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STORAGE RING LUMINOSITY AS A FUNCTION OF BEAM-TO-BEAM SPACE AND CROSSING ANGLE

(Abhängigkeit der Speicherring-Luminosität von der Taillenweite und vom Kreuzungswinkel)

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by

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Abstract

The storage ring luminosity as limited by the beam-to-beam space charge interaction is calculated, taking into account the crossing angle, the shape of the amplitude function in a low-β-region and the dependence of the Q-shift on the position of the particle within the bunch. It appears that there is an optimum value β_o for which the luminosity has a maximum. With increasing crossing angle and given beam size, the Q-shift falls off more rapidly than the luminosity. By adjusting the beam cross section for constant Q-shift, an increase of luminosity with increasing crossing angle is obtained.

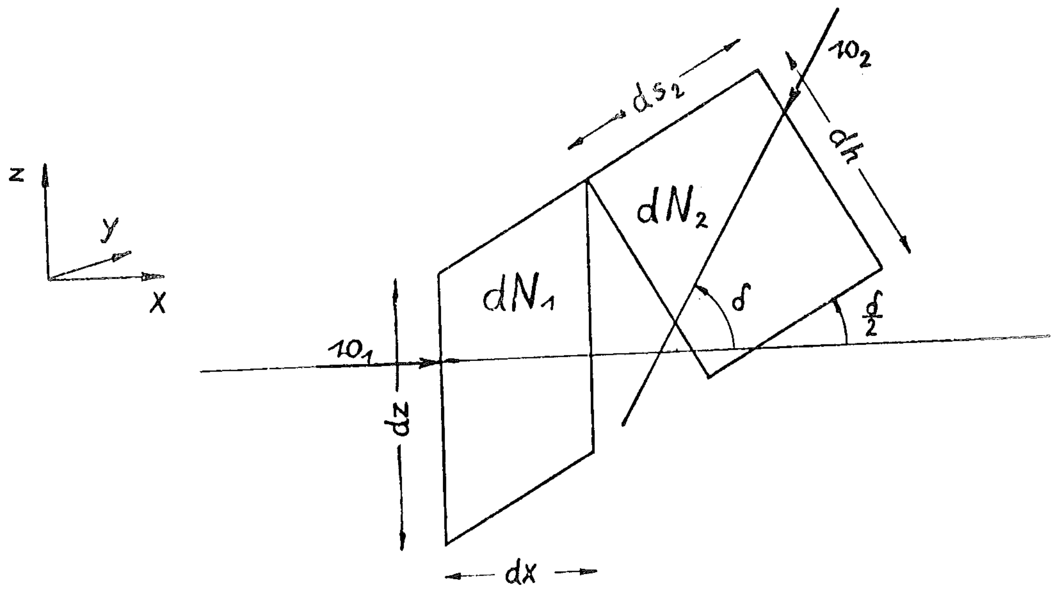
I. Introduction

The rate of events in storage ring experiments is proportional to the luminosity L. A course calculation of luminosity showed that it increased as the amplitude function β_o in the interaction zone decreased so that it appeared to be useful to make β_o as small as technical limitations would permit. In this study, luminosity will be calculated taking into account the exact changes in the amplitude function in the beam-to-beam space and it is shown that there is always an optimum β_o where luminosity reaches a maximum. This optimum β_o lies within the same order of magnitude as the technically feasible β_o for small bunch lengths, while for larger bunch lengths it is greater than the technically feasible β_o so that in the latter case a narrow beam-to-beam space does not increase luminosity rather it reduces it.

At the same time, the dependence of luminosity on the crossing angle of the two colliding particle beams will be observed. It is shown that for a constant emmittance luminosity decreases noticeably whenever the crossing angle δ reaches the order of magnitude of $\frac{\sqrt{\epsilon\beta_o}}{\sigma_x}$ where ϵ represents the emmittance in the vertical direction and σ_x the bunch length. The Q-shift resulting from the space charge effect decreases somewhat more rapidly than luminosity as the crossing angle increases. If the beam cross section is so changed that the Q-shift remains constant, we obtain an increase in luminosity with increasing crossing angle.

II. Luminosity

Two volumes with particle numbers dN_1 and dN_2 and the dimensions shown in the sketch yield dZ events ~~in random motion~~.



dZ is given by

$$dZ = \frac{\sigma}{dydh} \, dN_1 \, dN_2$$

where σ = the effective cross section

$$dN_1 = \rho_1^x \, dx \, dy \, dz$$

$$dN_2 = \rho_2^x \, ds_2 \, dy \, dh = \rho_2^x \, 2c \cos\frac{\delta}{2} \, dt \, dy \, dh$$

$$|10_1| = |10_2| = c$$

$\rho^x = \rho^x(x, y, z, t)$ = particle density

dt = the time required for a particle consisting of dN_1 to cross dN_2.

