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Construction of a Gauge-Invariant Cutoff of Quantum Electrodynamics from a Non-Local Gauge-Invariant Lagrangian.

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Construction of a Gauge-Invariant Cutoff of Quantum Electrodynamics from a Non-Local Gauge-Invariant Lagrangian.

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Abstract.

A non-local gauge-invariant Lagrangian is given, which describes gauge-invariant quantum electro-dynamics with a modified fermion propagator. Some examples are discussed.

1. Introduction.

In a paper {1} on a simple model for the modification of quantum electrodynamics (QED), McClure and Drell investigated the problem of gauge-invariance of QED with modified fermion propagator. In particular they showed that gauge-invariance of the amplitudes for pair production and g-2 is not guaranteed by merely changing the conventional one-photon vertex as required to satisfy the Ward-Takahashi equation {1} together with the fermion propagator. One must include an additional two-photon vertex to restore gauge-invariance for these S-matrix elements.
Now the question arises whether one can obtain gauge-invariance for all S-matrix elements by adding, in a similar manner, a certain number of suitably chosen multi-photon vertices. If this were possible a simple procedure for constructing these vertices in a more systematic way would be highly desirable. This procedure should prevent the treatment of every S-matrix element separately.
Kroll {4} solved this problem by giving a suitable solution of the Chang-Mani equations. In this paper an answer is sought to the following question: if we postulate a modified fermion propagator, can we give a gauge-invariant Lagrangian which describes the interaction between a fermion field and the electromagnetic field, and which corresponds to the Lagrangian of conventional QED? By "correspondence" we mean that the Lagrangian should approach the conventional form if the propagator does.
The form factors corresponding to the propagator and vertex modifications lead to a non-local interaction as an instrument for the treatment of the problem to be discussed. It is not intended to discuss the well-known problems of non-local interactions (form factors and convergence properties, unitarity of the S-matrix, etc.). Here the non-local interaction merely serves as a well-suited instrument for the examination of the problem of gauge-invariant modifications of QED. Neither will

the field equation which stands here for the Dirac equation be investigated further. It was discussed extensively by Pais and Uhlenbeck {3}.

Instead of investigating the most general form of a modified propagator we are here mainly concerned with the substantial features of the procedure. Therefore we shall confine ourselves to substituting the conventional propagator

$$S_F(p) = \frac{1}{\not p + m} \qquad \text{by} \qquad S_F'(p) = \frac{1}{F(p^2)} \cdot \frac{1}{\not p + m} \quad .$$

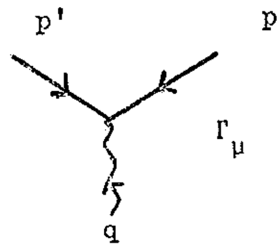
Generalizations should not be difficult.

As to mathematical exactness, we shall not apply the most rigorous standards, but the refinement of proofs should not involve any serious problems.

2. Construction of a Gauge-Invariant Lagrangian.

One is led to a non-local interaction by the fact that an interaction between fermion field and electromagnetic field with vertex {1}

$$\Gamma_\mu(p',p,q) = \delta(p' + p + q)\gamma_\mu \cdot C(p'^2,p^2)$$



with $C(p'^2,p^2)$ playing the rôle of a form factor can be described by the non-local interaction

$$H_i = \int d\xi d\zeta d\eta \cdot \psi(\xi)\tilde{C}\{(\xi-\eta)^2,(\zeta-\eta)^2\}\gamma_\mu A^\mu(\eta)\psi(\zeta) \quad . \qquad ^{1)}$$

Now the Ward-Takahashi equation requires a one-photon vertex similarly furnished with form factors when the fermion propagator is modified. Therefore one must try to find a Lagrangian which corresponds to the non-local gauge-invariant interaction with the propagator

1) $\tilde{C}$ is the Fourier transform of the function C.

(1) $$S'_F(p) = \frac{1}{\tilde{F}(p^2)} \cdot \frac{1}{\not p + m}$$

for the free fermion field.

2.1 The Lagrangian.

It is evident that we obtain the postulated propagator by substituting the Lagrangian for the Dirac field by [1]

$$L_D = \int d\xi d\zeta \cdot \bar{\psi}(\xi) \left\{ (i \quad -m) F \left((\xi-\zeta)^2\right) \right\} \psi(\zeta)$$

This term is not invariant against gauge transformations of the second kind

$$\psi(x) \to \psi'(x) = \psi(x) \cdot e^{ie\Lambda(x)}$$

$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x) \cdot e^{-ie\Lambda(x)}$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$$

but we can find a proper "interaction Lagrangian" L_i different from that of conventional QED with the sum $L = L_D + L_i$ being gauge-invariant as in conventional QED. Extending a remark of C. Bloch (2) concerning non-local meson-nucleon interaction we find

(2) $$L = \int d\xi d\zeta \cdot \bar{\psi}(\xi)\alpha(\xi,\zeta) \left\{ (i \not\partial_\xi - m) F(\xi,\zeta) \right\} \psi(\zeta) \quad .$$ [2]

---

1) For the moment we shall disregard the requirement of invariance of the Lagrangian against charge conjugation.
This question is discussed on page 11 f.

2) Here we must add the Lagrangian L_{em} for the free electromagnetic field to obtain the total Lagrangian for the interacting fields. We shall omit L_{em}, as L_{em} is gauge-invariant, and as it is not necessary to modify it.

α is a functional of the electromagnetic field having the form

(3) $\alpha(\xi,\zeta) = e^{-ie\int d^4x\cdot A^\mu(x)G_\mu(x;\xi,\zeta)}$

and $F(\xi,\zeta) \equiv F\{(\xi-\zeta)^2\}$.

The function G_μ in the exponent is not uniquely determined but has to fulfill two restrictive conditions. We request that

1. L is gauge-invariant and
2. for $F(\xi,\zeta) \to \delta(\xi-\zeta)$ the function L approaches the Lagrangian of conventional QED ("correspondence principle").

These conditions are equivalent to the following equations for G_μ:

(4) $\partial_x^\mu G_\mu(x;\xi,\zeta) = \delta(\xi-x) - \delta(\zeta-x)$

(5) $G_\mu(x;\xi,\xi) = 0$

(6) $G_\mu(x;\xi,\zeta)\overleftarrow{\not\partial}_\xi\Big|_{\zeta=\xi} = -\gamma_\mu\delta(x-\xi).$

Fourier transformation with respect to x gives [1)]

(4a) $is^\mu G_\mu(s;\xi,\zeta) = e^{is\zeta} - e^{is\xi}$

instead of (4).

Equations (4) and (4a) are equivalent to gauge invariance, while (5) and (6) guarantee correspondence with conventional QED.

