# Renormdynamics, Valence Quarks and Multiparticle Production

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Concise introduction in QCD renormdynamics with prediction of the pion-nucleon and low energy QCD fine structure constants and valence quark mechanism of the multiparticle production given.

## 1 Renormdynamics

Quantum field theory (QFT) and Fractal calculus (FC) provide Universal language of fundamental physics (see e.g. [13]). In QFT existence of a given theory means, that we can control its behavior at some scales (short or large distances) by renormalization theory [3]. If the theory exists, than we want to solve it, which means to determine what happens on other (large or short) scales. This is the problem (and content) of Renormdynamics. The result of the Renormdynamics, the solution of its discrete or continual motion equations, is the effective QFT on a given scale (different from the initial one).

### 1.1 p-adic convergence of perturbation theory series

Perturbation theory series (PTS) have the following qualitative form

$$f(g) = f_0 + f_1 g + ... + f_n g^n + ..., \ f_n = n!P(n)$$
$$f(x) = \sum_{n\geq 0} P(n)n!x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x}, \ \delta = x\frac{d}{dx} \tag{1}$$

So, we reduce previous series to the standard geometric progression series. This series is convergent for $|x| < 1$ or for $|x|_p < 1, \ x = p^k a/b, \ k \geq 1$. With proper nomalization of the expansion parametre, the coefficients of the series are rational numbers and if experimental data indicates for some prime value for $g$, e.g. in QED

$$g = \frac{e^2}{4\pi} = \frac{1}{137.0...} \tag{2}$$

then we can take corresponding prime number and consider p-adic convergence of the series. In the case of QED, we have

$$f(g) = \sum f_n p^{-n}, \ f_n = n!P(n), \ p = 137, \ |f|_p \leq \sum |f_n|_p p^n \tag{3}$$

In the Yukawa theory of strong interactions (see e.g. [1]), we take $g = 13$,

$$f(g) = \sum f_n p^n,\ f_n = n!P(n),\ p = 13, |f|_p \leq \sum |f_n|_p p^{-n} < \frac{1}{1-p^{-1}} \tag{4}$$

So, the series is convergent. If the limit is rational number, we consider it as an observable value of the corresponding physical quantity.

In $MSSM$ (see [10]) coupling constants unifies at $\alpha_u^{-1} = 26.3 \pm 1.9 \pm 1$. So,

$$23.4 < \alpha_u^{-1} < 29.2 \tag{5}$$

Question: how many primes are in this interval?

$$24,\ 25,\ 26,\ 27,\ 28,\ 29 \tag{6}$$

Only one!

Proposal: take the value $\alpha_u^{-1} = 29.0...$ which will be two orders of magnitude more precise prediction and find the consequences for the $SM$ scale observables.

Let us make more explicit the formal representation of (1)

$$\begin{aligned} f(x) &= \sum_{n\geq 0} P(n)n!x^n = P(\delta)\Gamma(1+\delta)\frac{1}{1-x}, \\ &= P(\delta)\int_0^\infty dte^{-t}t^\delta \frac{1}{1-x} = P(\delta)\int_0^\infty dt\frac{e^{-t}}{1+(-x)t},\ \delta = x\frac{d}{dx} \end{aligned} \tag{7}$$

This integral is well defined for negative values of $x$. The Mathematica answer for the corresponding integral is

$$I(x) = \int_0^\infty dt\frac{e^{-t}}{1+xt} = e^{1/x}\Gamma(0,1/x)/x,\ Im(x) \neq 0,\ Re(x) \geq 0,\ I(0) = 1 \tag{8}$$

where $\Gamma(a,z)$ is the incomplete gamma function

$$\Gamma(a,z) = \int_z^\infty dtt^{a-1}e^{-t} \tag{9}$$

For $x = 0.001,\ I(x) = 0.999$

## 1.2 The Goldberger-Treiman relation and the pion-nucleon coupling constant

The Goldberger-Treiman relation (GTR) [5] plays an important role in theoretical hadronic and nuclear physics. GTR relates the Meson-Nucleon coupling constants to the axial-vector coupling constant in $\beta$-decay:

$$g_{\pi N} f_\pi = g_A m_N \tag{10}$$

where $m_N$ is the nucleon mass, $g_A$ is the axial-vector coupling constant in nucleon $\beta$-decay at vanishing momentum transfer, $f_\pi$ is the $\pi$ decay constant and $g_{\pi N}$ is the $\pi - N$ coupling constant.

