Nonabelian Gauge Theories on Noncommutative Spaces

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Abstract

A formalism is presented where gauge theories for nonabelian groups can be constructed on a noncommutative algebra.

1 Introduction

Noncommutative coordinates have been with us almost as long as quantum field theory. After the awareness of singularities in such theories Heisenberg [1] proposed as early as 1930 in a letter to Peierls noncommuting space coordinates that lead to uncertainty relations for these variables and could help to eliminate the singular behaviour e.g. of the self-mass of the electron without having to introduce a smallest length. Heisenberg could not formulate this idea mathematically, in this letter he asked for the help of Peierls and Pauli. Thus this idea propagated via Pauli and Oppenheimer [2] and in 1943 H.S.Snyder, a Ph.D. student of Oppenheimer, published a paper on "Quantized Space Time" [3]. Snyder treated the coordinates as a Lie-algebra and based on quantum mechanics gave a profound discussion on the interpretation of such coordinates. Pauli found his work ingeneous, "however, it seems to be a failure for reasons of physics" [4].

The singularities are still with us. The fight against singularities in quantum field theory as it was called by Pais, however, proved rather successful. It led to gauge theories that are capable to describe all the present experimental facts found in our laboratories (standard model). It led to supersymmetry and in the more ambitious aim to formulate the law of gravity as quantum gravity it also led to string theory. Many more ideas have been created and many of them have proved to be "a failure" - so why not try to review the idea of noncommuting coordinates again. In a way supersymmetry has already gone in this direction by the use of superspace. Since Snyder's work our mathematical tools as well as the experimental art in high energy physics have seen an enormous improvement.

Recently noncommutative coordinates have emerged from string theory as well [6]. Their commutation relations are

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \tag{1}$$

with constant $\theta^{\mu\nu}$. This example of noncommutative coordinates has found most attention lately. The mathematical structure is exactly the same as the canonical structure in quantized phase space. Here we will always deal with the commutation relations [1] as an algebra for coordinates.

We do not see any sign of noncommutative coordinates at large distances. At small distances, much smaller than the confinement range, coordinates might not commute. This should be achieved e.g. by a more complicated x-dependent $\theta^{\mu\nu}$. We do not know a proper $\theta^{\mu\nu}(x)$ that could lead to such a property. A constant as well as any polynomial is certainly a bad approximation to such a function. We could, however, look at processes that take place at very short distances where a constant nevertheless is a good approximation. The disappearance of the noncommutativity should then occur at distances where the process has ceased to be possible. This is in agreement with our theoretical idea that noncommutativity should cure the ultraviolet behaviour of the theory, thus we expect the changes at very short distances.

Let me now show what modification of a gauge theory we have to expect as a consequence of [5]. We start from a gauge theory with an arbitrary non-Abelian gauge group. At this level the particle content of the theory can be fixed by choosing the appropriate representations. The step to the noncommutative modifications of the theory will be implemented by a power series expansion in $\theta^{\mu\nu}$.

The parameter $\theta^{\mu\nu}$ that parametrizes the noncommutativity will now enter the Lagrangian as a coupling constant. To first order in θ the Lagrangian changes as follows:

$$\begin{aligned}
& -\frac{1}{4}\mathrm{Tr}\int F_0^{\mu\nu}F_{0\mu\nu}\,\mathrm{d}x \\
\Rightarrow & -\frac{1}{4}\mathrm{Tr}\int F_0^{\mu\nu}F_{0\mu\nu}\,\mathrm{d}x \\
& +\frac{1}{8}\theta^{\kappa\lambda}\mathrm{Tr}\int F_{0\kappa\lambda}F^0_{\mu\nu}F^{0\mu\nu}\,\mathrm{d}x \\
& -\frac{1}{2}\theta^{\kappa\lambda}\mathrm{Tr}\int F^0_{\mu\kappa}F^0_{\nu\lambda}F^{0\mu\nu}\mathrm{d}x
\end{aligned} \tag{2}$$

$F_{0\mu\nu}$ is the usual field strength in a nonabelian gauge theory. We see that to first order triple vertices of the gauge field occur, giving rise to new phenomenological predictions.

For the coupling of the gague fields to the matter fields we obtain

$$\begin{aligned}
& \int \bar\Psi^0(\gamma^\mu D_\mu - m)\Psi^0\,\mathrm{d}x \\
\Rightarrow & \int \bar\Psi^0(\gamma^\mu D_\mu - m)\Psi^0\,\mathrm{d}x \\
& -\frac{1}{2}\theta^{\kappa\lambda}\int \bar\Psi^0 F_{0\kappa\lambda}(\gamma^\mu D_\mu - m)\Psi^0\,\mathrm{d}x \\
& -\frac{1}{4}\theta^{\kappa\lambda}\int \bar\Psi^0\gamma^\mu F^0_{\mu\kappa}D_\lambda\Psi^0\,\mathrm{d}x
\end{aligned} \tag{3}$$

Again new couplings of the gauge field to the matter fields occur.

