picture. The transition becomes one between D5 branes wrapped on the P^1 cycle of the resolved conifold and type IIB on the deformed conifold with fluxes. Even though it was on less firm ground as compared with the type IIA transition, the type IIB picture has been extensively extended to a large class of geometric transition dualities, for geometries which are more complicated than the conifold [14, 15, 16, 17, 19, 20, 21, 22, 23].

More recently, the type IIB geometric transition has become the backbone for a very powerful development, which relates the field theories and matrix models. This is the Dijkgraaf-Vafa procedure [6, 7, 8] which uses field theory, geometry and matrix model as a triangle, where the most important side of the triangle is the one connecting matrix models and field theories. The statement becomes that perturbative results in the matrix models (in the large N limit, when only planar diagrams contribute in matrix model) give information about the non-=perturbative effects in the field theory. The subject is now extensively studied and the list of references is becoming very large (some other important papers are [9, 10]).

In the next section I shall discuss our contribution to the study of the geometric transition, by using an approach based on brane configurations and their lift to M theory [19, 20, 21, 22]. In the last section I shall outline possible connections between our approach and other recent lines of research, and point out some possible future developments.

2 Geometric Transition from MQCD Dynamics

In this section we present our method on dealing with the geometrical transition, in the type IIB approach. The main procedure is to consider $\mathcal{N}=1$ geometries

$$xy - u\prod_{p=1}^{n}\left(u-\sum_{i=1}^{p}W_i'(v)\right)=0. \tag{1}$$

(where W_i are polynomials of v), wrap D5 branes on the different P^1 cycles and take the T-duality along the $U(1)$ direction of a natural $\mathbf{C}^*$ action on the $\mathcal{N}=1$ geometry (1) given by

$$\lambda\cdot(x,y,u,v)\to(\lambda x,\lambda^{-1}y,u,v)\ \text{ for } \lambda\in\mathbf{C}^*. \tag{2}$$

This T-duality gives the dictionary between the geometric engineering construction and the Hanany-Witten type brane construction [24].

The details of the T-duality map is as following

• the D5 branes on the P^1 cycles become D4 branes on intervals as $U(1)$ acts along the angular direction of the P^1's

• the NS branes appear when the $U(1)$ orbits degenerate. When the NS branes are projected onto the $u-v$ plane, they will be a collection of the holomorphic curves given by

$$u\prod_{p=1}^{n}\left(u-\sum_{i=1}^{p}W_i'(v)\right)=0, \tag{3}$$

• if we start in the field theory with an $\mathcal{N}=2$ which is deformed by some superpotential for the adjoint fields, there will be a number of gauge groups in the $\mathcal{N}=1$ theories. The above number of intersection points is in one-to-one correspondence with the $\mathcal{N}=1$ gauge groups.

2.1 The simplest example : The conifold

Consider an action S_c on the conifold given by $xy-uv=0$ as:

$$S_c:\ \ (e^{i\theta},x)\to x,\ \ (e^{i\theta},y)\to y\ \ (e^{i\theta},u)\to e^{i\theta}u,\ \ (e^{i\theta},v)\to e^{-i\theta}v, \tag{4}$$

The orbits of the action S_c degenerates along the union of two intersecting complex lines $y=u=v=0$ and $x=u=v=0$ on the conifold. This action can be lifted to the resolved conifold and deformed conifold. As discussed in [19], the T-dual picture for the D5 branes on the finite 2-cycle of the resolved conifold will be a brane configuration with D4 branes along the interval with two NS branes in the 'orthogonal' direction at the ends of the the interval, the length of the interval being the same as the size of the rigid P^1 [24, 25]. For the deformed conifold, by taking the T-duality, we obtain an NS brane along the curve $u=v=0$ with non-compact direction in the Minkovski space which is given by

$$xy=\mu \tag{5}$$

in the x-y plane.

The above is thus the brane constructions for a conifold, resolved conifold and deformed conifold. Now to study Vafa's duality we appeal to Witten's MQCD construction [26]. We denote the directions of the two NS5 branes along $x^{0,1,2,3,4,5}$ and $x^{0,1,2,3,8,9}$ respectively. The D4 branes are along $x^{0,1,2,3,7}$.

