DGLAP and BFKL equations in $N = 4$ SUSY

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The properties of the BFKL kernel in the next-to-leading approximation in QCD and in supersymmetric models are discussed. The maximal transcendality of anomalous dimensions in $N = 4$ SUSY is formulated. The explicit expressions for the anomalous dimensions up to four loops are given. Their asymptotic behavior at $j \to \infty$ and in the singular points $j = 1, 0, -1, ...$ is compared with predictions.

1 Introduction

The QCD scattering amplitude in the leading logarithmic approximation (LLA) has the Regge-type asymptotics with the gluon trajectory in one loop given below

$$\omega(-|q|^2) = -\frac{\alpha_c}{4\pi^2} N_c \int d^2k \frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_c}{2\pi} \ln \frac{|q^2|}{\lambda^2} \,. \tag{1}$$

In the coordinate representation the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation for the Pomeron wave function can be written as follows [2]

$$E\,\Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1, \vec{\rho}_2) \,, \;\; \Delta = -\frac{\alpha_s N_c}{2\pi} \min E \,, \tag{2}$$

where Δ is the Pomeron intercept. The BFKL Hamiltonian in the operator form is simple [3]

$$H_{12} = \ln |p_1 p_2|^2 + \frac{1}{p_1 p_2^*} \ln |\rho_{12}|^2 p_1 p_2^* + \frac{1}{p_1^* p_2} \ln |\rho_{12}|^2 p_1^* p_2 - 4\psi(1) \,, \tag{3}$$

where $\rho_{12} = \rho_1 - \rho_2$. It is invariant under the Möbius transformation [4] and has the property of the holomorphic separability [3]. The quantum numbers of the Möbius group are the anomalous dimension $\gamma = \frac{1}{2} + i\nu$ and the conformal spins n.

The Bartels-Kwiecinskii-Praszalowicz (BKP) equation [5] for the n-gluon composite states in the large-N_c limit has the duality symmetry [6], is integrable [3, 7] and equivalent to a Scgrödinger equation for the Heisenberg spin model [8]. To restore the s-channel unitarity one can use the effective field theory for Reggeized gluons [9]-[11].

2 DGLAP and BFKL dynamics in $N = 4$ SUSY

In the next-to-leading approximation the eigenvalue of the BFKL kernel is written below

$$\omega = \omega_0(n, \gamma) + 4\,\hat{a}^2\,\Delta(n, \gamma) \,, \;\; \hat{a} = g^2 N_c/(16\pi^2) \,. \tag{4}$$

In QCD $\Delta(n, \gamma)$ is a non-analytic function of the conformal spin $|n|$ [12, 13], but in $N = 4$ SUSY the Kroniker symbols are cancelled [13]. In this model we obtain for $\Delta(n, \gamma)$ the result

$$\Delta(n, \gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M) + \rho(M^*)}{2\hat{a}/\omega} \,, \; M = \gamma + \frac{|n|}{2} \,, \tag{5}$$

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$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\ \beta'(z) = \frac{1}{4}\left[\Psi'\Big(\frac{z+1}{2}\Big) - \Psi'\Big(\frac{z}{2}\Big)\right], \tag{6}$$

where all special functions have the maximal trancedentality property [13]

$$\phi(M) = 3\zeta(3) + \Psi^{''}(M) - 2\Phi(M) + 2\beta^{'}(M)\Big(\Psi(1) - \Psi(M)\Big) \tag{7}$$

and

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{\beta'(k+1)}{k+M} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1)\ -\ \frac{\Psi(k+1)-\Psi(1)}{k+M}\right). \tag{8}$$

For one loop anomalous dimension matrix in the case $N = 4$ the calculations were performed in Ref. [14]. In this model all twist-2 operators belong to the same supermultiplet and have the following anomalous dimension

$$\gamma^{(0)}_{uni}(j) = -4S_1(j-2)\,,\ \ S_r(j) = \sum_{i=1}^{j} \frac{1}{i^r}\,. \tag{9}$$

Note, that this function has the maximal transcedentality. It leads to an integrability of evolution equations for matrix elements of quasi-partonic operators in $N = 4$ SUSY [14].