As we do not know much about the renormalizability of the theory there is no notion of a "minimal coupling". We consider the Lagrangians as effective Lagrangians, in this case all gauge invariant couplings are allowed and should be considered.

This formalism can be applied to the standard model with the gauge group $SU(3) \times SU(2) \times U(1)$ [6].

The particle content is the same as in the ordinary standard model. For $\theta = 0$ the usual standard model is reproduced.

In general the new interactions will contribute to processes already present in the standard model. Due to the tensorial character of the "coupling constant" $\theta^{\mu\nu}$ Lorentz invariance will be violated for the new interactions. This makes these contributions different from the usual ones and it leads to signatures by which these interactions can be identified. The most interesting case is when the new terms lead to a process that is not allowed in a usual local and Lorentz invariant quantum field theory. An example is the $Z^0 \to \gamma\gamma$ decay [7]. Forbidden in conventional QFT, it can be obtained to first order in θ:

$$\begin{aligned} L_{Z^0\to\gamma\gamma} = {} & \frac{e}{8}\sin 2\theta_W \left(g'^2\kappa_1 + (g'^2 - 2g^2)\kappa_2\right)\theta^{\rho\tau}\Big[2Z^{\mu\nu}\left(2A_{\mu\rho}A_{\nu\tau} - A_{\mu\nu}A_{\rho\tau}\right) \\ & + 8Z_{\mu\rho}A^{\mu\nu}A_{\nu\tau} - Z_{\rho\tau}A_{\mu\nu}A^{\mu\nu}\Big], \end{aligned} \tag{4}$$

where the constants κ_1, κ_2 are defined in reference [8].

This leads to a partial width ζ in the rest frame of Z, spin averaged

$$\Gamma_{Z\to\gamma\gamma} = \frac{\alpha}{12}M_Z^5\sin^2(2\Theta_W)K^2(g,g')\left[\frac{7}{3}\vec{\Theta}_T^2 + \vec{\Theta}_S^2\right] \tag{5}$$

Details can be found in ref [8].

A similar argument holds for processes like

$$\Upsilon \to \gamma\gamma, \quad B \to K\gamma, \quad K \to \pi\gamma$$

Even a violation of the CPT-Theorem would be possible [8].

To construct this new class of deformed QFTs a few new ideas have to be incorporated.

1. Noncommutative coordinates [9] have to be accepted. The canonical case (1) is the simples example. As mentioned before we expect x-dependent θs. At the moment we only know how to treat examples that are linear or quadratic nx. These in a way are the first terms in a power series expansion of θ in x.

The constant case is created by Lie algebras. A particularly interesting case is the Lie-algebra [10]

$$[x^\mu, x^\nu] = i\left(a^\mu x^\nu - a^\nu x^\mu\right) \tag{6}$$

where a^μ is constant. This noncommutative space allows a deformed Lorentz group acting on it. This is the κ-deformation of the Lorentz group pioneered by Majid, Lukiersky and Rueckl [10].

The quadratic case is known in the context of quantum groups. These spaces are called quantum spaces [11] and have been thoroughly investigated within the last twenty years. The additional structure of a quantum group or a κ-deformed group gives rise to many restrictions of the gauge theory to be built by asking for a deformed Lorentz symmetry of the theory. An investigation of these models is therefore of great interest.

2. A star product [11] as it is known in the deformation quantization can be used to realize the algebra. In this formalism the objects of the algebra are functions of commuting variables, the noncommutativity is present in the noncommutative star product. For the canonical case this is

$$f \star g(x) = e^{\frac{i}{2}\frac{\partial}{\partial x^\mu}\theta^{\nu\mu}\frac{\partial}{\partial y^\mu}} f(x) \cdot g(x)\Big|_{y \to x} \tag{7}$$

From this star product follows immediately

$$[x^\mu \overset{\star}{,} x^\nu] = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu} \tag{8}$$

The star product originally defined for polynomials, extended to formal power series, has been extended to functions. These functions are the objects in physics, they are identified with fields.

3. Enveloping algebra valued [12] gauge transformations have to be used. A gauge theory is based on a Lie algebra:

$$[L^i, L^j] = i f_k^{ij} L^k \tag{9}$$

For the gauge transformation of fields the star product has to be used as well, or, which is the same, the gauge parameters depend on the noncommutative variables:

$$\delta\Psi = i\Lambda(x) \star \Psi \tag{10}$$

The commutator of two such transformations will only close if Λ is enveloping algebra valued

$$\Lambda(x) = \alpha_i^{(0)}(x) L^i + \alpha_{ij}^{(1)}(x)\ {:}L^i L^j{:} + \ldots + \alpha_{i_1 \ldots i_n}^{(n-1)}\ {:}L^{i_1} \cdots L^{i_n}{:} \tag{11}$$

The two dots indicate that a basis in the algebra is used. Completely symmetrical polynomials form such a basis.

