AdS/CFT Correspondence

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- J. Maldacena, "The large N limit of superconformal field theories and supergravity," hep-th/9711200 (2240 citations)
- S. Gubser, I. Klebanov, A. Polyakov, "Gauge theory correlators from noncritical string theory," hep-th/9802109 (1491 citations)
- E. Witten, "Anti-de Sitter space and holography," hep-th/9802150 (1636 citations)

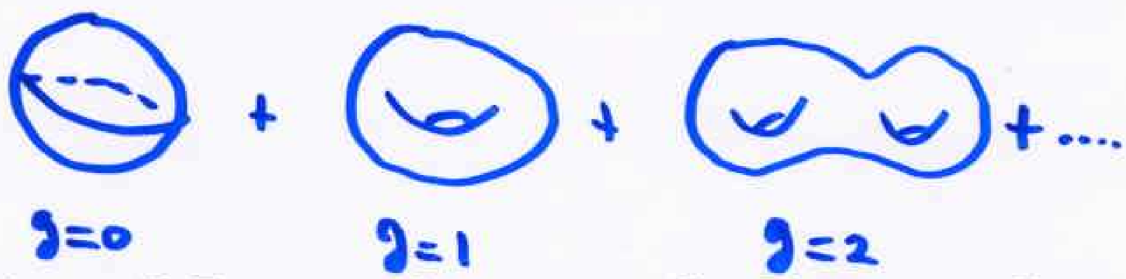
The AdS/CFT correspondence is related to two deep ideas:

1) Large N gauge theory is a string theory ('t Hooft).

$$Z = \sum_g N^{2-2g} f_g(\lambda)$$

with $\lambda = g_{YM}^2 N$ the 't Hooft coupling. This is similar to the loop expansion of string theory

$$Z = \sum_g g_s^{2g-2} Z_g$$



Large N Feynman diagrams are like Riemann surfaces with holes, and some dynamical mechanism "closes the holes" leaving a closed string theory (Ooguri, Vafa, hep-th/0205297).

2) The holographic principle ('t Hooft, Susskind).

Bekenstein-Hawking entropy of black holes

$$S = \frac{A}{G_N}$$

Thus, entropy is bounded by area rather than volume. This is a very peculiar property of (quantum) gravity, and suggests that quantum gravity in d dimensions can be equivalent to a local field theory in $d-1$ dimensions.

Ignoring the S^5, this is what happens here. String theory (a theory with quantum gravity) on AdS_5 is equivalent to $N = 4$ SYM (a theory without gravity) in four dimensions.

We can hope to:
Use string theory to learn about gauge theory.
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AdS_5 can be described as the space

$$-X_0^2 - X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 = L^2$$

in a space with metric

$$ds^2 = -dX_0^2 - dX_1^2 + dX_2^2 + dX_3^2 + dX_4^2 + dX_5^2$$

(Part of) AdS_5 is described by

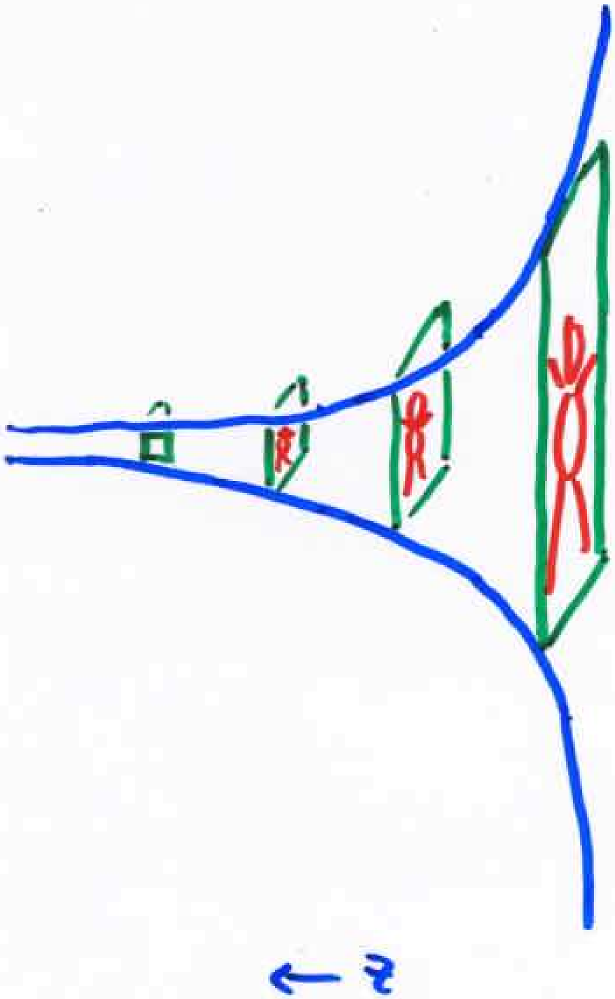
$$ds^2 = \frac{L^2}{z^2}(dz^2 - dt^2 + dx_1^2 + dx_2^2 + dx_3^2).$$