2.2 Gauge Invariance and Correspondence: Proof for Equations (4)-(6).

Gauge transformation of A_μ is equivalent to the substitution

$$\alpha(\xi,\zeta) \to \alpha(\xi,\zeta)\cdot e^{-ie\int dx\cdot\partial^\mu\Lambda(x)\cdot G_\mu(x;\xi,\zeta)}\ .$$

---

1) We define for any function $F(x_1,\ldots,x_n)$:

$$\tilde{F}(p_1,\ldots,p_n) = \int dx_1\ldots dx_n\cdot e^{i\sum p_i x_i}F(x_1,\ldots,x_n)$$

and therefore

$$F(x_1,\ldots,x_n) = \frac{1}{(2\pi)^{4n}}\int dp_1\ldots dp_n\cdot e^{-i\sum p_i x_i}\tilde{F}(p_1,\ldots,p_n)$$

Assuming that three-dimensional integrals of the type

$$\int d^4_x \cdot \partial^\mu_x \{ \Lambda(x) G_\mu(x;\xi,\zeta) \}$$

vanish, the additional factor can be written

$$e^{ie\int dx\cdot\Lambda(x)\,\partial^\mu_x G_\mu(x;\xi,\zeta)}.$$

If G_μ fulfills equation (4), this exponential function is annihilated by the factor

$$e^{-ie\Lambda(\xi)+ie\Lambda(\zeta)},$$

generated by gauge transformation of ψ and $\overline{\psi}$.
To obtain (5) and (6) we substitute $F(\xi,\zeta) = \delta(\xi-\zeta)$.
Now we can integrate by parts with respect to ζ. Assuming again that three-dimensional integrals shall vanish, we have

$$L = -\int d\xi\cdot \{\overline{\psi}(\xi)\alpha(\xi,\zeta)\}(i\overleftarrow{\not\partial}_\xi+m)\Big|_{\zeta=\xi}\cdot\psi(\xi)$$

with

$$\overline{\psi}(\xi)\alpha(\xi,\zeta)(i\overleftarrow{\not\partial}\xi+m) = \{i\partial^\mu_\xi\overline{\psi}(\xi)\}\,e^{-ieM(\xi,\zeta)}\cdot\gamma_\mu +$$

$$i\overline{\psi}(\xi)\,\{\partial^\mu_\xi e^{-ieM(\xi,\zeta)}\}\,\gamma_\mu + m\overline{\psi}(\xi)e^{-ieM(\xi,\zeta)}$$

(We write $\int dx\cdot A^\mu(x)G_\mu(x;\xi,\zeta) = M(\xi,\zeta)$).

Inserting the power series for the exponential function and anticipating the terms of order o in e, we get

$$\begin{aligned}\overline{\psi}(\xi)\alpha(\xi,\zeta)(i\overleftarrow{\not\partial}_\xi+m) &= \overline{\psi}(\xi)(i\overleftarrow{\not\partial}_\xi+m) +\\ &+\{i\partial^\mu_\xi\overline{\psi}(\xi)\}\,(e^{-ieM(\xi,\zeta)}-1)\,\gamma_\mu+\\ &+m\overline{\psi}(\xi)\,(e^{-ieM(\xi,\zeta)}-1)+\\ &+i\overline{\psi}(\xi)\,\{-ie\partial^\mu_\xi M(\xi,\zeta)+\frac{(-ie)^2}{2!}\left[(\partial^\mu_\xi M)M+M(\partial^\mu_\xi M)\right]+\ldots\}\,\gamma_\mu.\end{aligned}$$

We see that L approaches

$$L_{conv.} = -\int d\xi\{\overline{\psi}(\xi)(i\overleftarrow{\not\partial}_\xi+m)\psi(\xi)-\overline{\psi}(\xi)\cdot e\not A(\xi)\cdot\psi(\xi)\},$$

if in the last expression the second and third lines vanish for $\zeta = \xi$, that is

$$M(\xi,\xi) = o,$$

and the remaining term of the last line

$$e\bar{\psi}(\xi)\left[M(\xi,\zeta)\overleftarrow{\partial\!\!\!/}_{\xi}\right]_{\zeta=\xi}$$

becomes $-e\bar{\psi}(\xi)A\!\!\!/(\xi)$. A_μ being arbitrary, we infer equations (5) and (6). From this proof we see that there is an important restriction for (5) and (6): in this form they hold only if G_μ is not a functional of F. But if G_μ is a functional of F (we shall discuss an example) we can require them only in a restricted manner: if $F(\xi,\zeta) \to \delta(\xi-\zeta)$ then G_μ approaches a function G^o_μ, and equations (5), (6) hold for G^o_μ.

2.3 The Free Fermion Field.

For $e \to o$ we have the modified Dirac field

$$L_D = \int d\xi d\zeta \cdot \bar{\psi}(\xi) \left\{ (i\partial\!\!\!/_{\xi}-m)F(\xi,\zeta) \right\} \psi(\zeta)$$

with the field equations

$$\frac{\delta L_D}{\delta\bar{\psi}(x)} - \partial_\mu \frac{\delta L_D}{\delta\left(\partial_\mu\bar{\psi}(x)\right)} = \int d\xi \cdot F(x-\xi) \left\{ (i\partial\!\!\!/_{\xi}-m)\psi(\xi) \right\} = o.$$

This is a Dirac equation

$$(i\partial\!\!\!/_{x}-m)\psi_M(x) = o$$

for the smeared field operator

$$\psi_M(x) = \int d\xi \cdot F(x-\xi)\ \psi(\xi).$$

Transforming into momentum space we have

$$(7) \qquad (p\!\!\!/+m)\tilde{F}(p^2)u(p) = o$$

with $\tilde{F}$ being a scalar in spin space. The last equation means that the solutions of the conventional Dirac equation also fulfill the modified equation. Field equations of the type (7) are discussed extensively by Pais and Uhlenbeck {3}, and we shall not treat (7) any further. For the propagator for free fermions we infer from (7) the desired form

$$S'_F(p) = \frac{1}{\tilde{F}(p^2)} \cdot \frac{1}{\not{p}+m} .$$

Using the Lagrangian

$$L = \int d\xi d\zeta \cdot \bar{\psi}(\xi)\alpha(\xi,\zeta) \left\{ i\not{\partial}_\xi F_1\left((\xi-\zeta)^2\right) - mF_2\left((\xi-\zeta)^2\right) \right\} \psi(\zeta) \tag{2a}$$

instead of (2), one arrives at the most general form of the fermion propagator:

$$S'_F(p) = \frac{B_1(p^2)\not{p} + B_2(p^2)m}{p^2 - m^2}$$

To make things as clear as possible we shall continue to examine the simpler Lagrangian (2). The more general form (2a) is not expected to involve additional difficulties.

3. The Interaction.

3.1 The Vertices in Configuration Space and Momentum Space.

Inserting the power series for α we get the following perturbation expansion for the Lagrangian:

$$L = \sum_{n=o}^{\infty} L_n$$

with the terms

$$L_n = \frac{(-ie)^n}{n!} \int d\xi d\zeta d\eta_1 \ldots d\eta_n \cdot \bar{\psi}(\xi) A^{\mu_1}(\eta_1) \ldots A^{\mu_n}(\eta_n) \times \tag{8}$$

$$\times K_{\mu_1 \ldots \mu_n}(\xi,\zeta;\eta_1,\ldots,\eta_n)\psi(\zeta).$$

Thus we have a theory characterized by the vertices

$$K_{\mu_1 \ldots \mu_n}(\xi,\zeta;\eta_1,\ldots,\eta_n) = G_{\mu_1}(\eta_1;\xi,\zeta)\ldots G_{\mu_n}(\eta_n;\xi,\zeta)(i\not{\partial}_\xi - m)F(\xi,\zeta) \tag{9}$$

which suggests the application of Feynman rules as in conventional QED (for more details see page 12 f and (2)).