For commuting variables Lie algebra valued transformations close, it is consistent to put $\alpha^n = 0$ for $n \neq 1$. The transformation depends on a finite set of parameters $\alpha_i^0(x)$. The enveloping algebra valued transformations seem to depend on an infinite set of parameters. It is, however, possible to express these parameters in terms of the parameters $\alpha_i^0(x)$ and the usual Lie algebra valued gauge field $A_\mu(x) = A_\mu^i(x) T^i$ with the transformation properties

$$\begin{aligned} \delta A_\mu &= \partial_\mu \alpha^0 + i[\alpha^0, A_\mu] \\ \alpha_{i_1 \ldots i_r}^{(r-1)}(x) &= \Lambda_{i_1 \ldots i_r}^{r-1}\{\alpha_i^0(x), A_\mu^i, \partial\} \end{aligned} \tag{12}$$

This is part of the Seiberg–Witten map. We now denote a transformation that uses these parameters $\alpha^{(r)}$ as follows

$$\delta_{\alpha^0}\Psi = \alpha(x) \star \Psi \tag{13}$$

If we apply a second transformation we have to transform A_μ as well. It is now possible to find expressions $\alpha^{(r)}$ in terms of $\alpha^{(0)}$, A_μ and their derivatives such that

$$(\delta_{\alpha_0}\delta_{\beta_0} - \delta_{\beta_0}\delta_{\alpha_0})\Psi = \delta_{[\alpha_0, \beta_0]}\Psi \tag{14}$$

This equation determines $\alpha^{(r)}$ to a large extent. The problem is formulated in power series expansion in θ. To first order in θ we find

$$\alpha = \alpha_0 + \frac{1}{2}\theta^{\mu\nu} :\partial_\mu \alpha^0(x) A_\nu(x): + \dots \tag{15}$$

4. Covariant coordinates. The transformation law of $\hat{A}$ allows the construction of covariant coordinates. Due to the noncommutativity coordinates will not commute with the gauge transformation, a situation similar to derivatives in a usual gauge theory. Taking the idea from covariant derivatives one can try the following Ansatz

$$\hat{X}^\mu = \hat{x}^\mu + \hat{A}^\mu \tag{16}$$

$$\delta_{\alpha_0} \hat{X}^\mu \star \hat{\Psi} = i\alpha \star \hat{X}^\mu \star \hat{\Psi} \tag{17}$$

This defines the transformation law for $\hat{A}^\mu$.

A field strength can be introduced in complete analogy:

$$\hat{X}^\mu \hat{X}^\nu - \hat{X}^\nu \hat{X}^\mu = i\theta^{\mu\nu} + \hat{F}^{\mu\nu} \tag{18}$$

the field strength transforms tensorially

$$\delta_\alpha \hat{F}^{\mu\nu} = [\alpha \star \hat{F}^{\mu\nu}] \tag{19}$$

and can be used to construct invariant Lagrangians. This leads to the theories discussed at the beginning.

5. Seiberg–Witten map [13]. This is the most important new idea. One part is contained in the construction of the transformation parameter α, but it is also possible to express the new gauge field and the matter fields that transform as follows:

$$\begin{aligned} \delta\hat{\Psi} &= i\alpha \star \hat{\Psi} \\ \delta\hat{A}^\mu &= -i[x^\mu \overset{\star}{,} \alpha] + i[\alpha \overset{\star}{,} \hat{A}^\mu] \end{aligned} \tag{20}$$

in terms of fields that transform as usual

$$\begin{aligned} \delta\Psi &= i\alpha_0 \Psi \\ \delta A^\mu &= \partial_\mu \alpha_0 + i[\alpha_0, A^\mu] \end{aligned} \tag{21}$$

The new gauge potential is enveloping algebra valued.

$$\begin{aligned} \bar{\Psi} &= \Psi + \frac{1}{2}\theta_{\mu\nu} A_\nu \partial_\mu \Psi + \frac{1}{4}\theta^{\mu\nu} \partial_\mu A_\nu \Psi + \dots \\ \hat{A}_\mu &= A_\mu + \frac{1}{4}\theta^{\rho\nu}\{A_\nu(\partial_\rho A_\mu + F_{\rho\mu})\} + \dots \end{aligned} \tag{22}$$

This opens the way to construct gauge field theories with the star product without changing the particle content.

6. Invariant Integrals

An invariant action can only be defined if we know how to integrate. The integration must have the property

$$\int f \star g = \int g \star f \tag{23}$$

for constant θ the usual integral in x-space has this property .