If we lift this configuration to M-theory, all the branes in the picture become a *single* M5 brane with complicated world-volume structure. We define the complex coordinates:

$$x = x^4 + ix^5, \quad y = x^8 + ix^9, \quad t = exp(-R^{-1}x^7 + ix^{10}) \tag{6}$$

where R is the radius of the $11th$ direction, the world volume of the M5 corresponding to the resolved conifold is given by $R^{1,3} \times \Sigma$ and Σ is a complex curve defined, up to an undetermined constant ζ, by

$$y = \zeta x^{-1}, \quad t = x^N \tag{7}$$

Now if we consider the limit where the size of P^1 goes to zero, then the value of t on Σ must be constant because Σ is holomorphic and there is no non-constant holomorphic map into $\S^1$. Therefore the M5 curve make a transition from a "space" curve into a "plane" curve. From (7), we obtain N possible plane curves

$$\Sigma_k : \quad t = t_0, \quad xy = \zeta \exp 2\pi ik/N, \quad k = 0, 1, \ldots, N-1. \tag{8}$$

2.2 General $\mathcal{N} = 1$ geometries

It is possible to generalize the above construction to general geometries, which describe SU groups (in the absence of the orientifolds), SO/Sp theories (in the presence of orientifolds), gauge theories with matter (by putting D5 branes on some extra non-compact P^1 cycles).

If we start from a general $\mathcal{N} = 2$ A-D-E quiver theory, after deformation we get an $\mathcal{N} = 1$ A-D-E quiver theory which in the IR limit is equivalent to a pure $\mathcal{N} = 1$ gauge theory. So we expect to have gaugino condensation and mass gap as noticed in [16]. In the large N description, the theory lives on a geometry where the P^1 cycles have shrunk and S^3 cycles have grown, together with RR fluxes through the S^3 cycles and NS fluxes through their dual cycles have been created.

The $\mathcal{N} = 1$ geometry for the A_n quiver theory with general superpotential W for the adjoint field is the minimal resolution of a Calabi-Yau threefold defined in $\mathbf{C}^4$ by

$$X : xy - \prod_{p=0}^{n} \left(u - \sum_{i=0}^{p} W_i'(v) \right) = 0 \tag{9}$$

where W_0' is defined to be zero. The singularities are isolated and located at $x = y = 0$ and the intersection of any two curves in the $u - v$ plane defined by

$$u = \sum_{i=1}^{j-1} W_i'(v), \quad u = \sum_{i=1}^{k} W_i'(v). \tag{10}$$

The singularities can be resolved by successive blow-ups which replace each singular point by a P^1 cycle. Therefore we see that the number of P^1 cycles match the number of the Higgsed gauge groups. The resolved space is covered by $(n+1)$ three dimensional complex spaces $U_p, p = 0, \dots n$ with coordinates

$$u_p = \frac{\prod_{j=0}^{p}\left(u - \sum_{i=0}^{j} W_i'(v)\right)}{x}, \quad x_p = \frac{x}{\prod_{j=0}^{p-1}\left(u - \sum_{i=0}^{j} W_i'(v)\right)}, \quad v_p = v, \tag{11}$$

where $x_0 = x$. They blow down to the singular threefold (9) by

$$\sigma: \qquad \tilde{X} := U_0 \sqcup U_2 \sqcup \dots \sqcup U_n \to X, \tag{12}$$

$$U_p \ni (u_p, x_p, v_p) \mapsto \begin{cases} x = \begin{cases} x_0 \text{ if } p = 0, \\ x_p \prod_{j=0}^{p-1}\left(x_p u_p + \sum_{i=j+1}^{p} W_i'(v_p)\right) \text{ otherwise} \end{cases} \\ y = u_p \prod_{j=p+1}^{n}\left(x_p u_p - \sum_{i=p+1}^{j} W_i'(v_p)\right) \\ u = x_p u_p + \sum_{i=0}^{p} W_i'(v_p) \\ v = v_p \end{cases} \tag{13}$$

where $U_p \sqcup U_{p+1}$ means that the three spaces U_p, U_{p+1} are glued together by

$$x_{p+1} = u_p^{-1}, \ v_{p+1} = v_p, \ u_{p+1} = x_p u_p^2 - W_{p+1}'(v_p) u_p. \tag{14}$$

Thus the complex lines $\mathbf{C}^1$ defined by

$$W_{p+1}'(v_p) = 0, \quad x_p = 0 \tag{15}$$

in U_p together with the complex lines $\mathbf{C}^1$ in U_{p+1} defined by

$$W_{p+1}'(v_{p+1}) = 0, \quad u_{p+1} = 0 \tag{16}$$

form the P^1 cycles and there are no other P^1 cycles in $U_p \sqcup U_{p+1}$. This is a generalization of the A_1 quiver theory considered in [14]. While the P^1 in the resolution of the A_n singularity can move freely in the v-direction, the above P^1 cycles are frozen at

$$W_{p+1}'(v) = 0. \tag{17}$$

Hence the supersymmetry is broken from $\mathcal{N} = 2$ to $\mathcal{N} = 1$.