Without any further calculation we notice the following property: In a gauge-invariant theory the modified fermion propagator is connected with a one-photon vertex of the form

(9a) $$K_\mu(\xi,\zeta;\eta) = G_\mu(\eta;\xi,\zeta)\,(i\not\partial_\xi - m)F(\xi,\zeta).$$

G_μ is an arbitrary solution of (4) which guarantees the validity of the Ward-Takahashi equation for $S_F'(p)$ and K_μ. This means that the Ward-Takahashi equation couples the modification of the one-photon vertex to that of the propagator, but that this connection is not a definite one. With the propagator fixed we have a large degree of freedom to choose suitable vertices K_μ and functions G_μ respectively. If we want to maintain correspondence to conventional QED as described we have two additional equations (5), (6) for G_μ. They cause a further reduction of the number of candidates for one-photon vertices K_μ, but they cannot make G_μ definite. In addition we have a certain number - generally an infinite one - of multiphoton vertices $K_{\mu_1..\mu_n}$ (n=2,3,..) as a consequence of the requirement of gauge invariance. In momentum space we can write for the one-photon vertex

$$\tilde{K}_\mu(r,t;s) = \frac{1}{(2\pi)^4}\int dp\cdot\tilde{G}_\mu(s;r-p,t+p)\tilde{H}(p) \qquad 1)$$

$$H(\xi-\zeta) \equiv (i\not\partial_\xi - m)F(\xi,\zeta)\ .$$

Now we shall require that we have a momentum-conserving factor at this vertex. Then $\tilde{G}_\mu$ must be of the form

(10) $$\tilde{G}_\mu(s;r,t) = \delta(r+t+s)\tilde{g}_\mu(r,s)\ ,$$

therefore

(10a) $$G_\mu(\eta;\xi,\zeta) = \frac{1}{(2\pi)^4}\cdot g_\mu(\xi-\zeta,\eta-\zeta)\ .$$

As a consequence every vertex is furnished with a δ-function:

(11) $$\tilde{K}_{\mu_1\ldots\mu_n}(r,t;s_1,\ldots,s_n) = \frac{1}{(2\pi)^{4(2n-1)}}\cdot\delta(r+t+\sum_1^n s_i)\cdot\tilde{M}_{\mu_1\ldots\mu_n}(r,s_1,\ldots,s_n)$$

1) In the following r,t are fermion momenta, while s, s_i are momenta of photons.

and the "reduced vertex" $\tilde{M}_{\mu_1\ldots\mu_n}$ can be calculated by folding the vertex $\tilde{M}_{\mu_2,\ldots,\mu_n}$ with $\tilde{g}_{\mu_1}$:

$$\tilde{M}_{\mu_1\ldots\mu_n}(r,s_1,\ldots,s_n) = \int dp\cdot\tilde{g}_{\mu_1}(r-p,s_1)\tilde{M}_{\mu_2\ldots\mu_n}(p,s_2,\ldots,s_n)$$

(12)

$$\tilde{M}_{\mu}(r,s) = \int dp\cdot\tilde{g}_{\mu}(r-p,s)\tilde{H}(p) \ .$$

Transformation into the configuration space of the fermion with momentum r yields

(12a)
$$\tilde{M}_{\mu_1\ldots\mu_n}(x,s_1,\ldots,s_n) = (2\pi)^4 g_{\mu_1}(x,s_1)\tilde{M}_{\mu_2\ldots\mu_n}(x,s_2,\ldots,s_n) =$$
$$= (2\pi)^{4n}\cdot g_{\mu_1}(x,s_1)\ldots g_{\mu_n}(x,s_n)H(x).$$

For the reduced function g_μ we have equations

(4') $$\partial_x^\mu g_\mu(x,y) = (2\pi)^4\{\delta(x-y) - \delta(y)\}$$

resp. $$is^\mu\tilde{g}_\mu(r,s) = (2\pi)^8\{\delta(r) - \delta(r+s)\}$$

(5') $$g_\mu(x,y)\Big|_{x=o} = o$$

(6') $$g_\mu(x,y)\overleftarrow{\not\partial}_x\Big|_{x=o} = -(2\pi)^4\gamma_\mu\delta(y)$$

instead of (4)-(6). (4') and (12) are equivalent to the Chang-Mani equations {4}(generalizations of the Ward-Takahashi equation) for the reduced vertices

$$is_1^{\mu_1}\tilde{M}_{\mu_1\ldots\mu_n}(r,s_1,\ldots,s_n) = (2\pi)^8\{\tilde{M}_{\mu_2\ldots\mu_n}(r,s_2,\ldots,s_n) - \tilde{M}_{\mu_2\ldots\mu_n}(r+s_1, s_2,\ldots, s_n)\}$$

Generally we have an infinite number of multi-photon vertices, but from the "correspondence principle" (5), (6) (see also page 1) we can derive: if $\tilde{F}(p^2)$ is a polynomial [1] of p^2 with degree m, and G_μ is not a functional of F, all vertices $K_{\mu_1\ldots\mu_n}$ with n>m+1 vanish. For the p r o o f we transform

[1] Then we have derivative coupling.

(9) into momentum space. If $\tilde{F}(p^2)$ is a polynomial of p^2 then $F(x^2)$ has the form of a differential polynomial acting on the δ-function. This means that $H(x)$ has the form

$$H(x) \equiv (i\not{\partial}_x - m)F(x^2) = P_{m+1}(\partial_x)\delta(x)$$

with P_{m+1} being an invariant differential polynomial of degree m+1; thus we get

$$\tilde{K}_{(\mu_i)}(r,t;s_i) = \int d\xi d\zeta \cdot e^{ir\xi+it\zeta} \cdot G_{\mu_1}(s_1;\xi,\zeta)\ldots G_{\mu_n}(s_n;\xi,\zeta)P_{m+1}(\partial_\xi)\delta(\xi-\zeta) \ .$$

After integration by parts with respect to ξ the last expression becomes

$$\int d\zeta \ \{e^{ir\xi+it\zeta} \cdot G_{\mu_1}(s_1;\xi,\zeta)\ldots G_{\mu_n}(s_n;\xi,\zeta)Q_{m+1}(\overleftarrow{\partial}\xi) \Big|_{\xi=\zeta} \ .$$

The differential polynomial Q_{m+1} is generated from P_{m+1} by the process of integration. Once more we have assumed that three-dimensional integrals shall vanish.

If we perform the differentiation, only terms with every factor G_μ differentiated at least once are not equal to zero. That means that in the last integral the number of factors G_μ must be less than or equal to the order of the differential polynomial. But as derivative coupling would involve serious physical difficulties, we shall generally exclude polynomials for $\tilde{F}(p^2)$.

We have mentioned (page 8) that within the framework of this paper for a given fermion propagator the one-photon vertex is not uniquely determined. But it does not suffice to fix the one-photon vertex in addition to get the interaction definite; we have still much more freedom within the formalism. For illustration we show that we can easily construct gauge-invariant interactions without a one-photon vertex:

$$L' = \sum_{n=2}^{\infty} \frac{(-ie)^n}{n!} \int d\xi d\zeta d\eta_1 \ldots d\eta_n \cdot \bar{\psi}(\xi)A^{\mu_1}(\eta_1)\ldots A^{\mu_n}(\eta_n)K_{(\mu_i)}(\xi,\zeta;\eta_i)\psi(\zeta) \ .$$

Given an interaction L we can add L' without modifying propagator $S_F'(p)$ and vertex K_μ of L. To arrive at L' we can proceed in the following way: Let R_μ, S_μ be solutions of equs. (4)-(6), then

$$G_\mu = \frac{1}{2}(R_\mu + S_\mu)$$

is a third solution. That means:

$$\alpha_1(\xi,\zeta) = e^{-ie\int dxA^\mu(x)G_\mu(x;\xi,\zeta)}$$

and

$$\alpha_2(\xi,\zeta) = \frac{1}{2}\left\{e^{-ie\int dxA^\mu(x)R_\mu(x;\xi,\zeta)} + e^{-ie\int dxA^\mu(x)S_\mu(x;\xi,\zeta)}\right\}$$

give us gauge-invariant interactions L_1 and L_2 that will approach the conventional interaction for $F(\xi,\zeta)\to\delta(\xi-\zeta)$. The difference

$$L' = L_1 - L_2$$

is a gauge-invariant interaction with α in L substituted by

$$\alpha'(\xi,\zeta) = \alpha_1(\xi,\zeta) - \alpha_2(\xi,\zeta) .$$

L' has the desired form with vertices

$$K_{\mu_1\mu_2}\xi,\zeta;\eta_1,\eta_2) = -\frac{1}{4}\left\{R_{\mu_1}(\eta_1;\xi,\zeta)-S_{\mu_1}(\eta_1;\xi,\zeta)\right\}\left\{R_{\mu_2}(\eta_2;\xi,\zeta)-S_{\mu_2}(\eta_2,\xi,\zeta)\right\} \text{ etc.}$$

It should be possible to construct, in a similar way, interactions with all vertices vanishing up to a given order. This is equivalent to the statement that the Chang-Mani equations are not uniquely solvable.