Let me now go into more detail and show how in putting these new ideas together Lagrangians as we have discussed them in the beginning can be constructed.

2 The algebra

Let me first exhibit the algebraic structure of $\mathbb{R}^n$ and then generalise to noncommutative coordinates. The coordinates $x^1 \dots x^n \in \mathbb{R}^n$ are considered as elements of an associative algebra over $\mathbb{C}$. The algebra, freely generated by these elements, will be denoted by $\mathbb{C}\,[[x^1, \dots, x^n]]$. The two brackets indicate that formal power series are allowed in the algebra.

The elements of this algebra are then subject to relations that make them commutative:

$$\mathcal{R}: \quad x^i x^j - x^j x^i = 0. \tag{24}$$

These relations generate a two-sided ideal I_R; it consists of all the elements of the algebra $\mathbb{C}\,[[x^1, \dots, x^n]]$ that can be obtained from the relation (1) by multiplying (1) from the left and the right by all possible products of the coordinates. We factor out this ideal and obtain the desired algebra:

$$\mathcal{A}_x = \frac{\mathbb{C}\,[[x^1, \dots, x^n]]}{I_{\mathcal{R}}}. \tag{25}$$

The elements of this algebra are the polynomials and the formal power series in the commuting variables
$x^1, \dots, x^n \in \mathbb{R}$.

$$f(x^1, \dots, x^n) \in \mathcal{A}_x, \tag{26}$$
$$f(x^1, \dots, x^n) = \sum_{r_i=0}^{\infty} f_{r_1 \dots r_n} (x^1)^{r_1} \cdot \dots \cdot (x^n)^{r_n}.$$

Multiplication in this algebra is the pointwise multiplication of these functions.

This algebraic concept can be easily generalized to noncommutative coordinates. We consider algebras, freely generated by elements $\hat{x}^1, \dots \hat{x}^n$, again we call these elements coordinates, but now they are supposed to satisfy relations that make them noncommutative:

$$\mathcal{R}_{\hat{x},\hat{x}}: \quad [\hat{x}^i, \hat{x}^j] = i\theta^{ij}(\hat{x}). \tag{27}$$

Following L.Landau, noncommutativity carries a hat. Again the relations (4) generate an ideal and we define our algebra $\hat{\mathcal{A}}_{\hat{x}}$ as follows:

$$\mathcal{A}_{\hat{x}} = \frac{\mathbb{C}[[\hat{x^1}, \dots, \hat{x^n}]]}{I_{\mathcal{R}_{\hat{x},\hat{x}}}}, \tag{28}$$
$$\hat{f} \in \hat{\mathcal{A}}_{\hat{x}}.$$

We impose one more condition on the algebra. The vectorspace of the homogeneous polynomials of degree m, $\hat{V}^m_{\hat{x}}$ should have the same dimension as V^m_x. Algebras of this type are said to have the Poincare-Birkhoff-Witt property. In the following we shall consider such algebras only.

3 The $\star$ product

The vectorspaces V^m_x and $\hat{V}^m_{\hat{x}}$ are finite-dimensional, thus they are isomorphic. To establish an isomorphism we map a given basis of one space into a given basis of the other space. This then defines a vectorspace isomorphism between the vectorspaces $\hat{V}_{\hat{x}}$ and V_x.

We now change the algebra $\mathcal{A}_x$ to extend the above vector space isomorphism to an algebra isomorphism. For this purpose we have to change the multiplication law in $\mathcal{A}_x$. When we multiply two elements in $\hat{\mathcal{A}}_{\hat{x}}$ we can compute from the multiplication law in $\hat{\mathcal{A}}_{\hat{x}}$ the coefficient function of the product in a given basis. We define the product in the vectorspace V_x to be the element with the same coefficient function as it was calculated in $\hat{\mathcal{A}}_{\hat{x}}$. This multiplication rule we call $\star$ (star) product and this defines the algebra $\star\mathcal{A}_x$. The algebras $\hat{\mathcal{A}}_{\hat{x}}$ and $\star\mathcal{A}_x$ are isomorphic.

It is natural to use the elements of $\star\mathcal{A}_x$ as objects in physics. The pointwise product has to be replaced by the $\star$ product. In all the cases of interest the $\star$ product can be expressed with the help of a differential operator. This makes it possible to extend the $\star$ product to functions without referring to power series expansion. Thus we treat the elements $\mathcal{A}_x$ like ordinary fields but replace the pointwise product by the $\star$ product. This would be the starting point of deformation quantization. As we have based the concept on associative algebras, associativity of the $\star$ product is guaranteed.