We then blow-down the P^1 cycles to obtain a singular geometry, which is then deformed by a number of S^3 cycles. The deformations can be normalizable or non-normalizable, y counting the number of normalizable and non-normalizable S^3 cycles in the blown-down geometry, we obtained that there are $mn(n+1)/2$ normalizable and $mn(n+1)/2 - n$ non-normalizable, the first one corresponding to the gluino condensates in the $\mathcal{N} = 1$ theories and the latter to the vevs of the bifundamental fields. Of course, the appearing of the non-normalizable S^3's is completely fixed by the superpotential so by the expectation values of the adjoints and bifundamentals. Having all non-dynamical deformations fixed by expectation values of the adjoints and bifundamentals (in geometry, there are only conifold singularities in the blown-down geometry), the $\mathcal{N} = 1$ geometry (12) can go through the geometric transition where the rigid P^1's will disappear and will be replaced by the finite size of S^3.

We can also discuss the Seiberg duality which takes place in the presence of quark chiral superfields in the fundamental representation. At present, we only have an understanding for the massive fundamental quarks [20], which correspond to D5 branes wrapped on P^1 cycles, other that the exceptional one. Here we consider the case of massive fundamental fields, which will be integrated out in the infrared which changes the scale of the $\mathcal{N}=1$ theory so changes the flux on the S^3 cycles. If we have massless flavors, the story becomes more complicated and we do no have yet a clear understastind (see next section for some steps on this direction). By introducing orientifolds., one can describe the supersymmetry breaking for the SO/Sp groups and there is no further complication due to the presence of the orientifolds.

3 Relation to Other Approaches

Our results provide a general approach to deal with the strong coupling results in field theory, by using the brane configurations and transitions in MQCD. As brane configurations have been extensively studied in the literature, we expect that our method can be used for a large class of geometrical transitions.

One important question is whether we can compare our type IIA brane configuration construction with other type IIA constructions as the ones in [11, 12, 13]. This would relate our MQCD approach with M-theory curves to M-theory with G_2 manifolds. Moreover, one may be able to construct the various G_2 holonomy manifolds which have been used in M-theory lifts of the geometrical type IIA transitions.

Our type IIA picture has been obtained by taking one T-duality from type IIB picture. Hence we need to take two more T-dualities to obtain the mirror type IIA. It would be interesting to work out in details for the toric cases as in [11, 12, 13] to see how two type IIA picture appear and how their M-theory transitions are related. More generally, the geometry is not toric and there are no obvious three T-dualities one can take. In the case of degenerate superpotential with the same degree, the geometry become quasi-homogeneous and there is extra $U(1)$ action. Brane configurations have been extensively studied during the last years and in the present paper we extend them to more general cases. Therefore, after two extra T-dualities we could get a rich class of G_2 manifolds.

Another possibility is to relate our results to the new developments of Dijkgraaf-Vafa. In this case, we use the fact that geometrical transitions are the background of the matrix model/gauge theory correspondence [6, 7, 8]. The quantities to be compared are the Seiberg-Witten curve in field theory, the hyperelliptic curve obtained from the deformed geometry if this is seen as a S^2 fibration and the resolvent equation in the matrix model. Our previous discussion referred only to $SU(N_c)$, $\mathcal{N}=2$ SUSY theories without fundamentals, when the supersymmetry is broken to $\mathcal{N}=1$ SUSY by a superpotential for the adjoint field. It is interesting to include fundamental matter in the theory and to see the MQCD transition. The fundamental massless flavors would be described in the geometrical set-up by additional D5 wrapped on additional non-compact cycles of the conifold or to additional D7 branes. By a T-duality, in the brane configuration they become D6 brane for massless flavors or semi-infinite D4 branes for massive flavors.

By lifting it to MQCD and going through the transition, we should recover the moduli space of the field theory from the geometry. As the transition consist on closing the interval between the two NS branes (so going from a curved M5 brane to a plane M5 brane [26, 19, 20]), we expect to get a geometry described by an equation $xy = a$ in the (x, y) plane. But [27] tells us that the only case when the $xy \neq 0$ is when the mass of the $\mathcal{N} = 2$ adjoint field is not ∞. This is consistent with the fact that there is no supersymmetric vacua for the corresponding $\mathcal{N} = 1$ theory. In the case of infinite mass for the the adjoint fields, after integrating it out in the matrix theory we remain with decoupled integrals over the massless fundamental quarks (with no tree level superpotential) and, as discussed before, this gives the ADS superpotential, which removes any supersymmetric vacua. If the adjoint has a finite mass, then there are additional terms in the superpotential, which would give rise to stable supersymmetric vacua. It would be interesting to develop the discussion for the case of finite mass for the adjoint field.

As the MQCD approach should give useful information aboutb the field theory, the above mentioned connections are not unexpected, but it is very interesting to see how different methods come together to give very similar results.The importance of the different approaches to geometrical transitions is that we can even get results beyond the field theory, for the coupling with the gravity.

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