3.2 Charge Conjugation. Application of Feynman Rules {2}.

In general L is not invariant against charge conjugation if we form normal products, that means we have to use the invariant Lagrangian

(13) $$L' = \frac{1}{2}(L + L_c)$$

instead of (2) to obtain a theory that is invariant against charge conjugation. We get L_c from L by the process of substituting the field operators by the charge-conjugated ones: $\psi_c = C\overline{\psi}$ etc. The terms of the perturbation expansion of L_c can be written

$$(14) \qquad L_n^C = \frac{(-ie)^n}{n!} \int \frac{d\xi d\zeta}{d\eta_1 \ldots d\eta_n} \cdot \psi(\zeta) A^{\mu_1}(\eta_1) \ldots A^{\mu_n}(\eta_n) K^C_{\mu_1 \ldots \mu_n}(\xi,\zeta;\eta_1,\ldots,\eta_n)\overline{\psi}(\xi)$$

with vertices

$$(15) \qquad K^C_{\mu_1 \ldots \mu_n}(\xi,\zeta;\eta_1,\ldots,\eta_n) = (-1)^n \cdot C_T^{-1} K_{\mu_1 \ldots \mu_n}(\zeta,\xi;\eta_1 \ldots,\eta_n) C \quad .$$

If we want to apply the method of doubled graphs for calculating out-operators or the conventional Feynman rules for the computation of S-matrix elements {2} [1], the n-photon vertices in both L_n and L_n^c contribute characteristical terms of the form

$$\frac{1}{2}\psi_{IN}(\xi) K_{(\mu_i)}(\xi,\zeta;\eta_i) A_{IN}^{\mu_1}(\eta_1) \ldots A_{IN}^{\mu_n}(\eta_n) \psi_{IN}(\zeta) +$$

$$+ \frac{1}{2}\psi_{iN}(\zeta) K^C_{(\mu_i)}(\xi,\zeta;\eta_i) A_{iN}^{\mu_1}(\eta_1) \ldots A_{iN}^{\mu_n}(\eta_n) \overline{\psi}_{iN}(\xi) =$$

$$= \frac{1}{2} \{ K^{\sigma\rho}_{(\mu_i)}(\xi,\zeta;\eta_i) \overline{\psi}^{iN}_\sigma(\xi) \psi^{iN}_\rho(\zeta) +$$

$$+ {}^C K^{\rho\sigma}_{(\mu_i)}(\xi,\zeta;\eta_i) \psi^{iN}_\rho(\zeta) \overline{\psi}^{iN}_\sigma(\xi) \} \; A_{iN}^{\mu_1}(\eta_1) \ldots A_{iN}^{\mu_n}(\eta_n) \quad .$$

Writing this as a sum of normal products and using

$$: \psi^{iN}_\rho(\zeta) \overline{\psi}^{iN}_\sigma(\xi) : \; = - : \overline{\psi}^{iN}_\sigma(\xi) \psi^{iN}_\rho(\zeta) : \quad ,$$

we can describe the contribution of L' to a definite normal product by the vertex function

$$K'_{(\mu_i)}(\xi,\zeta;\eta_i) = \frac{1}{2} \{ K_{(\mu_i)}(\xi,\zeta;\eta_i) - {}^C K^T_{(\mu_i)}(\xi,\zeta;\eta_i) \} \quad ,$$

or, with (15) inserted, by

---

1) We are aware of certain difficulties concerning the unitarity of the S-matrix involved in using Feynman rules in this way {2}. In the spirit of the introduction we shall not enter into these problems here but will investigate them in future work.

$$(16)\quad K'_{(\mu_i)}(\xi,\zeta;\eta_i) = \frac{1}{2}\{K_{(\mu_i)}(\xi,\zeta;\eta_i)+(-1)^{n+1}\cdot CK^T_{(\mu_i)}(\zeta,\xi;\eta_i)C_T^{-1}\}$$

In special cases such as conventional QED L itself will yield C-invariant expressions after the formation of normal products and therefore correct vertex function and matrix elements. However, it is impossible to formulate this property as a simple additional equation for G_μ. The form of this equation depends on the transformation properties of G_μ in the space of the Dirac matrices γ_μ. If G_μ is a scalar, for instance, it has to be antimetric with respect to ξ and ζ.

4. Special Solutions of the Fundamental Equations (4) - (6).

4.1 Degeneration of $\int dxA^\mu(x)G_\mu(x;\xi,\zeta)$ to a Path Integral.

A special solution of (4a) having all the required properties is

$$(17a)\qquad G_\mu(s;\xi,\zeta) = \frac{(\xi-\zeta)_\mu}{is(\xi-\zeta)}\left(e^{is\zeta} - e^{is\xi}\right)$$

resp.

$$(17b)\qquad G_\mu(x;\xi,\zeta) = \frac{1}{(2\pi)^4}\cdot(\xi-\zeta)_\mu\int dk\cdot\frac{e^{ik(\zeta-x)}-e^{ik(\xi-x)}}{ik(\xi-\zeta)}$$

when transformed into the configuration space of all particles. We can write G_μ in the form

$$(17c)\qquad G_\mu(x;\xi,\zeta) = \int_\xi^\zeta dz_\mu\cdot\delta^{(4)}(x-z)\ ,$$

integrated over the straight line from ξ to ζ. To show that this representation of G_μ is identical with (17b), we insert in (17c) the Fourier transform of the δ-function and anticipate the integration with respect to z_μ.

In this special solution the electromagnetic field in the Lagrangian appears as a path integral

$$\int dxA^\mu(x)G_\mu(x;\xi,\zeta) = \int_\xi^\zeta dz_\mu A^\mu(z)\ ,$$

integrated over the straight line. Moreover, every function of the type (17c) with an arbitrary way of integration is a solution. It corresponds to an exponential function

$$\alpha(\xi,\zeta) = \exp\left\{-ie^{(\mathfrak{L})}\int_{\xi}^{\zeta} dx^{\mu}A_{\mu}(x)\right\}$$

in the Lagrangian.

Subsequently, only the simplest case of a rectilinear way of integration is considered [1]. We obtain the vertices

$$(18a)\quad K_{(\mu_i)}(\xi,\zeta;\eta_i) = l_{\mu_1}\cdots l_{\mu_n} I(\eta_1;\xi,\zeta)\cdots I(\eta_n;\xi,\zeta)(i\partial\!\!\!/_{\xi}-m)F(\xi,\zeta)$$

with

$$l_{\mu} = (\xi-\zeta)_{\mu}$$

$$I(\eta;\xi,\zeta) = \frac{1}{(2\pi)^4}\int dk\,\frac{e^{ik(\zeta-\eta)}-e^{ik(\xi-\eta)}}{ik(\xi-\zeta)}$$

or in momentum space:

$$(18b)\quad \tilde{K}_{(\mu_i)}(r,t;s_i) = i^n\delta(r+t+\sum s_i)\int dq\cdot R_n(r-q,s_i)\frac{\partial^n}{\partial q_{\mu_1}\cdots\partial q_{\mu_n}}(q\!\!\!/-m)\tilde{F}(q^2)$$

with

$$R_n(r,s_1,\ldots,s_n) = \int dk\cdot e^{ikr}\prod_{i=1}^{n}\frac{e^{iks_i}-1}{iks_i}\quad .$$

It is evident that vertices of higher order will vanish if $\tilde{F}(p^2)$ is a polynomial.