The conformal boundary of AdS_5 is at $z = 0$, and this is where the field theory lives. z sets the scale of the t, x_1, x_2, x_3 part of the metric, and z can quite literally be viewed as a scale of the dual field theory.

A free massive scalar field with equation of motion $(\Box + m^2)\phi = 0$ has solutions that behave as

$$z^{\Delta}, \qquad z^{4-\Delta} \quad \text{as } z \to 0$$

with $\Delta(\Delta - 4) = m^2$.


← z

Consider now a solution of the supergravity equations of motion with the boundary conditions that the fields behave as

$$\phi_i(z,t,\vec{x}) \sim \phi_i^0(t,\vec{x}) z^{4-\Delta}$$

for $z \to 0$.

The mapping between correlation functions of SYM theory and quantities in string theory is now given by

$$\exp(-\Gamma_{\text{sugra}}(\phi_i)) = \left\langle \exp\left(\int d^4x \phi_i^0 \mathcal{O}_i \right) \right\rangle$$

where the left hand side is the supergravity action evaluated on ϕ_i, and the right hand side is a generating function for correlation functions in SYM theory. The field $\mathcal{O}_i$ has scaling dimension Δ_i.

Mapping between parameters:

String theory on $AdS_5 \times S^5$ has
-a string coupling constant g_s
-a string length l_s, which is a dimensionful parameter that sets the size of the fluctuations of the string world-sheet
-the curvature radius L of AdS_5 and S^5

$N = 4$ $d = 4$ SYM theory with gauge group $SU(N)$ has, besides N, a dimensionless coupling constant g_{YM}^2. The theory is conformally invariant.

The identification reads

$$g_s = g_{YM}^2, \qquad (L/l_s)^4 = 4\pi g_{YM}^2 N = 4\pi\lambda$$

String theory has an expansion

$$Z = \sum_g g_s^{2g-2} Z_g(\lambda^{-1/2}) = \sum_g N^{2-2g} \tilde{Z}_g(\lambda^{-1/2})$$

and the supergravity approximation is valid for large N and large $\lambda = g_{YM}^2 N$.

The derivation of the AdS/CFT duality starts with a stack of N D3-branes.

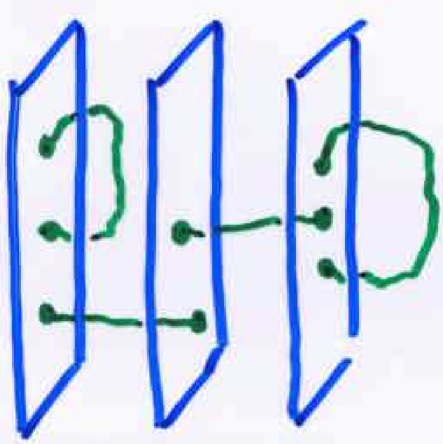
D-branes are dynamical objects in string theory on which open strings can end. These open strings carry gauge degrees of freedom, and give rise to a four-dimensional supersymmetric $SU(N)$ gauge theory living on the brane.