Since the days when the Goldberger-Treiman relation was discovered, the value of $g_A$ has increased considerably. Also, $f_\pi$ decreased a little, on account of radiative corrections. The main source of uncertainty is $g_{\pi N}$.

If we take

$$\alpha_{\pi N} = \frac{g_{\pi N}^2}{4\pi} = 13 \Rightarrow g_{\pi N} = 12.78 \tag{11}$$

experimental value for $f_\pi$ from pion decay and neutron mass

$$f_\pi = \frac{130}{\sqrt{2}} = 91.9MeV, \ m_N = 940MeV, \tag{12}$$

from (10), we find

$$g_A = \frac{f_\pi g_{\pi N}}{m_N} = \frac{91.9 \times \sqrt{52\pi}}{940} = 1.2496 \simeq 1.25 = \frac{5}{4} \tag{13}$$

In an old version of the unified theory [7], for the $\alpha_{\pi N}$ the following value were found

$$\alpha_{\pi N} = 4\pi(1 - \frac{m_\pi^2}{3m_p^2}) = 12.5 \tag{14}$$

Determination of $g_{\pi N}$ from $NN, N\bar{N}$ and $\pi N$ data by the Nijmegen group [16] gave the following value

$$g_{\pi N} = 13.05 \pm .08, \ \Delta = 1 - \frac{g_A m_N}{g_{\pi N} f_\pi} = .014 \pm .009, \ 13.39 < \alpha_{\pi N} < 13.72 \tag{15}$$

This value is consistent with assumption $g_{\pi N} = 13 \Rightarrow \alpha_{\pi N} = 13.45$

Due to the smallness of the u and d quark masses, $\Delta$ is necessarily very small, and its determination requires a very precise knowledge of the $g_{\pi N}$ coupling ($g_A$ and $f_\pi$ are already known to enough precision, leaving most of the uncertainty in the determination of $\Delta$ to the uncertainty in $g_{\pi N}$).

## 2 Renormdynamics of QCD

QCD is the theory of the strong interactions with, as only inputs, one mass parameter for each quark species and the value of the QCD coupling constant at some energy or momentum scale in some renormalization scheme. This last free parameter of the theory can be fixed by $\Lambda_{QCD}$, the energy scale used as the typical boundary condition for the integration of the Renormdynamic (RD) equation for the strong coupling constant. This is the parameter which expresses the scale of strong interactions, the only parameter in the limit of massless quarks. While the evolution of the coupling with the momentum scale is determined by the quantum corrections induced by the renormalization of the bare coupling and can be computed in perturbation theory, the strength itself of the interaction, given at any scale by the value of the renormalized coupling at this scale, or equivalently by $\Lambda_{QCD}$, is one of the above mentioned parameters of the theory and has to be taken from experiment.

The RD equations play an important role in our understanding of Quantum Chromodynamics and the strong interactions. The beta function and the quarks mass anomalous dimension are among the most prominent objects for QCD RD equations. The calculation of the one-loop $\beta$-function in QCD has lead to the discovery of asymptotic freedom in this model and to the establishment of QCD as the theory of strong interactions [8, 6, 15].

The MS-scheme [9] belongs to the class of massless schemes where the $\beta$-function does not depend on masses of the theory and the first two coefficients of the $\beta$-function are scheme-independent.

The Lagrangian of QCD with massive quarks in the covariant gauge is

$$L = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{q}_n(i\gamma D - m_n)q_n - \frac{1}{2\xi}(\partial A)^2 + \partial^\mu \bar{c}^a(\partial_\mu c^a + gf^{abc}A^b_\mu c^c)$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu,\ (D_\mu)_{kl} = \delta_{kl}\partial_\mu - igt^a_{kl}A^a_\mu, \tag{16}$$

$A^a_\mu, a = 1, ..., N_c^2 - 1$ are gluon; $q_n, n = 1, ..., n_f$ are quark; $c^a$ are ghost fields; $\xi$ is gauge parameter; $t^a$ are generators of fundamental representation and $f^{abc}$ are structure constants of the Lie algebra

$$[t^a, t^b] = if^{abc}t^c, \tag{17}$$

we consider an arbitrary compact semi-simple Lie group G. For QCD, $G = SU(N_c), N_c = 3$.

The RD equation for the coupling constant is

$$\dot{a} = \beta(a) = -\beta_2 a^2 - \beta_3 a^3 - \beta_4 a^4 - \beta_5 a^5 + O(a^6),$$
$$a = \frac{\alpha_s}{4\pi} = (\frac{g}{4\pi})^2,\ \int_{a_0}^a \frac{da}{\beta(a)} = t - t_0 = \ln\frac{\mu^2}{\mu_0^2}, \tag{18}$$

$\mu$ is the 't Hooft unit of mass, the renormalization point in the MS-scheme.