3 Two and three loop results

Using maximal transcedentality hypothesis [13] and QCD results [15], one can calculate also the anomalous dimensions in two and three loops [16, 17]

$$\gamma_{uni}(j) = \hat{a}\gamma^{(0)}_{uni}(j) + \hat{a}^2\gamma^{(1)}_{uni}(j) + \hat{a}^3\gamma^{(2)}_{uni}(j) + \dots, \tag{10}$$

where

$$\frac{1}{8}\,\gamma^{(1)}_{uni}(j+2) = 2S_1(j)\left(S_2(j) + S_{-2}(j)\right) - 2S_{-2,1}(j) + S_3(j) + S_{-3}(j)\,, \tag{11}$$

and

$$\begin{aligned}\frac{1}{32}\,\gamma^{(2)}_{uni}(j+2) = {} & 24\,S_{-2,1,1,1} - 12\,\left(S_{-3,1,1} + S_{-2,1,2} + S_{-2,2,1}\right) + 6\,\left(S_{-4,1} + S_{-3,2} + S_{-2,3}\right) \\ & -3\,S_{-5} - 2\,S_3\,S_{-2} - S_5 - 2\,S_1^2\left(3\,S_{-3} + S_3 - 2\,S_{-2,1}\right) - S_2\left(S_{-3} + S_3 - 2\,S_{-2,1}\right) \\ & -S_1\left(8\bar{S}_{-4} + \bar{S}_{-2}^2 + 4S_2\bar{S}_{-2} + 2S_2^2\right) - S_1\left(3S_4 - 12\bar{S}_{-3,1} - 10\bar{S}_{-2,2} + 16\bar{S}_{-2,1,1}\right)\,.\end{aligned} \tag{12}$$

Here the corresponding harmonic sums are defined below

$$S_a(j) = \sum_{m=1}^{j} \frac{1}{m^a}\,,\ \ S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j} \frac{1}{m^a}\,S_{b,c,\cdots}(m)\,, \tag{13}$$

$$S_{-a}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a},\ S_{-a,b,\dots}(j) = \sum_{m=1}^{j} \frac{(-1)^m}{m^a}\,S_{b,\dots}(m)\,. \tag{14}$$

4 Weak and strong coupling regimes

The above results are in an agreement with the BFKL prediction [13] for the singularities at $j \to 1$

$$\gamma_{uni}^{N=4}(j) = \hat{a}\frac{4}{\omega} - 32\zeta_3\,\hat{a}^2 + 32\zeta_3\,\hat{a}^3\frac{1}{\omega} - \frac{16\hat{a}^4}{\omega^4}\left(32\zeta_3 + \frac{\pi^4}{9}\omega\right). \tag{15}$$

Note, that recently the four-loop result $\gamma_{uni}^{(3)}(j)$ was calculated with the use of the asymptotic Bethe ansatz [18]. It turned out, that the obtained expression has the singularity in $j = 1$ incompatible with the BFKL prediction. A simple modification of the four loop result taking into account the wrapping effect gives an agreement with the BFKL equation and the following non-linear equation for $j + 2r = \omega \to 0$ $(r = 0, 1, 2, ...)$

$$\omega\gamma_{uni} = \gamma_{uni}^2 + 16\hat{a}^2(S_2 + \zeta_2 - S_1^2) + 4\,\hat{a}\left(1 - \omega S_1 - \omega^2(S_2 + \zeta_2) + \gamma^2(S_2 + S_{-2})\right),$$

generalizing the resummation of the double logarithmic terms $\sim \alpha/\omega^2$ (cf. [19]).

Further, the universal anomalous dimension at large j

$$\gamma_{uni}^{N=4} = a(z)\,\ln j\,,\ z = \frac{\alpha N_c}{\pi} = 4\hat{a} \tag{16}$$

can be found from our results up to three loops

$$a(z) = -z + \frac{\pi^2}{12}\,z^2 - \frac{11}{720}\pi^4 z^3 + \dots. \tag{17}$$

It is remarkable, that using the AdS/CFT correspondence [20] between the superstring model on the anti-de-Sitter space and the $N = 4$ supersymmetric Yang-Mills theory A. Polyakov with collaborators calculated the coefficient $a(z)$ in the strong coupling limit [21]

$$\lim_{z\to\infty} a(z) = -z^{1/2} + \frac{3\ln 2}{4\pi} + \dots. \tag{18}$$

In Ref. [16] the resummation of the perturbative expansion of $a(z)$ was suggested, which reproduces approximately the three-loop result and the strong coupling limit.