A detailed examination of the one-photon vertex in this example will be postponed to Section 6. There we shall show that, unlike the one-photon vertices of the following examples, this $\tilde{K}_{\mu}$ is different from the conventional value even for the two fermions on-shell. This is equivalent to introducing a form factor for the electron that modifies, for instance, the matrix element for e-p-scattering.

As for (16) we find here

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1) This path seems to be the "natural" one in the sense that it is the only path that can be described in a relativistic invariant way.

$$K'_\mu(\xi,\zeta;\eta) = K_\mu(\xi,\zeta;\eta) \quad .$$

4.2 A Simple Model without Electron Form Factor.

Another simple solution is

$$(20) \quad G_\mu(\zeta;\xi,\zeta) = \gamma_\mu \frac{1}{i\not s} \left(e^{is\zeta} - e^{is\xi}\right) .$$

Within the class of solutions of (4)

$$G_\mu(s;\xi,\zeta) = \frac{\alpha_\mu(s)}{i(s\alpha)} e^{is\zeta} - \frac{\beta_\mu(s)}{i(s\beta)} e^{is\xi}$$

the solution (20) is uniquely determined by the "correspondence principle". The one-photon vertex used in the determination of S-matrix elements by application of Feynman rules becomes

$$(21) \quad \begin{aligned} \tilde{K}'_\mu(r,t;s) &= (2\pi)^4 \cdot i\gamma_\mu \delta(r+t+s) \cdot \frac{1}{2} \{\tilde{F}(r^2) + \tilde{F}(t^2)\} + \\ &+ (2\pi)^4 \cdot i\delta(r+t+s) \cdot \frac{\tilde{F}(t^2) - \tilde{F}(r^2)}{2s^2} \{\gamma_\mu \not s(\not r - m) - (\not t + m)\not s \gamma_\mu\} . \end{aligned}$$

For the two-photon vertex we find

$$\tilde{K}_{\mu\nu}(r,t;p,q) = (2\pi)^4 \delta(r+t+p+q) \cdot \gamma_\mu \frac{1}{i\not p} \gamma_\nu \frac{1}{i\not q} \times$$

$$\times \{(\not r - m)\tilde{F}(r^2) - (\not r - m + \not q)\tilde{F}((r+q)^2) - (\not r - m + \not p)\tilde{F}((r+p)^2) -$$

$$- (\not t + m)\tilde{F}(t^2)\}$$

resp.

$$\tilde{K}'_{\mu\nu}(r,t;p,q) = (2\pi)^4 \delta(r+t+p+q) \times$$

$$\times \left\{ \frac{\tilde{F}(r^2) - \tilde{F}((r+p)^2) - \tilde{F}((r+q)^2) - \tilde{F}(t^2)}{2p^2 q^2} \{(\not t + m)\not q \gamma_\nu \not p \gamma_\mu - \gamma_\mu \not p \gamma_\nu \not q(\not r - m)\} + \right.$$

$$+ \frac{\tilde{F}(r^2) - \tilde{F}((r+\not p)^2)}{2p^2} \cdot \gamma_\nu \not p \gamma_\mu - \frac{\tilde{F}(t^2) - \tilde{F}((t+p)^2)}{2p^2} \gamma_\mu \not p \gamma_\nu +$$

$$\left. + \frac{\tilde{F}(r^2) - \tilde{F}((r+q)^2)}{2p^2 q^2} \cdot \not p \not q \gamma_\nu \not p \gamma_\mu - \frac{\tilde{F}(t^2) - \tilde{F}((r+p)^2)}{2p^2 q^2} \cdot \gamma_\mu \not p \gamma_\nu \not q \not p \right\} ,$$

taking into consideration the requirement of C-invariance according to (16). If we apply this vertex to pair production the terms of the first line do not contribute on account of the factors $(\not{t}+m)$, $(\not{r}-m)$.

4.3 The Vertex of McClure and Drell {1}.

Here things are somewhat more complicated. The one-photon vertex used by McClure and Drell {1} to satisfy the Ward-Takahashi equation can be written {4} :

$$\tilde{K}_\mu(r,t;s) = (2\pi)^4 i\delta(r+t+s) \times$$

(22)

$$\times\left\{(\not{t}+m)\tilde{F}(t)^2 \cdot \frac{\not{t}\gamma_\mu-\gamma_\mu\not{r}}{t^2-r^2} + \frac{\not{t}\gamma_\mu-\gamma_\mu\not{r}}{t^2-r^2}(\not{r}-m)\tilde{F}(r^2)\right\}$$

(As in 5.1 : $\tilde{K}'_\mu = \tilde{K}_\mu$).

To reconstruct (22) and to build up higher-order vertices, we have to find a solution of (4) - (6) with the property

(9a) $\quad G_\mu(\eta;\xi,\zeta)H(\xi-\zeta) = K_\mu(\xi,\zeta;\eta)$,

K_μ is the Fourier-transformed of (22) [1)]

A function G_μ which fulfills (9a) is

$$G_\mu(\eta;\xi,\zeta) = \frac{1}{(2\pi)^8}\int drdt\; e^{ir(\xi-\eta)+it(\zeta-\eta)} \times$$

(23a)

$$\times\left\{\frac{\not{t}\gamma_\mu-\gamma_\mu\not{r}}{t^2-r^2}\tilde{H}(-r) - \tilde{H}(t)\frac{\not{t}\gamma_\mu-\gamma_\mu\not{r}}{t^2-r^2}\right\}H^{-1}(\xi-\zeta)$$

or

$$i\tilde{G}_\mu(s;p;q) = \delta(p+q+s)\int dxdz\cdot R_\mu(x,s)H(z-x)H^{-1}(z)e^{ipz} -$$

(23b)

$$-\delta(p+q+s)\int dxdz\cdot H(z+x)R_\mu(x,s)H^{-1}(z)e^{iqz}$$

with

---

1) At first sight one might take

$$(2\pi)^4 i\delta(r+t+s)\frac{\not{t}\gamma_\mu-\gamma_\mu\not{r}}{t^2-r^2}(\not{r}-m)\tilde{F}(r^2)$$

for a suitable vertex $\tilde{K}_\mu$, as it yields (22) for $\tilde{K}'_\mu$. But for this simpler expression the Ward-Takahashi equation would not hold.

$$R_\mu(x,s) = \int dr \, \frac{\gamma_\mu \not{r} + (\not{s}+\not{r})\gamma_\mu}{(s+r)^2 - r^2} \, e^{-irx} \quad .$$

Now we have to show that this function is a solution of the fundamental equations. Equation (4) brings no difficulties, but for the proof of (5) and (6) we remark that now G_μ is a functional of F. If $F(\xi,\zeta) \to \delta(\xi-\zeta)$ the function iG_μ approaches

$$iG^{(o)}_\mu(\eta;\xi,\zeta) = -\gamma_\mu \delta(\zeta-\eta) \frac{\delta(\xi-\zeta)}{(i\not{\partial}_\xi - m)\delta(\xi-\zeta)} \quad .$$

Using the relation

$$\frac{d}{dx} \delta^{(1)}(x) = - \frac{\delta^{(1)}(x)}{x}$$

for the δ-function in one dimension, we see at once that

$$G^{(o)}_\mu(\eta;\xi,\xi) = o$$

and in a similar way we can show

$$iG^{(o)}_\mu(\eta;\xi,\zeta)\overleftarrow{\not{\partial}}_\xi \Big|_{\xi\to\zeta} = -\gamma_\mu \delta(\xi-\eta) \lim_{x=o} \frac{\delta(x)}{(i\not{\partial}_x - m)\delta(x)} \overleftarrow{\not{\partial}}_x =$$

$$= -i\gamma_\mu \delta(\xi-\eta).$$

The calculation of the two-photon vertex will be rather complicated, and we do not expect to get the same $\tilde{K}_{\mu\nu}$ as McClure and Drell on account of the ambiguity of the two-photon vertex with the one-photon vertex fixed.