To calculate the $\beta$-function we need to calculate the renormalization constant $Z$ of the coupling constant, $a_b = Za$, where $a_b$ is the bare (unrenormalized) charge.

The expression of the $\beta$-function can be obtained in the following way

$$0 = d(a_b\mu^{2\varepsilon})/dt = \mu^{2\varepsilon}(\varepsilon Za + \frac{\partial(Za)}{\partial a}\frac{da}{dt})$$
$$\Rightarrow \frac{da}{dt} = \beta(a,\varepsilon) = \frac{-\varepsilon Za}{\frac{\partial(Za)}{\partial a}} = -\varepsilon a + \beta(a),\ \beta(a) = a\frac{d}{da}(aZ_1) \tag{19}$$

where

$$\beta(a,\varepsilon) = \frac{D-4}{2}a + \beta(a) \tag{20}$$

is $D-$dimensional $\beta-$function and $Z_1$ is the residue of the first pole in $\varepsilon$ expansion

$$Z(a,\varepsilon) = 1 + Z_1\varepsilon^{-1} + ... + Z_n\varepsilon^{-n} + ... \tag{21}$$

Since $Z$ does not depend explicitly on $\mu$, the $\beta$-function is the same in all MS-like schemes, i.e. within the class of renormalization schemes which differ by the shift of the parameter $\mu$.

For quark anomalous dimension, RD equation is

$$\dot{b} = \gamma(a) = -\gamma_1 a - \gamma_2 a^2 - \gamma_3 a^3 - \gamma_4 a^4 + O(a^5),$$
$$b(t) = b_0 + \int_{t_0}^t dt\gamma(a(t)) = b_0 + \int_{a_0}^a da\gamma(a)/\beta(a). \tag{22}$$

To calculate the quark mass anomalous dimension $\gamma(g)$ we need to calculate the renormalization constant $Z_m$ of the quark mass $m_b = Z_m m,\ m_b$ is the bare (unrenormalized) quark mass. Than we find the function $\gamma(g)$ in the following way

$$\begin{aligned}
&0 = \dot{m}_b = \dot{Z}_m m + Z_m \dot{m} = Z_m m((\ln Z_m)^{\cdot} + (\ln m)^{\cdot}) \\
&\Rightarrow \gamma(a) = -\frac{d\ln Z_m}{dt} = \dot{b} = -\frac{d\ln Z_m}{da}\frac{da}{dt} = -\frac{d\ln Z_m}{da}(-\varepsilon a + \beta(a)) \\
&= a\frac{dZ_{m1}}{da},\ b = -\ln Z_m = \ln\frac{m}{m_b},
\end{aligned} \tag{23}$$

where RD equation in $D-$dimension is

$$\dot{a} = -\varepsilon a + \beta(a) = \beta_1 a + \beta_2 a^2 + ... \tag{24}$$

and $Z_{m1}$ is the coefficient of the first pole in the $\varepsilon-$expantion of the $Z_m$ in $MS$-scheme

$$Z_m(\varepsilon, g) = 1 + Z_{m1}(g)\varepsilon^{-1} + Z_{m2}(g)\varepsilon^{-2} + ... \tag{25}$$

Since $Z_m$ does not depend explicitly on $\mu$ and $m$, the $\gamma_m$-function is the same in all MS-like schemes.

## 2.1 Reparametrization and general method of solution of the RD equation

RD equation,

$$\dot{a} = \beta_1 a + \beta_2 a^2 + ... \tag{26}$$

can be reparametrized,

$$a(t) = f(A(t)) = A + f_2 A^2 + ... + f_n A^n + ... = \sum_{n\geq 1} f_n A^n, \tag{27}$$