4 Gauge theory

In this context it is possible to formulate a gauge theory [3, 4, 5]. We start from a Lie algebra:

$$[T^a, T^b] = i f^{ab}_c T^c. \tag{29}$$

In a usual gauge theory on commutative spaces the fields will span a representation of this Lie algebra and they will transform under the usual gauge transformation with Lie algebra valued parameters:

$$\delta_{\alpha^0}\psi(x) = i\alpha^0(x)\psi(x). \tag{30}$$

$$\begin{aligned}(\delta_{\alpha^0}\delta_{\beta^0} - \delta_{\beta^0}\delta_{\alpha^0})\psi &= -(\beta^0\alpha^0 - \alpha^0\beta^0)\psi \\ &= i(\alpha^0 \times \beta^0)\psi = \delta_{\alpha^0\times\beta^0}\psi, \\ \alpha^0 \times \beta^0 &\equiv \alpha^0_a\beta^0_b f^{ab}_c T^c.\end{aligned} \tag{31}$$

The commutator of two such transformations remains Lie algebra valued.

For a theory on non-commutative spaces we start with fields that are elements of $\star\mathcal{A}_x$. Gauge transformations have to be defined with the $\star$ product:

$$\delta_\alpha\psi(x) = i\alpha(x)\star\psi(x) \tag{32}$$

The star product of functions is not commutative. The commutator of two Lie algebra valued transformations does not reproduce a Lie algebra valued parameter. Thus we shall assume that the infinitesimal transformation parameters are enveloping algebra valued [2]:

$$\alpha(x) = \alpha^0_a(x)T^a + \alpha^1_{ab}(x) : T^aT^b : + \ldots + \alpha^{n-1}_{a_1\ldots a_n}(x) : T^{a_1}\cdot\ldots\cdot T^{a_n} : + \cdots \tag{33}$$

We have adopted the :: notation for a basis in the enveloping algebra of the Lie algebra. Completely symmetrized products could serve as a basis:

$$\begin{aligned}: T^a : &= T^a, \\ : T^aT^b : &= \frac{1}{2}(T^aT^b + T^bT^a) \text{ etc.}\end{aligned} \tag{34}$$

The commutator of two transformations is certainly enveloping algebra valued.

$$(\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha)\psi = [\alpha \overset{\star}{,} \beta]\star\psi. \tag{35}$$

The disadvantage of this approach is that infinitely many parameters $\alpha^n(x)$ have to be introduced.

It is a surprise that it is possible to define gauge transformations where all the parameters $\alpha^n(x)$ depend on the finite set of parameters $\alpha^0(x)$ (Lie algebra valued) and in addition on the gauge potential $a(x)$ of a usual gauge theory and on their derivatives. The gauge potential $a(x)$ has the usual transformation properties:

$$\begin{aligned}\delta a_i &= \partial_i\alpha^0 + i[\alpha^0, a_i], \\ \delta a_{i,a} &= \partial_i\alpha^0_a - \alpha^0_b f^{bc}_a a_{i,c}.\end{aligned} \tag{36}$$

We will call the new type of transformation parameters $\Lambda_{\alpha^0}(x)$. The new transformations are supposed to close under a commutator into a transformation characterized by $(\alpha^0\times\beta^0)$:

$$\begin{aligned}\delta_{\alpha^0}\psi(x) &= i\Lambda_{\alpha^0(x)}(x)\star\psi(x), \\ (\delta_{\alpha^0}\delta_{\beta^0} - \delta_{\beta^0}\delta_{\alpha^0})\psi &= \delta_{\alpha^0\times\beta^0}\psi, \\ (\alpha^0\times\beta^0)_a &= \alpha^0_b\beta^0_c f^{bc}_a.\end{aligned} \tag{37}$$

These equations define $\Lambda_{\alpha^0}(x)$. We shall see that all $\alpha_n(x)$ in (10) can be defined in terms of $\alpha^0(x)$ and the gauge potential $a(x)$. The transformation property (14) then holds as a consequence of (13). The solution of this problem, however, is not unique, this will be seen in the following.

As a consequence of the a dependence of Λ^0_α we have to transform Λ^0_α under the second variation in the commutator. This changes equation (12) and this is the reason why the new approach works.

5 Constant θ

To illustrate this approach we restrict it to the algebra where $\theta^{\mu\nu}$ is a constant. In this case we obtain in a fully symmetrized basis the following $\star$ product:

$$(f \star g)(x) = e^{\frac{i}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial y^j}} f(x)g(y)\Big|_{y\Rightarrow x} \tag{38}$$
$$= \int d^n y\ \delta^n(x-y) e^{\frac{i}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial y^j}} f(x)g(y).$$

We expand in θ.

$$\Lambda_{\alpha^0} = \alpha^0_a T^a + \theta^{ij}\Lambda^1_{\alpha^0,ij} + \dots, \tag{39}$$

The $\star$ product has to be expanded as well. Finally we expand the defining equation for Λ_{α^0}

$$(\delta_{\alpha^0}\delta_{\beta^0} - \delta_{\beta^0}\delta_{\alpha^0})\psi = i(\delta_{\alpha^0}\Lambda_{\beta^0} - \delta_{\beta^0}\Lambda_{\alpha^0}) \star \psi + [\Lambda_{\alpha^0} \stackrel{\star}{,} \Lambda_{\beta_0}] \star \psi, \tag{40}$$
$$= \delta_{\alpha^0 \times \beta^0}\psi = i\Lambda_{\alpha^0\times\beta^0} \star \psi$$

The zeroeth order in the expansion of (40) defines α^0 as Lie algebra valued.

