The Pancharatnam phase in two flavor neutrino oscillations

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We show existence of a topological phase in the phenomena of neutrino oscillations in vacuum and in matter for the minimal case of two flavors and CP conserving situation.

1 Introduction

In the ultra-relativistic limit, the Dirac equation for two flavor neutrinos (antineutrinos) can be reduced to a Schrödinger form written in terms of a two-component vector of positive (negative) energy probability amplitude. The two neutrino flavor states can be mapped to a two-level quantum system with distinct energy eigenvalues, $E_i \simeq p + m_i^2/2p$ in the ultrarelativistic limit under the assumption of equal fixed momenta (or energy). The mapping of the two flavor neutrino Hamiltonian to that of a two level quantum system straightforwardly leads to the identification of the topological component in the total phase, which is the central result of this article.

2 The Hamiltonian for the two flavor neutrinos

Ignoring the term proportional to the Identity, the neutrino Hamiltonian (both in vacuum and matter) can be cast in exactly the same form given by

$$\mathbb{H}_\nu \quad = \quad \frac{\omega}{2}\left[(\sin\vartheta)\sigma_x - (\cos\vartheta)\sigma_z\right] \ , \tag{1}$$

where $\omega = \delta m^2/2p$ and the mixing angle Θ is replaced by $\vartheta/2$ [1]. The mass eigenstates $|\,\vartheta, +\,\rangle$ and $|\,\vartheta, -\,\rangle$ are orthogonal antipodal points on the Poincaré sphere which always lie on the great circle formed by the intersection of the $x - z$ plane with the Poincaré sphere. And neutrino oscillations can be viewed as the neutrino flavor state precessing about the line joining the stationary mass eigenstates (analogous to elliptic axis) induced by the time-evolution operator $e^{-i\mathbb{H}_\nu t}$ on the Poincaré sphere. In the language of neutrino optics, both vacuum and matter exhibit elliptic birefringence property with different elliptic axes.

[1] In defining the Poincaré sphere, it is useful to work with half angles $\vartheta/2$ as it allows for a mapping of the entire set of states on to a two-dimensional sphere $\mathbb{S}_2$ as ϑ changes from 0 to 4π.

3 The Pancharatnam phase

In the context of two level system, if we consider three rays on the Poincaré sphere represented by $|\,\mathfrak{A}\,\rangle$, $|\,\mathfrak{B}\,\rangle$ and $|\,\mathfrak{C}\,\rangle$ such that the neighbouring ones are non-orthogonal, then the phase of the complex number $\langle\,\mathfrak{A}\,|\,\mathfrak{C}\,\rangle\langle\,\mathfrak{C}\,|\,\mathfrak{B}\,\rangle\langle\,\mathfrak{B}\,|\,\mathfrak{A}\,\rangle$ is given by $\Omega/2$ [1, 2] where Ω is equal to half the solid angle subtended by the geodesic triangle $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ on the Poincaré sphere at its center. Pancharatnam's phase reflects the curvature of projective Hilbert space (ray space) and is independent of any parameterization or slow variation. Thus it can also appear in situations where the Hamiltonian is constant in time.

4 Applying Pancharatnam's idea to neutrino oscillations

In order to probe effects related to quantum phases (dynamical or geometric), one usually needs a split beam experiment. It is impossible to design a split-beam experiment in physical space for neutrinos due to their feeble interaction strength. However, the fact that neutrinos are produced and detected as flavor states allows us to think of the time evolution of neutrinos as a split-beam experiment in energy space as illustrated in [3]. Let us consider a neutrino created as a flavor state $|\,\nu_\alpha\,\rangle$,

$$|\,\nu_\alpha\,\rangle = \nu_{\alpha+}|\,\vartheta_1, +\,\rangle + \nu_{\alpha-}|\,\vartheta_1, -\,\rangle \ , \tag{2}$$

where $|\,\vartheta_1, \pm\,\rangle$ are the eigenstates of $\mathbb{H}_\nu(\vartheta_1)$. Evolving the mass eigenstates adiabatically from $|\,\vartheta_1, \pm\,\rangle$ to $|\,\vartheta_2, \pm\,\rangle$ due to a slow enough variation of background density such that no mixing between the two eigenstates is ensured under time evolution leads to

$$\begin{aligned} |\,\vartheta_1, \pm\,\rangle &\rightarrow e^{-i\mathcal{D}_\pm}|\,\vartheta_2, \pm\,\rangle \quad \text{with} \\ \mathcal{D}_\pm &= \pm\frac{1}{2}\int_0^t \sqrt{(\omega\sin\vartheta)^2 + (V_C - \omega\cos\vartheta)^2}dt' + \int_0^t \left(p + \frac{m_1^2 + m_2^2}{4p} + \frac{V_C}{2} + V_N\right) dt' \end{aligned}$$

