Recent Developments in Small-x Physics

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Recent theoretical progress in understanding high-energy scattering beyond the mean field approximation is reviewed. The role of Lorentz invariance and pomeron loops in the evolution, the relation between high-energy QCD and statistical physics and results for the saturation momentum and the scattering amplitude are discussed.

1   Introduction

The high-energy scattering of a dipole off a nucleus/hadron in the *mean field approximation* is decribed by the BK-equation [2]. The main results following from the BK-equation are the geometric scaling behaviour of the scattering amplitude and the roughly powerlike energy dependence of the saturation scale which are both nicely supported by the HERA data.

The recent progress consists in understanding small-x dynamics (near the unitarity limit) *beyond the mean field approximation*, i.e., beyond the BK-equation. A first step beyond the mean field approximation was done in [3] where the BFKL evolution in the scattering process was enforced to satisfy natural requirements as unitarity limits and Lorentz invariance. The result was a correction to the saturation scale and the breaking of the geometric scaling at high energies. Afterwards a relation between high-energy QCD and statistical physics was found [4] which has clarified the physical picture of, and the way to deal with, the dynamics beyond the BK-equation. It has been understood that *gluon number fluctuations* from one scattering event to another and the *discreteness* of gluon numbers, both ignored in the BK evolution and also in the Balitsky-JIMWLK equations [5], lead to the breaking of the geometric scaling and to the correction to the saturation scale, respectivelly. New evolution equations [6, 7, 8], which describe Pomeron loops, have been proposed to account for the above effects. Very recently possible effects of Pomeron loops on various observables [9, 10, 11] have been studied in case they become important in the range of collider energies.

In the following sections I will show the recent developments in some detail by considering equations and results in and beyond the mean field approximation.

2   Mean field approximation

Consider the high-energy scattering of a dipole of transverse size r off a target (hadron, nucleus) at rapidity $Y = \ln(1/x)$. The rapidity dependence of the T-matrix in the mean field approximation is given by the BK-equation which has the schematic structure (transverse dimensions are suppressed)

$$\partial_Y T = \alpha_s \left[T - T\, T \right] . \tag{1}$$

The linear part of the BK equation, i.e., the BFKL equation, gives the growth of T with rapidity Y whereas the non-linear term TT tames the growth of T in such a way that the unitarity limit $T \leq 1$ is satisfied.

One of the main results following from the BK-equation is the *geometric scaling* behaviour of the T-matrix [12]

$$T(r, Y) = T(r^2 Q_s^2(Y)) , \qquad (2)$$

where $Q_s(Y)$ is the so-called *saturation momentum* defined such that $T(r \simeq 1/Q_s, Y)$ be of $\mathcal{O}(1)$. Eq. (2) means that the T-matrix scales with a single quantity $r^2 Q_s^2(Y)$ rather than depending on r and Y separatelly. This behaviour implies a similar scaling for the DIS cross section, $\sigma^{\gamma^* p}(Y, Q^2) = \sigma^{\gamma^* p}(Q^2/Q_s^2(Y))$, which is supported by the HERA data.

Another important result that can be extracted from the BK-equation is the rapidity dependence of the saturation momentum (leading-Y contribution)[13],

$$Q_s^2(Y) = Q_0^2 \operatorname{Exp}\left[\frac{2\alpha_s N_c}{\pi} \frac{\chi(\lambda_0)}{1 - \lambda_0} Y\right] , \qquad (3)$$

where $\chi(\lambda)$ is the BFKL kernel and $\lambda_0 = 0.372$.

The shape of the T-matrix resulting from the BK-equation is preserved in the transition region from weak to strong scattering, $0 < T < 1$, with rising Y (front of the travelling wave): The saturation region at $r \gg 1/Q_s(Y)$ where $T \simeq 1$ however widens up, including smaller and smaller dipoles, due to the growth of the saturation momentum. The situation changes, as we will, once gluon number fluctuations are taken into account.