$$\begin{aligned}
&\dot{A} = b_1 A + b_2 A^2 + ... = \sum_{n\geq 1} b_n A^n, \\
&\dot{a} = \dot{A} f'(A) = (b_1 A + b_2 A^2 + ...)(1 + 2f_2 A + ... + n f_n A^{n-1} + ...) \\
&= \beta_1(A + f_2 A^2 + ... + f_n A^n + ...) + \beta_2(A^2 + 2f_2 A^3 + ...) + ... \\
&+\beta_n(A^n + n f_2 A^{n+1} + ...) + ... \\
&= \beta_1 A + (\beta_2 + \beta_1 f_2)A^2 + (\beta_3 + 2\beta_2 f_2 + \beta_1 f_3)A^3 + \\
&... + (\beta_n + (n-1)\beta_{n-1} f_2 + ... + \beta_1 f_n)A^n + ... \\
&= \sum_{n,n_1,n_2\geq 1} A^n b_{n_1} n_2 f_{n_2} \delta_{n,n_1+n_2-1} \\
&= \sum_{n,m\geq 1; m_1,...,m_k\geq 0} A^n \beta_m f_1^{m_1} ... f_k^{m_k} f(n, m, m_1, ..., m_k), \\
&f(n, m, m_1, ..., m_k) = \frac{m!}{m_1! ... m_k!}\delta_{n,m_1+2m_2+...+km_k}\delta_{m,m_1+m_2+...+m_k},
\end{aligned} \tag{28}$$

$$\begin{aligned}
&b_1 = \beta_1,\ b_2 = \beta_2 + f_2\beta_1 - 2f_2 b_1 = \beta_2 - f_2\beta_1, \\
&b_3 = \beta_3 + 2f_2\beta_2 + f_3\beta_1 - 2f_2 b_2 - 3f_3 b_1 = \beta_3 + 2(f_2^2 - f_3)\beta_1,
\end{aligned}$$

$$b_4 = \beta_4 + 3f_2\beta_3 + f_2^2\beta_2 + 2f_3\beta_2 - 3f_4b_1 - 3f_3b_2 - 2f_2b_3, ...$$
$$b_n = \beta_n + ... + \beta_1 f_n - 2f_2b_{n-1} - ... - nf_nb_1, ... \tag{29}$$

so, by reparametrization, beyond the critical dimension ($\beta_1 \neq 0$) we can change any coefficient but $\beta_1$.

We can fix any higher coefficient with zero value, if we take

$$f_2 = \frac{\beta_2}{\beta_1},\ f_3 = \frac{\beta_3}{2\beta_1} + f_2^2,\ ...\ , f_n = \frac{\beta_n + ...}{(n-1)\beta_1}, ... \tag{30}$$

In the critical dimension of space-time, $\beta_1 = 0$, and we can change by reparametrization any coefficient but $\beta_2$ and $\beta_3$.

From the relations (29), in the critical dimenshion ($\beta_1 = 0$), we find that, we can define the minimal form of the RD equation

$$\dot{A} = \beta_2 A^2 + \beta_3 A^3, \tag{31}$$

We can solve (31) as implicit function,

$$u^{\beta_3/\beta_2} e^{-u} = ce^{\beta_2 t},\ u = \frac{1}{A} + \frac{\beta_3}{\beta_2} \tag{32}$$

then, as in the noncritical case, explicit solution will be given by reparametrization representation (27) [14].

If we know somehow the coefficients $\beta_n$, e.g. for first several exact and for others asymptotic values (see e.g. [11]) than we can construct reparametrization function (27) and find the dynamics of the running coupling constant. This is similar to the action-angular canonical transformation of the analytic mechanics (see e.g. [4]).

Statement: The reparametrization series for $a$ is p-adically convergent, when $\beta_n$ and $A$ are rational numbers.

## 2.2 Reparametrization of the anomalous dimensions

Let us take the the anomalous dimension of some quantity

$$\gamma(a) = \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + ... \tag{33}$$

and make reparametrization

$$a = f(A) = A + f_2A^2 + f_3A^3 + ... \tag{34}$$

$$\gamma(a) = \gamma_1(A + f_2A^2 + f_3A^3 + ...) + \gamma_2(A^2 + 2f_2A^3 + ...) + \gamma_3(A^3 + ...) + ...$$
$$= \Gamma_1 A + \Gamma_2 A^2 + \Gamma_3 A^3 + ...$$
$$\Gamma_1 = \gamma_1,\ \Gamma_2 = \gamma_2 + \gamma_1 f_2,\ \Gamma_3 = \gamma_3 + 2\gamma_2 f_2 + \gamma_1 f_3, ... \tag{35}$$

When $\gamma_1 \neq 0$, we can take $\Gamma_n = 0,\ n \geq 2$, if we define $f_n$ as

$$f_2 = -\frac{\gamma_2}{\gamma_1},\ f_3 = -\frac{\gamma_3 + 2\gamma_2 f_2}{\gamma_1} = -\frac{\gamma_3 - 2\gamma_2^2/\gamma_1}{\gamma_1}, ... \tag{36}$$