The perturbative calculations of the anomalous dimension at large j are in agreement with the recent papers [22, 23], where integral equations was derived from the integrability of the model. One can rewrite the Eden-Staudacher integral equation [22] as a set of linear equations [24]

$$a_{n,\epsilon} = \sum_{n'=1}^{\infty} K_{n,n'}(\epsilon)\ (\delta_{n',1} - a_{n',\epsilon})\ ,\ K_{n,n'}(\epsilon) =$$

$$2n\sum_{R=0}^{\infty}(-1)^R\,\frac{2^{-2R-n-n'}}{\epsilon^{2R+n+n'}}\zeta(2R+n+n')\,\frac{(2R+n+n'-1)!\,(2R+n+n')!}{R!\,(R+n)!\,(R+n')!\,(R+n+n')!}\,, \tag{19}$$

where the function $a(z)$ is expressed in terms of $a_{1,\epsilon}$

$$a(z) = \frac{2(1 - a_{1,\epsilon})}{\epsilon^2}\ ,\ \epsilon = \frac{1}{g\sqrt{2}}\,. \tag{20}$$

We can easiliy prove, that the maximal transcedentality property for $a(z)$ is valid in all orders of the perturbation theory and the coefficients in front of the products of the corresponding ζ-functions are integer numbers [24]. It is possibly to show [24], that the asymptotic behaviour of $a(z)$ in the case of the Beisert-Eden-Staudacher equation [22] in the agreement with the AdS/CFT prediction [21].

$$\lim_{g\to\infty} \gamma_{sing} = \frac{2}{\epsilon}\,\frac{I_1(2\epsilon^{-1})}{I_0(2\epsilon^{-1})} \approx 2\sqrt{2}\,g\, - \frac{1}{2} \tag{21}$$

Note, that the intercept of the BFKL Pomeron in the strong coupling limit was calculated in Refs. [17] and [25]

References

[1] Slides:
`http://indico.cern.ch/contributionDisplay.py?contribId=188&sessionId=8&confId=9499`

[2] V.S. Fadin, E.A. Kuraev, L.N. Lipatov, Phys. Lett. **B60** 50 (1975);
Ya.Ya. Balitsky, L.N. Lipatov, Yad. Fiz. **28** 1597 (1978).

[3] L.N. Lipatov, Phys.Lett. **B309** 394 (1993).

[4] L.N. Lipatov, JETP **90** 1536 (1986).

[5] J. Bartels, Nucl.Phys. **B175** 365 (1980);
J. Kwiecinski, M. Praszalowicz, *Phys.Lett.*B **B94** 413 (1980).

[6] L.N. Lipatov, Nucl.Phys. **B548** 328 (1999).

[7] L.N. Lipatov *High energy asymptotics of multi-colour QCD and exactly solvable lattice models*, Padova preprint DFPD/93/TH/70, hep-th/9311037, unpublished.

[8] L.N. Lipatov, JETP Lett. **59** 596 (1994);
L.D. Faddeev, G.P. Korchemsky, Phys. Lett. **B342** 311 (1995).

[9] L.N. Lipatov, Nucl.Phys. **B365** 614 (1991).

[10] L.N. Lipatov, Nucl.Phys. **B452** 369 (1995).

[11] E. Antonov, I. Cherednikov, E. Kuraev, L. Lipatov, Nucl.Phys. **B721** 111 (2005).

[12] V. Fadin, L. Lipatov, Phys. Lett. **B429** 127 (1998);
M. Ciafaloni, G. Camici, Phys. Lett. **B430** 349 (1998).

[13] A. Kotikov, L. Lipatov, Nucl.Phys. **B582** 19 (2000), Nucl.Phys. **B661** 19 (2003).

[14] L. Lipatov, talk at "Perspectives in Hadronic Physics", Proc. of Conf. ICTP. Triest, Italy, May 1997.

[15] S. Moch, J.A.M. Vermaseren, A. Vogt, Nucl.Phys. **B688** 101 (2004).

[16] A. Kotikov, L. Lipatov, V. Velizhanin, Phys. Lett. **B557** 114 (2003).

[17] A. Kotikov, L. Lipatov, A. Onishchenko, V. Velizhanin, Phys. Lett. **B595** 521 (2004);
Phys. Lett. **B632** 754 (2006).

[18] A.V. Kotikov, L.N. Lipatov, A. Rej, M. Staudacher, V.N. Velizhanion, arXiv:hep-th/0704.3586v.

[19] R. Kirschner, L.N. Lipatov, Phys.Rev. **D26** 1202 (1982.)

[20] J.M. Maldacena, Adv. Theor. Math. Phys. **2** 231 (1998).

[21] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Nucl.Phys. **B636** 99 (2002).

[22] B. Eden, M. Staudacher, J. Stat. Mech. **0611** P014 (2006).

[23] N. Beisert, B. Eden, M. Staudacher, J. Stat. Mech. **0701** P021 (2007).

[24] A.V. Kotikov, L.N. Lipatov, Nucl. Phys. **B769** 217 (2007).

[25] R.C. Brower, J. Polchinski, M.J. Strassler, C-I Tan, preprint hep-th/0603115.