In first order we obtain

$$\theta^{ij}((\delta_{\alpha^0}\Lambda^1_{\beta^0,ij} - \delta_{\beta^0}\Lambda^1_{\alpha^0,ij}) - i([\alpha^0, \Lambda^1_{\beta^0,ij}] - \tag{41}$$
$$- [\beta^0, \Lambda^1_{\alpha^0,ij}])) + \frac{1}{2}\partial_i\alpha^0_a\partial_j\beta^0_b : T^aT^b := \theta^{ij}\Lambda^1_{\alpha^0\times\beta^0,ij}.$$

A closer look shows that this is an inhomogeneous linear equation for Λ^1. The inhomogeneous term is known, it contains α^0 and β^0 only. A particular solution of (17) is:

$$\theta^{ij}\Lambda^1_{\alpha^0,ij} = \frac{1}{2}\theta^{ij}(\partial_i\alpha^0_a)a_{j,b} : T^aT^b : . \tag{42}$$

Any solution of the homogeneous part of (17) can be added to (18).

We can proceed order by order in θ, the structure of the equations will always be the same. It will be an inhomogeneous linear equation, the homogeneous remains the same, the inhomogeneous part will contain known quantities only. This way we obtain Λ_{α^0} in a θ expansion.

$$\Lambda_{\alpha^0} = \alpha^0_a T^a + \frac{1}{2}\theta^{ij}(\partial_i\alpha^0_a)a_{j,b} : T^aT^b : + \dots \tag{43}$$

Such a construction of the transformation parameter first occured in the context of the Seiberg–Witten map [6].

6 Covariant coordinates

In a usual gauge theory we would procede with the definition of covariant derivatives [14]. Derivatives, however, are not a natural concept for algebras. It is more natural to introduce covariant coordinates. Based on such a concept gauge theories can be developed as well.

It is obvious that coordinates do not commute with gauge transformations, it is also natural to introduce covariant coordinates in analogy to covariant derivatives:

$$X^i = x^i + A^i(x), \tag{44}$$
$$\delta_{\alpha^0} X^i \star \psi = i\Lambda_{\alpha^0} \star X^i \star \psi.$$

This leads to the following transformation law for the gauge potential:

$$\delta A^i = -i[x^i \stackrel{\star}{,} \Lambda_{\alpha^0}] + i[\Lambda_{\alpha^0} \stackrel{\star}{,} A^i]. \tag{45}$$

To satisfy such a transformation law we have again to assume that $A(x)$ is enveloping algebra valued. In general, this would imply infinitely many gauge fields. For the restricted gauge transformations Λ_{α^0} it is possible to construct a gauge potential that depends on the Lie algebra valued potential $a(x)$ and its derivatives only. The transformation law (13) for $a(x)$ will imply the transformation law for $A(x)$. This is the main achievement of the Seiberg–Witten map (22).

The construction of gauge fields that transform to tensorial follows the usual concept as we know it from covariant derivatives. An obvious definition is

$$X^\mu X^\nu - X^\nu X^\mu - i\theta^{\mu\nu}(X) = \hat{F}^{\mu\nu} \tag{46}$$

It is chosen in such a way that $\tilde{F}^{\mu\nu}$ vanishes for a vanishing gauge potential A^μ.

The tensorial transformation law of $\tilde{F}^{\mu\nu}$ follows directly from (22):

$$\delta_\alpha \tilde{F}^{\mu\nu} = [\Delta_{\alpha 0} \star \tilde{F}^{\mu\nu}] \tag{47}$$

It should be noted, however, that the trace in the representation space of the Lie algebra of a tensor is not an invariant because the star product is not commutative.

7 The integral

An invariant action can be constructed only if the integral has its trace property:

$$\int f \star g = \int g \star f. \tag{48}$$

Integration is not a natural concept in an algebra. It is supposed to be a linear map from $\mathcal{A}_{\hat{x}}$ into $\mathbb{C}$.

$$\int \; : \; \mathcal{A}_{\hat{x}} \to \mathbb{C}, \tag{49}$$
$$\int (c_1 \hat{f} + c_2 \hat{g}) = c_1 \int \hat{f} + c_2 \int \hat{g},$$

In addition the trace property is required:

$$\int \hat{f}\hat{g} = \int \hat{g}\hat{f}. \tag{50}$$

This is equivalent to (25).