So, we get the exact value for the anomalous dimension

$$\gamma(A) = \gamma_1 A = \gamma_1 f^{-1}(a) = \gamma_1(a + \gamma_2/\gamma_1 a^2 + \gamma_3/\gamma_1 a^3 + ... :) \tag{37}$$

### 2.3 QCD, parton model, valence quarks and $\alpha_s = 2$

While it has been well established in the perturbative regime at high energies, QCD still lacks a comprehensive solution at low and intermediate energies, even 40 years after its invention. In order to deal with the wealth of non-perturbative phenomena, various approaches are followed with limited validity and applicability. This is especially also true for lattice QCD, various functional methods, or chiral perturbation theory, to name only a few. In neither one of these approaches the full dynamical content of QCD can yet be included. Basically, the difficulties are associated with a relativistically covariant treatment of confinement and the spontaneous breaking of chiral symmetry, the latter being a well-established property of QCD at low and intermediate energies. As a result, most hadron reactions, like resonance excitations, strong and electroweak decays etc., are nowadays only amenable to models of QCD. Most famous is the constituent-quark model (CQM), which essentially relies on a limited number of effective degrees of freedom with the aim of encoding the essential features of low- and intermediate-energy QCD.

The CQM has a long history, and it has made important contributions to the understanding of many hadron properties, think only of the fact that the systematization of hadrons in the standard particle-data base follows the valence-quark picture. Namely the $Q$ dependence of the nucleon form factor corresponds to three-constituent picture of the nucleon and is well described by the simple equation [2], [12]

$$F(Q^2) \sim (Q^2)^{-2} \tag{38}$$

It was noted [17] that parton densities given by the following solution

$$\begin{aligned}
&M_2(Q^2) = \frac{3}{25} + \frac{2}{3}\omega^{32/81} + \frac{16}{75}\omega^{50/81}, \\
&\bar{M}_2(Q^2) = M_2^s(Q^2) = \frac{3}{25} - \frac{1}{3}\omega^{32/81} + \frac{16}{75}\omega^{50/81}, \\
&M_2^G(Q^2) = \frac{16}{25}(1 - \omega^{50/81}), \\
&\omega = \frac{\alpha_s(Q^2)}{\alpha_s(m^2)},\ Q^2 \in (5, 20)GeV^2,\ b = 9,\ \alpha_s(Q^2) \simeq 0.2
\end{aligned} \tag{39}$$

of the Altarelli-Parisi equation

$$\begin{aligned}
&\dot{M} = AM, \\
&M^T = (M_2, \bar{M}_2, M_2^s, M_2^G), \\
&M_2 = \int_0^1 dxx(u(x) + d(x)),\ \bar{M}_2 = \int_0^1 dxx(\bar{u}(x) + \bar{d}(x)), \\
&M_2^s = \int_0^1 dxx(s(x) + \bar{s}(x)),\ M_2^G = \int_0^1 dxxG(x), \\
&A = -a(Q^2)\begin{pmatrix} 32/9 & 0 & 0 & -2/3 \\ 0 & 32/9 & 0 & -2/3 \\ 0 & 0 & 32/9 & -2/3 \\ -32/9 & -32/9 & -32/9 & 2 \end{pmatrix},\ a = (\frac{g}{4\pi})^2,\ \dot{M} = Q^2\frac{dM}{dQ^2}
\end{aligned} \tag{40}$$

with the following "valence quark" initial condition at a scale $m$

$$M_2(m^2) = 1,\ \bar{M}_2(m^2) = M_2^s(m^2) = M_2^G(m^2) = 0, \tag{41}$$

and

$$\alpha_s(m^2) = 2, \tag{42}$$

gives the experimental values

$$M_2 = 0.44,\ \bar{M}_2 = M_2^s = 0.04,\ M_2^G = 0.48 \tag{43}$$

So, for valence quark model (VQCD), $\alpha_s(m^2) = 2$. We have seen, that for $\pi\rho N$ model $\alpha_{\pi\rho N} = 3$, and for $\pi N$ model $\alpha_{\pi N} = 13$. It is nice that $\alpha_s^2 + \alpha_{\pi\rho N}^2 = \alpha_{\pi N}$. This relation can be seen, e.g., by considering pion propagator in the low energy $\pi N$ model and in superposition of higher energy VQCD and $\pi\rho N$ models.

Note that $g = 5$ corresponds to the

$$\alpha_s = \frac{g^2}{4\pi} = 1.989 \simeq 2 \tag{44}$$

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