3 Beyond the mean field approximation

3.0.1 *Lorentz invariance plus Unitarity*

Let's start with an elementary dipole of size r_1 at rapidity $y = 0$ and evolve it using the BFKL evolution up to $y = Y$. The number density of dipoles of size r_2 at Y in this dipole, $n(r_1, r_2, Y)$, obeys a completeness relation

$$n(r_1, r_2, Y) = \int \frac{d^2 r}{2\pi r^2} \ n(r_1, r, Y/2) \ n(r, r_2, Y/2) \qquad (4)$$

where on the right hand side the rapidity evolution is separated in two successive steps, $y = 0 \to y = Y/2 \to y = Y$. With

$$T(r_1, r_2, Y) \simeq c \ \alpha_s^2 \ r_2^2 \ n(r_1, r_2, Y) \qquad (5)$$

eq.(4) can be approximately rewritten in terms of the T-matrix as

$$\left(\frac{1}{r_2^2} \ T(r_1, r_2, Y)\right) \simeq \frac{1}{2c\alpha_s^2} \int d\rho \ \left(\frac{1}{r^2} \ T(r_1, r, Y/2)\right) \left(\frac{1}{r_2^2} \ T(r, r_2, Y/2)\right) \qquad (6)$$

where $\rho = \ln(r_0^2/r^2)$. In Ref. [3] it was realized that the above completeness relations, or, equivalently, the Lorentz invariance, is satisfied by the BK evolution only by violating unitarity limits. This can be illustrated as follows: Suppose that r_2 is close to the saturation line, $r_2 \simeq 1/Q_s(Y/2)$, so that the left hand side of Eq.(6) is large. On the right hand side of Eq.(6) it turns out that $T(r_1, r, Y/2)/r^2$ is typically very small in the region of ρ which dominates the integral. This means that $T(r, r_2, Y/2)/r_2^2$ must be typically very large and must violate unitarity, $T(r, r_2, Y/2) \gg 1$, in order to satisfy (6).

The simple procedure used in Ref. [3] to solve the above problem was to limit the region of the ρ-integration in Eq.(6) by a boundary $\rho_2(Y/2)$ so that $T(r, r_2, Y/2)/r_2^2$ would never violate unitarity, or $T(r_1, r, Y/2)/r^2$ would always be larger than α_s^2. The main consequence of this procedure, i.e., BK evolution plus boundary correcting it in the weak scattering region, is the following scaling behaviour of the T-matrix near the unitarity limit

$$T(r, Y) = T\left(\frac{\ln(r^2 Q_s^2(Y))}{\alpha_s Y/(\Delta \rho)^3}\right) \tag{7}$$

and the following energy dependence of the saturation momentum

$$Q_s^2(Y) = Q_0^2 \, \text{Exp}\left[\frac{2\alpha_s N_c}{\pi} \frac{\chi(\lambda_0)}{1 - \lambda_0} Y \left(1 - \frac{\pi^2 \chi''(\lambda_0)}{2(\Delta \rho)^2 \chi(\lambda_0)}\right)\right] \tag{8}$$

with

$$\Delta \rho = \frac{1}{1 - \lambda_0} \ln \frac{1}{\alpha_s^2} + \frac{3}{1 - \lambda_0} \ln \ln \frac{1}{\alpha_s^2} + \text{const.} \ . \tag{9}$$

Eq.(7) shows a breaking of the geometric scaling which was the hallmark of the BK equation (cf. Eq.(2)) and Eq.(8) shows the correction to the saturation momentum due to the evolution beyond the mean field approximation (cf. Eq.(3)).

3.0.2 Statistical physics - high density QCD correspondence

The high energy evolution can be viewed also in another way which is inspired by dynamics of reaction-diffusion processes in statistical physics [4]. To show it, let's consider an elementary target dipole of size r_1 which evolves from $y = 0$ up to $y = Y$ and is then probed by an elementary dipole of size r_2, giving the amplitude $\bar{T}(r_1, r_2, Y)$. It has become clear that the evolution of the target dipole is *stochastic* leading to random dipole number realizations inside the target dipole at Y, corresponding to different events in an experiment. The physical amplitude, $\bar{T}(r_1, r_2, Y)$, is then given by averaging over all possible dipole number realizations/events, $\bar{T}(r_1, r_2, Y) = \langle T(r_1, r_2, Y) \rangle$, where $T(r_1, r_2, Y)$ is the amplitude for dipole r_2 scattering off a particular realization of the evolved target dipole at Y.