8 Gauge theory for constant θ

For constant θ the usual integral in x-space will have the trace property. This can be shown by a direct calculation.

Let us have a look at this formalism for constant $\theta^{\mu\nu}$: The Seiberg–Witten map:

$$\begin{aligned} A^i(x) &= \theta^{ij} V_j, \\ V_j(x) &= a_{j,a} T^a - \frac{1}{2}\theta^{ln} a_{l,a}(\partial_n a_{j,b} + F_{nj,b} : T^a T^b : + \ldots, \\ F_{nj,b} &= \partial_n a_{j,b} - \partial_j a_{n,b} + f_b^{cd} a_{n,c} a_{j,d}. \end{aligned} \tag{51}$$

The field strength:

$$\tilde{F}_{ij} = F_{ij,a}T^a + \theta^{ln}(F_{il,a}F_{jn,l} - \frac{1}{2}a_{l,a}(2\partial_n F_{ij,b} + a_{n,c}F_{ij,d}f_e^{cd})) : T^a T^b : + \ldots . \tag{52}$$

The Lagrangian:

$$L = \frac{1}{4}\mathrm{Tr} F_{ij} \star F^{ij}. \tag{53}$$

The invariant action:

$$\begin{aligned} W &= \frac{1}{4}\int \mathrm{Tr} F_{ij} \star F^{ij} \\ &= \frac{1}{4}\int \mathrm{Tr} F_{ij} F^{ij}. \end{aligned} \tag{54}$$

New coupling terms arise, $\theta^{\mu\nu}$ appears as a coupling constant, it is a Lorentz tensor and the interaction term breaks Lorentz invariance. This was to be expected because the defining relation (1) already breaks Lorentz invariance.

These new terms in the Lagrangian will give rise to new interactions. Due to the breaking of Lorentz invariance interaction terms will occur that are forbidden in a Lorentz invariant theory. A good example is the $Z^0 \to \gamma\gamma$ decay. From (53) we find the following interaction terms that contribute to this decay if the gauge theory is based on the standard model.

$$\begin{aligned} \mathcal{L}_{Z_{\gamma\gamma}} = {} & \frac{e}{8}\sin 2\theta_W \left(g'^2\kappa_1 + (g'^2 - 2g^2)\kappa_2\right)\theta^{kl} \\ & \times \Big(2(-\partial_i Z_k + \partial_k Z_i)\partial_j A_l(\partial^i A^j - \partial^j A^i) \\ & + (\partial_i A_k \partial_j A_l + \partial_k A_i \partial_l A_j - 2\partial_k A_i \partial_j A_l)(-\partial^i Z^j + \partial^j Z^i) \\ & + (-2\partial_k Z_i \partial_l A_j + 2\partial_j Z_l \partial_k A_i + 2\partial_i Z_j \partial_k A_l + \partial_k Z_l \partial_i A_j)(\partial^i A^j - \partial^j A^i)\Big) \end{aligned} \tag{55}$$

This expression is gauge invariant under the usual Lie algebra valued gauge transformation. It contributes to the branching ratio of the Z^0 decay.

We still have to learn how the gauge potential couples to the matter fields. This will be done via covariant derivatives.

$$\begin{aligned} \mathcal{D}_i \star \psi &= (\partial_i - iV_i) \star \psi, \\ \delta_{\alpha^0}\mathcal{D}_i \star \psi &= i\Lambda_{\alpha^0} \star \mathcal{D}_i \star \psi. \end{aligned} \tag{56}$$

9 Derivatives

First we have to define derivatives. In general, the star product will depend on the coordinates, when we differentiate it the coordinate dependence of the $\star$ product will contribute as well. Nevertheless, we demand a Leibniz rule of the type

$$\partial_\mu \star (f \star g) = (\partial_\mu f) \star g + \mathcal{O}_\mu^\nu(f) \star \partial_\nu g. \tag{57}$$

From the associativity of the $\star$ product follows that $O_\mu^\nu(f)$ has to be an algebra homomorphism.

It is easier to define derivatives for $\hat{\mathcal{A}}_{\hat{x}}$. A general procedure was outlined in ref. [12]. We first extend the algebra by algebraic elements $\hat{\partial}$ and consider the algebra $\mathbb{C}\left[[\hat{x}^1, \ldots, \hat{x}^n, \ldots, \hat{\partial}^1, \ldots, \hat{\partial}^n]\right]$. This algebra has to be divided by the ideal $I_{\hat{x},\hat{x}}$ as before. Then we have to construct a derivative, based on a Leibniz rule that is a map in $\mathbb{C}\left[[\hat{x}^1, \ldots, \hat{x}^n, \ldots, \hat{\partial}^1, \ldots, \hat{\partial}^n]\right] / I_{\hat{x},\hat{x}}$. This leads to consistency relations for the Leibniz rule The Leibniz rule can now be interpreted as a relation and the respective ideals can be constructed and factored out. Finally this has to be supplemented by $\hat{\partial}, \hat{\partial}$ relations. We treat these relations as usual and after dividing by the respective ideal we arrive at an algebra that we call $\hat{\mathcal{A}}_{\hat{x},\hat{\partial}}$.