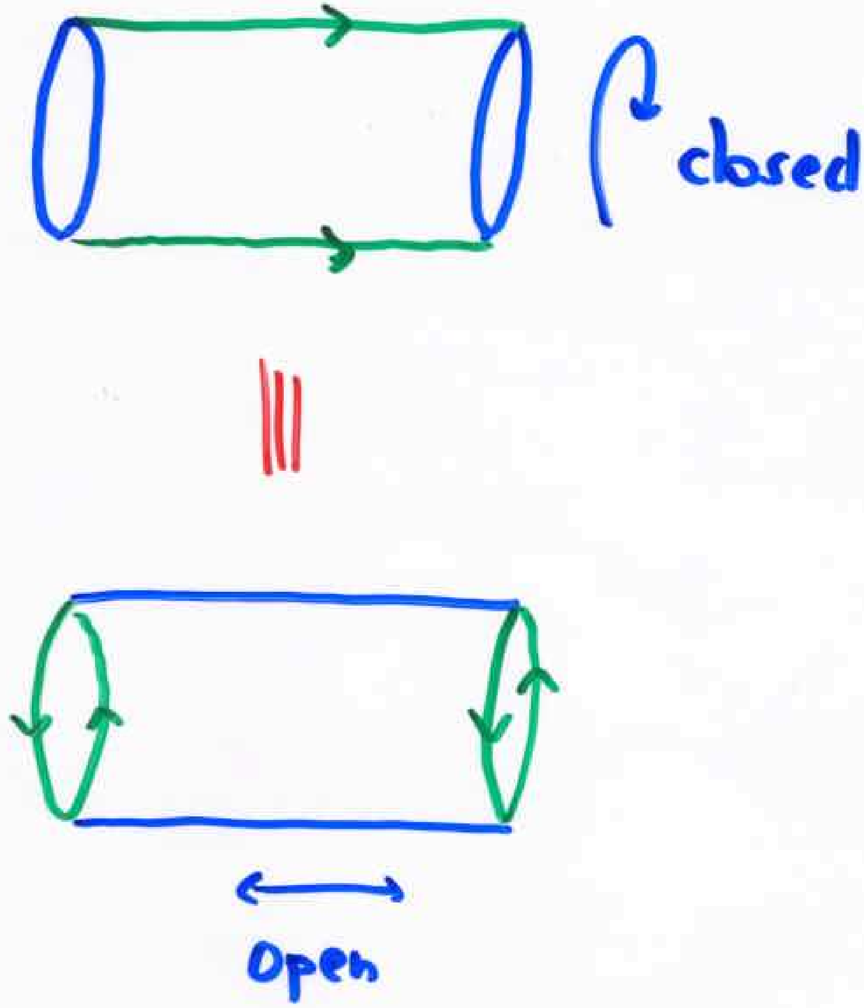
(poor analogy: pointlike magnetic monopoles on which magnetic field lines can end versus the 't Hooft-Polyakov monopole).

At the same time, D-branes can be described from the closed string point of view as a non-trivial black hole-like solution of the field equations of the closed string. The gauge degrees of freedom arise as collective modes of this solution.

By taking a certain limit of both descriptions, the first one reduces to pure gauge theory, the second one to string theory on $AdS_5 \times S^5$.

The AdS/CFT correspondence is closely related to open/closed string duality.




closed
Open

Tests of the AdS/CFT correspondence

The correspondence states that for every gauge-invariant operator in the gauge theory there should be a corresponding closed string field on $AdS_5 \times S^5$, whose mass is related to the scaling dimension of the operator.

Difficult to test in full generality, because we don't know the spectrum of SYM at strong coupling. We know the spectrum of a subset of the operators at strong coupling, so-called BPS operators.

If Q is a hermitian supersymmetry operator that satisfies $Q^2 = \Delta + J$, with J some $U(1)$ generator, then a state annihilated by Q must have $0 = |Q|\psi\rangle|^2 = (\Delta + J)|\psi\rangle|^2$ so $\Delta = -J$. The eigenvalues of J are quantized, and therefore Δ cannot depend on the coupling constant and the weak coupling answer is equal to the strong coupling answer.

In string theory, there are massless and massive fields in ten-dimensions. The dimension of operators corresponding to massive fields scales as $(L/l_s) \sim (g_{YM}^2 N)^{1/4}$. These are not BPS and cannot be compared.

The massless fields have $g_{YM}^2 N$ independent scaling dimensions and should correspond to the BPS operators.