The mean field descripton breaks down at low target dipole occupancy due to the *discreteness and the fluctuations of dipole numbers*. Because of discreteness the dipole occupancy can not be less than one for any dipole size. Taking this fact into account by using the BK equation with a cutoff when T becomes of order α_s^2 [4], or the occupancy of order one (see Eq.(5)), leads exactly to the same correction for the saturation momentum as given in Eq.(3). The latter cutoff is essentially the same as, and gives a natural explanation of, the boundary used in Ref.[3] and briefly explained in the previous section.

The dipole number fluctuations in the low dipole occupancy region result in fluctuations of the saturation momentum from event to event, with the strength

$$\sigma^2 = \langle \rho_s^2 \rangle - \langle \rho_s \rangle^2 = \text{const.} \frac{\alpha_s Y}{(\Delta \rho)^3} \tag{10}$$

extracted from numerical simulations of statistical models. The averaging over all events with random saturation momenta, in order to get the physical amplitude, causes the breaking of the geometric scaling and replaces it by the scaling law

$$\langle T(r, Y) \rangle = T\left(\frac{\ln(r^2 Q_s^2(Y))}{\sqrt{\alpha_s Y/(\Delta \rho)^3}}\right) \ . \tag{11}$$

This equation differs from Eq.(7) since Eq.(7) misses dipole number fluctuations.

3.0.3 Pomeron loop equations

It was always clear that the BK equation does not include fluctuations. However, it took some time to realize that also the Balitsky-JIMWLK equations do miss them. As soon as this became clear (first Ref. in [7]), the so-called Pomeron loop equations [6, 7] have been constructed, aiming at a description of fluctuations. They can be written (schematic way, transverse dimensions ignored) as a stochastic equation of Langevin-type,

$$\partial_Y T = \alpha_s \left[T - TT + \alpha_s \sqrt{T} \; \nu \right] \tag{12}$$

or as a hierarchy of coupled equations of averaged amplitudes, the first two of them reading

$$
\begin{aligned}
\partial_Y \langle T \rangle &= \alpha_s \left[\langle T \rangle - \langle TT \rangle \right] \\
\partial_Y \langle TT \rangle &= \alpha_s \left[\langle TT \rangle - \langle T\,T\,T \rangle + \alpha_s^2 \langle T \rangle \right] \; .
\end{aligned}
\tag{13}
$$

The last term in Eq.(12), containing a non-Gaussian noise ν, is new as compared with the BK equation and accounts for the fluctuations in the dipole number. Eq.(13) reduces to the BK equation in the mean field approximation, $\langle T\,T \rangle = \langle T \rangle \langle T \rangle$. The hierarchy in Eq.(13), as compared with the Balitsky-JIMWLK hierarchy, involves in addition to linear BFKL evolution and pomeron mergings, also pomeron splittings, and therefore *pomeron loops*. The three pieces of evolution are represented by the three terms in the second equation in Eq.(13), respectivelly, in the case where two dipoles scatter off a target.

It isn't yet clear at which energy fluctuation effects start becoming important. The results shown in the previous sections, Eq.(8) and Eq.(11), are valid at asymptotic energies. A solution to the evolution equations, which is not yet available because of their complexity, would have helped to better understand the subasymptotics. However, using the methods outlined in the previous subsections, phenomenological consequences of fluctuations in the fixed coupling case have been studied, for example for DIS and diffractive cross sections [10], forward gluon production in hadron-hadron collisions [11] and for the nuclear modification factor $R_p A$ [9], in case fluctuations become important in the range of LHC energies.

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