4.4 The Modification of Kroll {4}.

Using equation (12) we can define a linear operator $D_\mu(s)$:

$$(24) \quad \tilde{M}_{\mu_1 \ldots \mu_n}(r,s_1,\ldots,s_n) = \int dp \cdot \tilde{g}_{\mu_1}(p,s_1) \tilde{M}_{\mu_2 \ldots \mu_n}(r-p,s_2,\ldots,s_n) =$$

$$= D_{\mu_1}(s_1) \tilde{M}_{\mu_2 \ldots \mu_n}(r,s_2,\ldots,s_n)$$

As $\tilde{g}_\mu$ satisfies (5'), (6'), we can easily prove

(25a) $D_\mu(s)\alpha = o$

(25b) $D_\mu(s)\not{x} = (2\pi)^8 \cdot i\gamma_\mu$

if α is constant in momentum space, but possibly a Dirac matrix. From the definition of $D_\mu(s)$ we get

$$D_\mu(s)\alpha = \int dp \cdot \tilde{g}_\mu(p,s)\alpha = \int dp\tilde{g}_\mu(p,s)e^{-ipx}\Big|_{x=o} \cdot \alpha =$$

$$= (2\pi)^4 g_\mu(x,s)\Big|_{x=o} \cdot \alpha = o$$

and

$$D_\mu(s)\not{x} = \int dp \cdot \tilde{g}_\mu(p,s)(\not{x}-\not{p}) = -\int dp\tilde{g}_\mu(p,s)\not{p} =$$

$$= -i\int dpe^{-ipx} \cdot \tilde{g}_\mu(p,s)\overleftarrow{\not{\partial}}_x\Big|_{x=o} = -i(2\pi)^4 g_\mu(x,s)\overleftarrow{\not{\partial}}_x\Big|_{x=o} = i(2\pi)^8\gamma_\mu \ .$$

Starting from the propagator, $D_\mu(s)$ generates the vertices in the same way as the operator $d_\mu(k)$ does in the paper of Kroll. If we require the additional equation

(26) $D_\mu(s)\{\tilde{F}(\not{x})G(\not{x})\} = \{D_\mu(s)\tilde{F}(\not{x})\}\tilde{G}(\not{x}) + \tilde{F}(\not{x}+\not{s})\{D_\mu(s)\tilde{G}(\not{x})\}$

to be satisfied by the operator $D_\mu(s)$ with $\tilde{F}(\not{x})$, $\tilde{G}(\not{x})$ out of a certain class of non-commuting functions {4}, the operator $-i/(2\pi)^8 \cdot D_\mu(s)$ has the same properties as $d_\mu(k)$ and we get the Kroll modification as a special case of the more general type, characterized by equations (5) and (6). Instead of (26) we can write an integral equation for $\tilde{g}_\mu$:

(27) $\int dp \cdot \tilde{g}_\mu(p,s)\tilde{F}(\not{x}-\not{p})\tilde{G}(\not{x}-\not{p}) = \int dp \cdot \tilde{g}_\mu(p,s)\tilde{F}(\not{x}-\not{p})\tilde{G}(\not{x}) +$

$+ \tilde{F}(\not{x}+\not{s}) \int dp \cdot \tilde{g}_\mu(p,s)\tilde{G}(\not{x}-\not{p})$.

The modification characterized by (25a,b) and (27) is a special case of that described by equations (5) and (6). This means that we can derive these equations from the integral equation.

Moreover g_μ is uniquely determined by the integral equation together with (25a,b). Choosing suitable functions for $\tilde{F}$, $\tilde{G}$ we can derive from these equations all partial derivatives of $g_\mu(x,y)$ with respect to the components of x at x = o. Thus we can give for $g_\mu(x,y)$ a power series in x_ν which shall converge within a certain circle around x = o. We shall briefly outline the calculation in the Appendix.

Physically the minimal modification of Kroll is distinguished from the others included in (2) by the fact that in the absence of closed loops the propagator modification has no influence upon the theory. But generally we have also to expect modifications in S-matrix elements without closed loops, as there is no reason to prefer the minimal modification.

5. The One-Photon Vertex with the Fermions On-Shell.

In this section we shall resume the more detailed discussion of the one-photon vertices of the examples discussed in section {4}. On this occasion also certain characteristics of the formalism itself will be revealed.

5.1 Current-Conservation and Boundary Value $\tilde{F}(m^2)$.

From (12) and (4') we get

$$s^\mu \tilde{K}_\mu(r,t;s) = -i(2\pi)^4 \delta(r+t+s)\{(\not{t}+m)\tilde{F}(t^2)+(\not{r}-m)\tilde{F}(r^2)\} ,$$

assuming we have free fermions with $\bar{u}(t)(\not{t}+m) = (\not{r}-m)u(r) = o$, the continuity equation

$$\bar{u}(t)s^\mu \tilde{K}_\mu(r,t;s)u(r) \doteq o$$

is satisfied.

Moreover, with both fermions on-shell:

$$s^\mu \tilde{K}_\mu(r,t;s) = i\not{s}(2\pi)^4 \delta(r+t+s)\tilde{F}(m^2) .$$

If, in addition, we put the photon on-shell ($s^2 = o$), we have to require that the last expression is the same as for the conventional vertex

$$s^\mu \cdot i\gamma_\mu(2\pi)^4 \delta(r+t+s) .$$

Thus we get the additional condition for $\tilde{F}$

$$\tilde{F}(m^2) = 1 .$$

5.2 The Kroll Modification {4}.

The fact that for the fermions on-shell the modified vertices $\tilde{K}_\mu$ are identical with the conventional vertex

$$(2\pi)^4 \cdot i\gamma_\mu \delta(r+t+s)$$

is obvious for the Drell vertex (22) and for the simple vertex (21) in 4.2 .

The Kroll modification has the same property: it does not influence the one-photon vertex with the fermions on-shell as for processes without closed loops it reproduces the conventional S-matrix elements {4}.

However, we think that it should also allow some test of the formalism to give a direct proof. For this case we shall assume that $\tilde{F}(p^2)$ can be developed into a power series at $p^2 = m^2$, that is, that $\tilde{F}(p^2)$ is an integral function [1]:

$$\tilde{F}(p^2) = \sum_{n=o}^{\infty} \frac{1}{n!} \tilde{F}^{(n)}(m^2)(p^2-m^2)^n$$

with

$$\tilde{F}^{(n)}(m^2) = \frac{d^n}{(dp^2)^n} \tilde{F}(p^2)\Big|_{p^2=m^2} \quad .$$

Then, from (12) we get for any model:

$$\tilde{M}_\mu(r,s) = \sum_{n=o}^{\infty} \frac{\tilde{F}^{(n)}(m^2)}{n!} \int dp \cdot \tilde{g}_\mu(r-p,s)(\not{p}-m)(p^2-m^2)^n \quad .$$

If, proceeding to the Kroll case, we insert in (27)

$$\tilde{G}(\not{r}) = (\not{r}-m)(r^2-m), \; \tilde{F}(\not{r}) = r^2-m^2$$

we find for $r^2 = m^2$ and $n > o$:

$$\int dp \cdot \tilde{g}_\mu(r-p,s)(\not{p}-m)(p^2-m^2)^{n+1} =$$

$$= \{(r+s)^2-m^2\} \int dp \cdot \tilde{g}_\mu(r-p,s)(\not{p}-m)(p^2-m^2)^n,$$

---

1) We have not examined ways for reducing this assumption.