as the dynamical phases, relevant both for the vacuum case ($V_C = V_N = 0$) and in the presence of varying matter density profile and t is the time of flight of the neutrino. $V_C = \sqrt{2}G_F n_e = 7.6 \times 10^{-14} Y_e \rho$ eV and $V_N = -\sqrt{2}G_F n_n/2 = -3.8 \times 10^{-14} Y_n \rho$ eV are the respective effective potentials due to coherent forward scattering of neutrinos with electrons (via charged current interactions) and neutrons (via neutral current interactions). The oscillation probability for transition $\nu_\alpha \rightarrow \nu_\beta$ is given by

$$\begin{aligned} \mathcal{P}(\nu_\alpha \rightarrow \nu_\beta) &= |\mathcal{A}(\nu_\alpha \rightarrow \nu_\beta)|^2 = \langle\,\nu_\alpha\,|\,\vartheta_1, +\,\rangle\langle\,\vartheta_2, +\,|\,\nu_\beta\,\rangle\langle\,\nu_\beta\,|\,\vartheta_2, +\,\rangle\langle\,\vartheta_1, +\,|\,\nu_\alpha\,\rangle \\ &+ \langle\,\nu_\alpha\,|\,\vartheta_1, -\,\rangle\langle\,\vartheta_2, -\,|\,\nu_\beta\,\rangle\langle\,\nu_\beta\,|\,\vartheta_2, -\,\rangle\langle\,\vartheta_1, -\,|\,\nu_\alpha\,\rangle \\ &+ [\langle\,\nu_\alpha\,|\,\vartheta_1, -\,\rangle e^{i\mathcal{D}_-}\langle\,\vartheta_2, -\,|\,\nu_\beta\,\rangle\langle\,\nu_\beta\,|\,\vartheta_2, +\,\rangle e^{-i\mathcal{D}_+}\langle\,\vartheta_1, +\,|\,\nu_\alpha\,\rangle + \text{c.c.}] \ . \end{aligned} \tag{3}$$

By closely inspecting the form of cross term appearing in the neutrino oscillation probability, we note that it is related to the interference term resulting from the two path interferometer [3]. Upon dropping the dynamical phase, we have $\langle\,\nu_\alpha\,|\,\vartheta_1, -\,\rangle\langle\,\vartheta_2, -\,|\,\nu_\beta\,\rangle\langle\,\nu_\beta\,|\,\vartheta_2, +\,\rangle\langle\,\vartheta_1, +\,|\,\nu_\alpha\,\rangle$ which can be viewed as a series of closed loop quantum collapses with intermediate adiabatic evolutions given by $|\,\nu_\alpha\,\rangle \rightarrow |\,\vartheta_1, +\,\rangle \rightarrow |\,\vartheta_2, +\,\rangle \rightarrow |\,\nu_\beta\,\rangle \rightarrow |\,\vartheta_2, -\,\rangle \rightarrow |\,\vartheta_1, -\,\rangle \rightarrow |\,\nu_\alpha\,\rangle$ that essentially covers a great circle in the $x - z$ plane as is shown in Fig. 1(a). This closed trajectory subtends a solid angle of $\Omega = 2\pi$ at the center of the great circle. Hence the phase of the

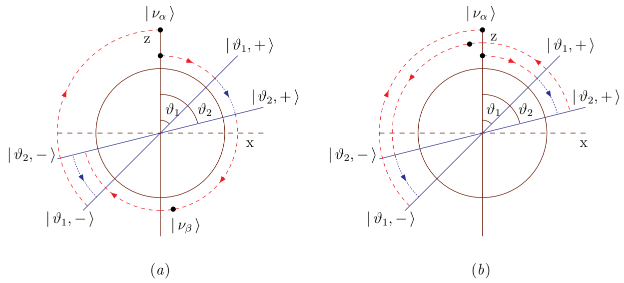

Figure 1: Direction of collapse processes (dashed, red) and adiabatic evolutions (dotted, blue) on the great circle in $x-z$ plane of the Poincaré sphere for appearance and survival probability.

interference term will be π (half the solid angle) due to Pancharatnam's prescription. For the case when $\alpha = \beta$, i.e. survival probability, it is easy to see that the collapses do not enclose the origin and therefore the interference term will not pick up any phase. This case is depicted in Fig. 1(b).

5 Conclusion

Our study provides the first clear prediction that a topological phase exists at the probability level even in the minimal case of two flavors and CP conservation. This is connected to the presence of an effective flux line of strength π (0) at the origin of ray space which is same as degeneracy point associated with the null Hamiltonian. It is shown that the topological phase is quite robust since it remains irrespective of evolution of neutrinos through vacuum or matter. The topological phase is built into the structure of the leptonic mixing matrix. The impact of CP violation is studied in [4] where it is shown that the phase can become geometric (different from π or 0) if the CP phase also varies as a function of time (space).

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