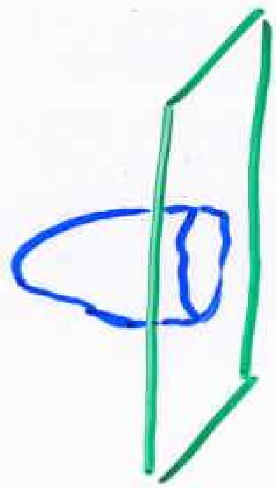
There is indeed a precise agreement between the two sets of operators. A first necessary condition for this is that the symmetries match. $AdS_5 \times S^5$ has isometry group $SO(2,4) \times SO(6)$, whereas $N = 4$ SYM has conformal group $SO(2,4)$ and R-symmetry $SO(6)$.

correlation functions

Correlations functions have been considered in considerable detail. The results are quite encouraging. Some correlators satisfy non-renormalization theorems, and those can be compared. AdS/CFT has also suggested new non-renormalization theorems, which remain to be proven in field theory, for extremal correlation functions of the type

$$\langle \mathcal{O}_\Delta(x)\mathcal{O}_{\Delta_1}(x_1)\ldots\mathcal{O}_{\Delta_n}(x_n)\rangle$$

where $\Delta = \sum_i \Delta_i$.



Wilson loops

A Wilson loop on the boundary corresponds to a fundamental string worldsheet on AdS which ends on that boundary (Maldacena).

From this one finds that the quark-quark energy with separation r is

$$E = -\frac{4\pi^2(2g_{YM}^2 N)^{1/2}}{\Gamma(1/4)^4 r}$$

Notice the power of $(g_{YM}^2 N)$ that appers.

The fundamental string is the QCD string, as in the original approach to string theory!

finite temperature

So-called near-extremal D-branes can be used to compute the free energy at finite temperature.

We get

$$F_{sugra} = -\frac{\pi^2}{8} N^2 V T^4 \qquad F_{SYM} = -\frac{\pi^2}{6} N^2 V T^4$$

Is there a smooth interpolation between these results or a phase transition?

More generally, we can consider the situation where the boundary R^4 is replaced by $S^3 \times S^1$, so that imaginary time is periodic (Witten).

There are two five-manifolds that solve the closed string equations of motion and have $S^3 \times S^1$ as conformal boundary: a finite temperature version of Euclidean AdS, and a Schwarzschild black holes in AdS.

The first dominates the supergravity path integral at low temperatures. The expectation value of a Wilson loop vanishes, and the center of the gauge group is unbroken, as in a confining phase.

The second solution has nonzero expectation value for the Wilson loop, and the center is broken, as in a deconfining phase.

The confinement-deconfinement transition is mapped to topology change in gravity!

glue balls, string tension

Take $N = 4$ SYM at finite temperature and send $T \to \infty$ limit. Finite T breaks supersymmetry and one may hope to study non-supersymmetric QCD_3 in this way.

Glueball masses are computed using field equations in the bulk, and string tensions using the Wilson loop.

There is considerable disagreement with lattice results (Caselle). The string theory has additional states with mass $\sim T \sim \Lambda_{QCD}$, which is also the mass scale of glueballs. No way as of yet to remove the cutoff and reach a weak coupling limit.

The string of AdS/CFT is different from the strong coupling lattice result which in turn is different from the continuum lattice result. The latter two are separated by a phase transition. Are there any such phase transitions in AdS/CFT?

Other Solutions

There are many other solutions involving AdS_p for various values of p. These are dual to suitable conformal field theories. In particular, there is type IIB string theory on $AdS_3 \times S^3 \times T^4$, which is dual to a $1+1$ CFT. The latter are relatively well-understood, and it is precisely this CFT that featured in the first counting of microstates of a black hole by Strominger and Vafa.

One way to reduce the supersymmetry is by dividing by a discrete group. If G is a discrete subgroup of $SO(6)$, we can consider $AdS_5 \times S^5/G$, which is dual to a gauge theory with less supersymmetry (typical of quiver type, or moose type, or deconstruction type).

Another possibility is to deform the gauge theory by a relevant operator. In some cases, one can solve for the corresponding supergravity solutions; these are then dual to RG flows that flow between a UV and an IR fixed point.

For such flows one can prove a c-theorem, where c is a quantity that appears in the conformal anomaly of the dual theory.