which vanishes for the second fermion on-shell with $(r+s)^2 = m^2$. Thus we have reduced the power series for $\tilde{M}_\mu$ to the first two terms. The integrals occurring there are computed in the Appendix, and we get for the complete vertex

$$\tilde{K}_\mu(r,t;s) = (2\pi)^4 \cdot i\gamma_\mu \delta(r+t+s)\tilde{F}(m^2) +$$
$$+ (2\pi)^4 \cdot \frac{i}{2}(r_\mu - t_\mu)\delta(r+t+s)\tilde{F}'(m^2)(\not{r}-m) \quad \text{for } r^2 = t^2 = m^2.$$

For computing S-matrix elements we have expressions like

$$\bar{u}(t)\tilde{K}_\mu(r,t;s)u(r).$$

The second term in K_μ giving no contribution, we can substitute the vertex by

$$\tilde{K}_\mu(r,t;s)\Big|_{r^2=t^2=m^2} = (2\pi)^4 i\gamma_\mu\delta(r+t+s) .$$

5.3 The One-Photon Vertex for Example 4.1 : Degeneration of $\int dx \cdot A^\mu(x)G_\mu(x;\xi,\zeta)$ to a Path Integral.

This example yields a modified vertex $\tilde{K}_\mu$ even with both fermions on-shell.

For arbitrary momenta the reduced vertex can be written (see (18b)) :

$$\tilde{M}_\mu(r,s) = i(2\pi)^4 \int dq \cdot R_1(r-q,s)\frac{\partial}{\partial q_\mu}(\not{q}-m)\tilde{F}(q^2)$$

with

$$R(r-q,s) = \int dk \cdot e^{ik(r-q)} \cdot \frac{e^{iks}-1}{iks} .$$

If we assume $s_o = o$ for the moment we find

$$R_1(r-q,s) = \frac{(2\pi)^3}{s_o} \cdot \{\eta(r_o-q_o+s_o)-\eta(r_o-q_o)\}\delta^{(3)}\left(\vec{q}-\vec{r}+\vec{s}\cdot\frac{r_o-q_o}{s_o}\right)$$

with

$$\eta(r_o) = \int_{-\infty}^{+\infty} dx \cdot \frac{e^{irx}}{ix} = \begin{cases} +\pi & \text{for } r_o > o \\ o & \text{for } r_o = o \\ -\pi & \text{for } r_o < o \end{cases} .$$

When integrating in $\tilde{M}_\mu$ we have to substitute

$$q_\mu \to r_\mu - s_\mu \cdot \frac{r_o - q_o}{s_o} .$$

The singularity at $s_o = 0$ is a spurious one; it will cancel out later in the calculation.

Now the integrand in $\tilde{M}_\mu$ gives two terms:

$$\frac{\partial}{\partial q_\mu}(\not{q}-m)\tilde{F}(q^2) = \gamma_\mu \tilde{F}(q^2) \pm (\not{q}-m)\cdot 2q_\mu F'(q^2) \begin{cases} + \text{ for } \mu = o \\ - \text{ for } \mu = 1,2,3 \end{cases} ,$$

the first of which contributes the integral

$$I = \frac{i}{s_o}\gamma_\mu (2\pi)^7 \int_{-\infty}^{+\infty} dq_o \{\eta(r_o - q_o + s_o) - \eta(r_o - q_o)\} \tilde{F}\left\{\left(r - s\frac{r_o - q_o}{s_o}\right)^2\right\} .$$

Corresponding to the physical situation we have $r_o > o$, and if we put $s_o > o$ for the moment on account of

$$\eta(r_o + s_o - q_o) - \eta(r_o - q_o) = \begin{cases} 2\pi \text{ for } r_o < q_o < r_o + s_o \\ \pi \text{ for } q_o = r_o, \quad q_o = r_o + s_o \\ o \text{ for } q_o < r_o, \quad q_o > r_o + s_o \end{cases}$$

the domain of integration is restricted to $r_o \leqq q_o \leqq r_o + s_o$. Substituting by 2π the value π at the limits of the domain of integration (this does not change the value of the integral) we can write

$$I = \frac{i\gamma_\mu}{s_o}(2\pi)^8 \int_{r_o}^{r_o+s_o} dq_o \cdot \tilde{F}\left\{\left(r - s\frac{r_o - q_o}{s_o}\right)^2\right\} .$$

By substitution of the variable λ defined by

$$q_o = r_o + \lambda s_o \quad (o \leqq \lambda \leqq 1),$$

the integral takes an invariant form with the singularity at $s_o = o$

vanished, namely

$$I = i\gamma_\mu (2\pi)^8 \int_0^1 d\lambda \cdot \tilde{F}\{(r+\lambda s)^2\} .$$

For $s_o = o$ we get the same expression.
Dealing with the second term of the integrand in $\tilde{M}_\mu$ in the same manner, we find without difficulty

$$\tilde{K}_\mu(r,t;s) = (2\pi)^4 \cdot i\gamma_\mu \delta(r+t+s) \cdot D(r,s) +$$

(29)

$$+ (2\pi)^4 \cdot i\delta(r+t+s)\left\{(\not{r}-m)\frac{\partial}{\partial r_\mu} + (\not{t}+m)\frac{\partial}{\partial t_\mu}\right\} D(r,s)$$

with

$$D(r,s) = \int_0^1 d\lambda \cdot \tilde{F}\{(r+\lambda s)^2\} . \tag{30}$$

These formulae hold for any momenta of the three particles involved. If we form matrix elements for processes with both fermions on-shell the second term does not contribute. As to the first term we can transform $D(r,s)$. On account of $r^2 = t^2 = m^2$ the integrand $\tilde{F}$ has the argument

$$m^2 + s^2\lambda(\lambda-1) = \left(\frac{q}{2}\right)^2 .$$

As the integrand is symmetrical to $\lambda = \frac{1}{2}$ we can write for $D(r,s)$ in this special case, after substituting q for λ:

$$D_o(s^2) = \frac{1}{2\sqrt{s^2}} \int_{4m^2-s^2}^{4m^2} dq^2 \frac{\tilde{F}\left(\frac{q^2}{4}\right)}{\sqrt{q^2+s^2-4m^2}} \tag{30'}$$

and the one-photon vertex is now

$$\tilde{K}_\mu^{(o)}(r,t;s) = (2\pi)^4 i\gamma_\mu \delta(r+t+s) D_o(s^2) \quad (r^2=t^2=m^2) . \tag{29'}$$

For $\tilde{F}(q^2) \equiv 1$ we have $D(r,s) = D_o(s^2) = 1$, but with $\tilde{F}$ arbitrary we expect $D_o(s^2) \neq 1$.

The expression $D_o(s^2)$ has the meaning of a form factor for the electron; it yields a modified matrix element in the case of e-p-scattering, for instance.

Of course,

$$D_o(s^2) \to \tilde{F}(m^2) = 1,$$

if we also put the photon on-shell.

The experimental analysis of this model will be much more involved than, for instance, that of the Drell-McClure model. Here we get a correction factor for the conventional cross section for pair production in an external field that is formed from the function $\tilde{F}(q^2)$ and its "mean value" $D(r,s)$. With the model of Drell and McClure things are much simpler as it yields a correction factor formed in a rational way from $\tilde{F}$.