In more detail the generalized Leibniz rule is supposed to have the form:

$$\hat{\partial}_i(\hat{f}\hat{g}) = (\hat{\partial}_i \hat{f})\hat{g} + O_i^l(\hat{f})\hat{\partial}_l \hat{g}. \tag{58}$$

From the law of associativity in $\hat{\mathcal{A}}_{\hat{x}}$ follows that the map 0 has to be an algebra homomorphism

$$O_j^i(\hat{f}\hat{g}) = O_l^i(\hat{f}) O_j^l(\hat{g}). \tag{59}$$

If we define the Leibniz rule on the linear coordinates we can generalize it to all elements.

In the ${}^\star\mathcal{A}_x$ version of the algebra the Leibniz rule takes the form of equation (34). This rule can be found as follows: $\hat{\partial}$ introduces a map on the basis of $\hat{\mathcal{A}}_{\hat{x}}$, this map defines a map in ${}^\star\mathcal{A}_x$. This map has finally to be expressed with ordinary x-derivatives. This then leads to (34).

For constant $\theta^{\mu\nu}$ where the $\star$ product does not depend on x the $\partial\star$ derivatives are just the ordinary x-derivatives.

Covariant derivatives are then defined as usual:

$$\begin{aligned} \mathcal{D}_i \star \psi &= (\partial_i - iV_i) \star \psi, \\ \delta_{\alpha^0} \mathcal{D}_i \star \psi &= i\Lambda_{\alpha^0} \star \mathcal{D}_i \star \psi. \end{aligned} \tag{60}$$

The vector potential has to be enveloping algebra valued. Again, it can be expressed in terms of a_μ by a Seiberg–Witten map. Therefore we expect that A_μ and V_μ are related.

For constant $\theta^{\mu\nu}$ we find:

$$A^i(x) = \theta^{ij} V_j \tag{61}$$

Covariant derivatives exist for $\theta^{\mu\nu} = 0$. From (34) follows that A^μ vanishes in this case, coordinates are already covariant.

10 Gauge couplings to matter fields

The matter field ψ that transforms like

$$\delta_{\alpha^0}\psi(x) = i\Lambda_{\alpha^0(x)}(x) \star \psi(x) \tag{62}$$

can be expressed in terms of a field ψ^0 that transforms with a Lie algebra valued parameter and the Lie algebra valued vector potential a. The transformation property (35) will be a consequence of (7) and (13).

For constant θ we find:

$$\psi = \psi^0 - \frac{1}{2}\theta^{\mu\nu} a^l_\mu T^l \partial_\nu \psi^0 + \dots \tag{63}$$

This now leads to the Lagrangian

$$\int \bar{\psi} \star (\gamma^\mu D_\mu \star - m)\, \psi d^4x = \int \bar{\psi}^0 (\gamma^\mu D_\mu - m)\, \psi^0 d^4 x - \frac{1}{4}\theta^{\mu\lambda} \int \bar{\psi}^0 F^0_{\mu\lambda} (\gamma^\mu D_\mu - m)\, \psi^0 - \frac{1}{4}\theta^{\sigma\lambda} \int \bar{\psi}^0 \gamma^\mu F^0_{\mu\sigma} D_\lambda \psi^0 d^4x + \cdots \tag{64}$$

The fields ψ^0 and $F^{\mu\nu 0}$ transform like the usual gauge fields with a Lie algebra valued parameter. $F^{\mu\nu 0}$ is just the usual field strength of a gauge theory. Accordingly, $D_\mu \psi^0$ is the usual covariant derivative with the field a_μ as a gauge potential.

11 Conclusion

Such a theory based on noncommutative coordinates should only be relevant for a region with very high energy density, thus for very short distances, i.e. well inside the confinement range. For larger distances we know that physics is described very well with commuting coordinates. $\theta^{\mu\nu}(x)$ will be a complicated function, we treat this function in a power series expansion and start with constant $\theta^{\mu\nu}$. This has a chance to be relevant for processes that take place at very short distances where the constant $\theta^{\mu\nu}$ might be dominant. The higher order contribution on the expansion become relevant at distances where the process has already occured. Such a process will not be sensitive to the functional behaviour of $\theta^{\mu\nu}(x)$ and the constant $\theta^{\mu\nu}$ approximation might be a good approximation [14, 13]. To find such a process demands physical intuition.

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