Actually, we made a simplification when defining the correspondence. One really has to choose a cutoff $z = z_0 \neq 0$, evaluate everything on this space with a cutoff, and finally take the cutoff to zero.

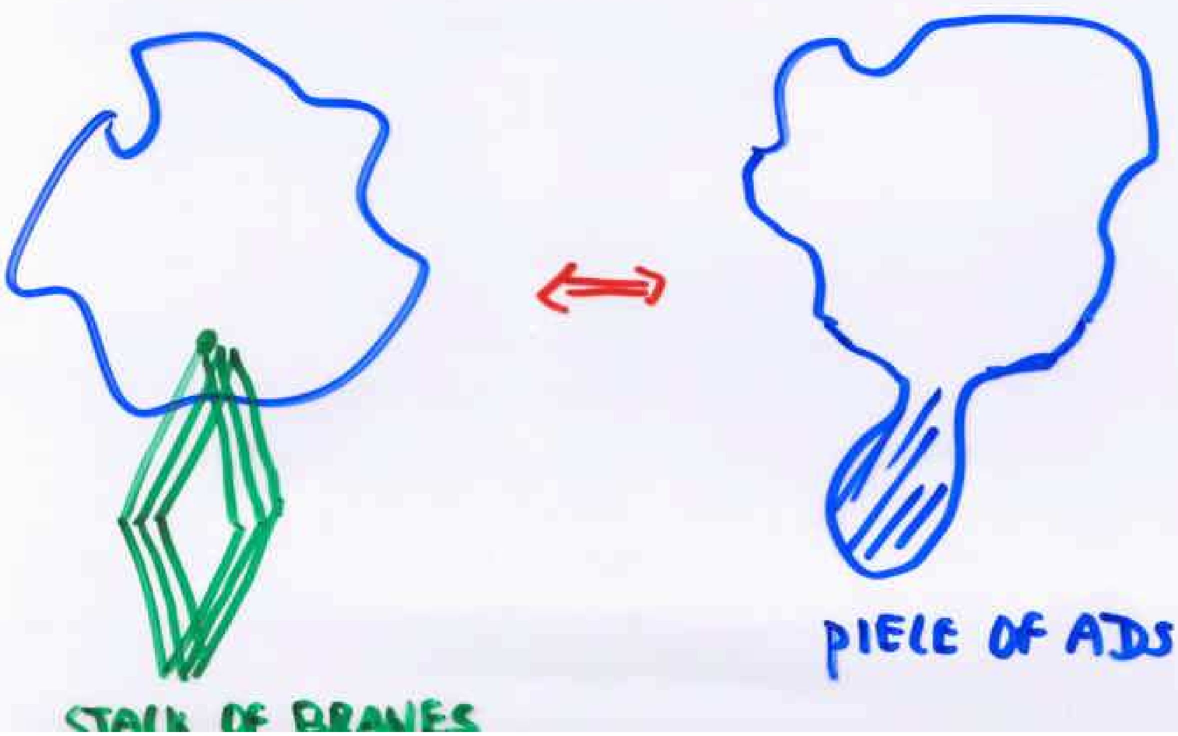
In this process, the supergravity action diverges. It can be rendered finite by subtracting local counterterms, exactly as in standard renormalization of field theories.

In this way, the fifth dimension is literally identified with a scale in the dual field theory, and one can consider a coordinate dependent cutoff.

Changing the cutoff is now a geometric procedure, and described by the equations of motion of supergravity. When written in Hamilton-Jacobi form, these equations turn into the Callan-Symanzik renormalization group equations of the field theory.

One can also consider the AdS/CFT correspondence with a finite cutoff. In that case, the dual theory is a field theory with a cutoff, coupled to gravity. One can choose either description, the most weakly coupled one will typically be the most useful.

This is important for brane world scenarios and generic string compactifications. The latter contain typically branes, and whether one uses the actual brane description or a dual supergravity description depends on the number of branes, their coupling constants and the curvature of the ambient geometry.



The Klebanov-Strassler solution

An interesting $N = 1$ theory can be made by starting with a conifold singularity

$$zw - uv = 0$$

and putting N D3 branes and M wrapped D5 branes at the singularity.