6. Summary.

We have developed a formalism which allows to construct gauge-invariant QED with given fermion propagator of the form

$$(1) \qquad S'_F(p) = \frac{1}{\tilde{F}(p^2)} \quad \frac{1}{\not p + m}$$

in a relatively simple way. Postulating a gauge-invariant Lagrangian we have made allowance for gauge invariance of the modified theory. Necessarily the Lagrangian has to be a non-local one. With the propagator (1) given we have the vertices

$$K_{(\mu_i)}(\xi,\zeta;\eta_i) = (2\pi)^{-4n}\cdot g_{\mu_1}(\xi-\zeta,\eta_1-\zeta)\ldots g_{\mu_n}(\xi-\zeta,\eta_n-\zeta)(i\not\partial_\xi-m)F\{(\xi-\zeta)^2\}$$

for this non-local interaction. Requiring that the Lagrangian is invariant against charge conjugation and using doubled graphs for computing out-operators or Feynman rules for computing S-matrix elements, the vertices $K_{(\mu_i)}$ of the interaction are substituted by

$$(16) \qquad K'_{(\mu_i)}(\xi\zeta;\eta_i) = \frac{1}{2}\{K_{(\mu_i)}(\xi,\zeta;\eta_i)+(-1)^{n+1}\cdot CK^T_{(\mu_i)}(\zeta,\xi;\eta_i)C_T^{-1}\}.$$

Functions g_μ suited for generating the vertices from the propagator have to fulfill equations (4'), (5'), and (6') which imply gauge invariance and correspondence to the conventional theory as explained in the introduction. They are by no means uniquely determined by these equations, that is, with a given fermion propagator we have great variety of interactions that yield a gauge-invariant theory with this propagator. This manifold also contains interactions distinguished by higher-order vertices only.

Generally we need an infinite number of multi-photon vertices to guarantee gauge invariance, but for $\tilde{F}(p^2)$ being a polynomial the higher-order vertices vanish. We have not discussed the most general form (2a) of a fermion propagator as generalizition should not be difficult.

Special models contained in the variety of solutions of the fundamental equations (4') - (6') are the vertex used by McClure and Drell {1} and the modification discussed by Kroll {4}. We have extensively discussed how to arrive at them by specification. In 4.1 and 5.3 we have examined another example that yields, contrary to the other solutions discussed, a form factor for the electron, that is, for this example we have a modified one-photon vertex even for the two fermions on-shell. In this context we have pointed to certain complications as they can occur in experimental analysis of a possibly modified QED.

We have not investigated the problems of the non-local interaction itself. As stressed in the introduction it has merely served as an aid for the examination of the problem of gauge invariance of a QED with modified propagator.

Appendix

Determination of $g_\mu(x,y)$ for the Minimal Modification.

If we insert

$$\tilde{G}(\not{x}) = \not{x}^n$$

$$\tilde{F}(\not{x}) = \gamma_\nu$$

in (27), we get

$$\int dp\cdot\tilde{g}_\mu(p,s)\gamma_\nu(\not{x}-\not{p})^n = \gamma_\nu\int dp\cdot\tilde{g}_\mu(p,s)(\not{x}-\not{p})^n .$$

For further treatment it is favorable to distinguish the two cases n even and n odd. Then we get

$$\text{(28a)}\quad \gamma_\nu\cdot\int dp\cdot\tilde{g}_\mu(p,s)(\not{x}-\not{p})^{2m}\cdot\gamma_\nu = 4\int dp\cdot\tilde{g}_\mu(p,s)(\not{x}-\not{p})^{2m}$$

$$\text{(28b)}\quad \gamma_\nu\int dp\cdot\tilde{g}_\mu(p,s)(\not{x}-\not{p})^{2m+1}\cdot\gamma_\nu = -2\int dp\cdot\tilde{g}_\mu(p,s)(\not{x}-\not{p})^{2m+1}$$

$$(m = 0,1,2,\dots) .$$

Together with

$$\text{(4')}\quad \frac{i}{(2\pi)^8}\cdot s^\mu\tilde{g}_\mu(p,s) = \delta(p) - \delta(p+s)$$

and using relations of the form

$$(r\partial_x)g_\mu(x,s)\Big|_{x=0} = -\frac{1}{(2\pi)^4}\int dp\cdot\tilde{g}_\mu(p,s)(rp)$$

$$g_\mu(x,s)\overleftarrow{\not{\partial}}^n\Big|_{x=0} = \frac{(-i)^n}{(2\pi)^4}\int dp\cdot\tilde{g}_\mu(p,s)\not{p}^n$$

etc.,

these equations are sufficient to compute every partial derivative of g_μ. By the way we notice that the computation of $g_\mu(x,s)\overleftarrow{\not{\partial}}_x^n\Big|_{x=0}$ is equivalent to that of $\overrightarrow{\not{\partial}}_x^n g_\mu(x,s)\Big|_{x=0}$ with the differential operator acting from the left. If we insert

$$F(x) = \not{\partial}_x^n\delta(x)$$

in the Fourier transform of (27) with G(x) arbitrary we infer

$$\overrightarrow{\not{\partial}}_x^n g_\mu(x,s)\Big|_{x=o} = (-1)^{n+1} g_\mu(x,s)\overleftarrow{\not{\partial}}_x^n\Big|_{x=o} \quad .$$

For $\underline{m = o}$ equation (28a) is fulfilled with

$$\int dp\tilde{g}_\mu(p,s) = o$$

and with this (28b) becomes

$$\gamma_\nu \int dp\cdot\tilde{g}_\mu(p,s)\not{p}\gamma_\nu = -2\int dp\cdot\tilde{g}_\mu(p,s)\not{p} \quad .$$

Considering the transformation properties of the integral in the space of the 16 Dirac matrices 1, γ_μ, $\gamma_\mu\gamma_\nu$,...., we infer

$$\int dp\cdot\tilde{g}_\mu(p,s)\not{p} = \gamma_\mu f(s) \quad .$$

The function f(s) is determined by (4'):

$$f(s) = -i(2\pi)^4 \quad ,$$

and we have finally

$$\int dp\cdot\tilde{g}_\mu(p,s)\not{p} = -i\gamma_\mu(2\pi)^8 = \int dp\cdot\not{p}\tilde{g}_\mu(p,s) \quad .$$

A short calculation leads to

$$\frac{1}{(2\pi)^8}\int dp\cdot\tilde{g}_\mu(p,s)(rp) = -ir_\mu \quad .$$

In a similar way we get for $\underline{m = 1}$ from (28a):

$$\frac{1}{(2\pi)^8}\int dp\cdot\tilde{g}_\mu(p,s)p^2 = is_\mu \quad .$$

For m fixed both the equations (28a,b) contain a certain number of relations for the computation of partial derivatives. For m = 1, for instance, (28b) yields the 5 relations:

$$\gamma_\nu\int dp\cdot\tilde{g}_\mu(p,s)\begin{Bmatrix} p^2\not{r} \\ (rp)\not{r} \\ \not{p} \\ \not{p}^3 \\ (rp)\not{p}\end{Bmatrix}\gamma_\nu = -2\cdot\int dp\cdot\tilde{g}_\mu(p,s)\begin{Bmatrix} p^2 \\ (rp)\not{r} \\ \not{p} \\ \not{p}^3 \\ (rp)\not{p}\end{Bmatrix}$$

To split up (28b) in this manner we make use of the fact that terms of the same order in r on the left and the right sides, respectively, must be equal to themselves.

The first three relations are fulfilled by the expressions already computed, while the remaining two equations lead to

$$\frac{1}{(2\pi)^8}\int dp\cdot\tilde{g}_\mu(p,s)\not{p}^3 = -i\gamma_\mu s^2$$

resp.

$$\frac{1}{(2\pi)^8}\int dp\cdot\tilde{g}_\mu(p,s)(rp)\not{p} = i\gamma_\mu(rs) \ ,$$

if we use (4') to determine functions such as f(s) on page 27.

Literature.

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