The resulting gauge theory has gauge group $SU(N) \times SU(N+M)$, two chiral superfields A_i in the $(N, \overline{N+M})$ representation, and two chiral superfields B_i in the $(\overline{N}, N+M)$. In addition, there is a superpotential

$$W \sim \epsilon^{ij}\epsilon^{kl}\text{tr}(A_i B_k A_j B_l)$$

The field theory goes running gauge couplings, and once a coupling gets strong the theory admits a Seiberg-dual description. This, in turn has running couplings, and the process continues. The process continues indefinitely in the UV, and continues in the IR until e.g. a gauge group $SU(M)$ is reached.

This "duality cascade" is nicely reproduced in the dual description, where N and M become non-trivial functions of the radial coordinate z

In addition, these supergravity solutions are non-singular, and exhibit:

- confinement
- glueballs and baryons with a mass scale that emerges through dimensional transmutation
- gluino condensates that break the $\mathbf{Z}_{2M}$ chiral symmetry to $\mathbf{Z}_2$
- domain walls separating different vacua

The hard scattering of glueballs at high-energy in confining gauge theories can be understood using AdS/CFT correspondence. Although string scattering is soft, the warping of the geometry compensates for this. (Polchinski, Strassler)

Actually, strong gravity processes, including black hole formation, play an important role in the string description of such processes, and may saturate the Froissart bound $\sigma \sim \log^2 E$. (Giddings)

One can also study other aspects of the parton model (Polchinski, Susskind)

Another recent development: pp waves (Berenstein, Maldacena, Nastase).

Idea: take a scaling limit of the AdS/CFT duality. In this "Penrose limit" the AdS space reduces to a pp-wave background of string theory which are exactly solvable sigma models. In particular, we know the free string spectrum.

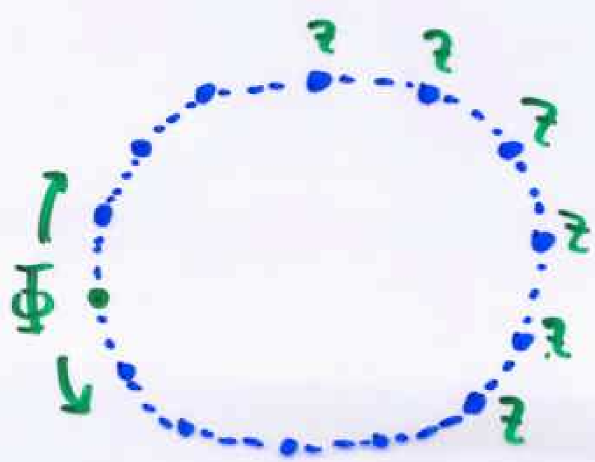
From the field theory point of view, this is a large N limit, where $\Delta + J$ scales as L^2, but $\Delta - J$ remains of order one. (Here J is the generator of some distinguished global $U(1)$ symmetry).

Amazingly, it seems one can recover the complete string spectrum from the gauge theory.

ground state: $\mathrm{tr}(Z^J)$

string states $\sim \sum_k e^{2\pi i k/J} \mathrm{tr}(Z^i \Phi Z^{J-i})$

Realizes the "string bit" model.



The strong form of the AdS/CFT correspondence:

There is an exact equivalence between type IIB string theory on $AdS_5 \times S^5$ and $d = 4$ $N = 4$ supersymmetric Yang-Mills (SYM) theory.

Hard to prove because we don't have a definition of non-perturbative type IIB string theory; even at string tree level we do not (yet) know how to solve the theory.

From this point of view, $N = 4$ SYM theory can be viewed as the definition of non-perturbative string theory on this particular background.

The weakest form of the AdS/CFT correspondence:

The large $g_{YM}^2 N$, large N limit of $N = 4$ $d = 4$ SYM theory is equivalent to classical type IIB supergravity on $AdS_5 \times S^5$.

This statement has been very well tested by now.

Outlook

- Identify the dual string theory directly for more realistic gauge theories.
- Learn more about black holes and quantum gravity using field theory methods.
- Find holography for flat space.
- Understand the dual description of asymptotically free theories.
- Find a dual of $N = 1$ QCD.
- Solve string theory on AdS at tree level, in particular at strong coupling (= free SYM at large N).
- And many, many more