The rate of events is

$$\frac{dZ}{dt} = \sigma dL$$

dL represents the differential luminoisty, i.e. the luminosity in the space element whose dimensions are dx, dy, dz. It turns out to be

$$dL(x,y,z,t) = 2\ c\ \cos\frac{\delta}{2}\rho_1^x(x,y,z,t)\rho_2^x(x,y,z,t)\ dx\ dy\ dz$$

The distribution of the particles in the bunch satisfied as a close approximation a Gaussian distribution [1,2]

$$\rho_1^x(x',y',z',t) = \frac{A}{\sigma_z}\exp\left(-\frac{(x'-ct)^2}{2\sigma_x^2} - \frac{y'^2}{2\sigma_y^2} - \frac{z'^2}{2\sigma_z^2}\right)$$

$$\rho_2^x(x'',y'',z'',t) = \frac{A}{\sigma_z}\exp\left(-\frac{(x''+ct)^2}{2\sigma_x^2} - \frac{y''^2}{2\sigma_y^2} - \frac{z''^2}{2\sigma_z^2}\right)$$

$$A = \frac{N}{k(2\pi)^{2/3}\sigma_x\sigma_y}\ ; \quad k = \text{number of bunches}$$

$$\sigma_i(i = x,y,z) = \text{standard deviation}$$

The coordinate systems x', y', a' and x", y", z" are always rotated about half the crossing angle $\phi = \frac{\delta}{2}$ with respect to the coordinate system x, y, z.

$$x' = x\cos\phi + z\sin\phi,\ y' = y,\ z' = -x\sin\phi + z\cos\phi$$

$$x'' = x\cos\phi - z\sin\phi,\ y'' = y,\ z'' = x\sin\phi + z\cos\phi$$

The standard deviation σ_z depends on x since

$$\sigma_z = \sqrt{\varepsilon\beta}$$

ε = emmittance in the vertical direction

β = amplitude function for the vertical direction

In the field-free space we obtain the amplitude function β from the equation $y'' = 0$ and the relationship $y = \sqrt{\varepsilon\beta}\, e^{\pm i\Psi}$ [3]. It becomes

$$\beta = \beta_o + \frac{x^2}{\beta_o}$$

and

$$\sigma_z^2 = \varepsilon\, \beta_o \left(1 + \frac{x^2}{\beta_o^2}\right)$$

The total luminosity for an interaction zone is finally obtained by integrating over the entire space and by averaging over time.

$$L = k \cdot f \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dL(x,y,z,t)\, dt$$

where f = the rotational frequency of the bunches.

The integrations over y, z and t can be carried out directly and there remains the simple integral

$$L = \frac{fN^2}{k 4\pi\sqrt{\pi}\, \sigma_x \sigma_y \sqrt{\varepsilon\, \beta_o}} \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2 \cos^2\phi}{\sigma_x^2} - \frac{x^2 \sin^2\phi}{\varepsilon\, \beta_o \left(1 + \frac{x^2}{\beta_o^2}\right)}\right\} \frac{1}{\sqrt{1 + \frac{x^2}{\beta_o^2}}}\, dx.$$

Substituting

$$\eta = \frac{x \cos\phi}{\sigma_x}, \quad B = \sin^2\phi \cos^2\phi \frac{\beta_o^3}{\sigma_x^2 \varepsilon}, \quad C = \cos^2\phi \frac{\beta_o^2}{\sigma_x^2}$$

we obtain

$$L = \frac{fN^2}{k4\pi\sigma_y\sqrt{\varepsilon\,\sigma_x}} \cdot Z\left(\frac{\beta_o}{\sigma_x}\right)$$

$$Z\left(\frac{\beta_o}{\sigma_x}\right) = \frac{2}{\sqrt{\pi}}\sqrt{\frac{\beta_o}{\sigma_x}}\,\exp\left(-\frac{B}{C}\right)\int_0^\infty \exp\left(-\eta^2 + \frac{B}{C+\eta^2}\right)\frac{d\eta}{\sqrt{C+\eta^2}}$$

The function A was calculated as a function of $\frac{\beta_o}{\sigma_x}$ for various values of the parameters $\frac{\varepsilon_z}{\sigma_x}$ and ϕ on an IBM 1044. Fig. 1 shows the result for $\frac{\varepsilon}{\sigma_x} = 3.66 \cdot 10^{-8}$. The curves approach zero asynptotically in both directions. Fig. 2 includes a correction factor, namely the value by which the luminosity

$$L_o = \frac{fN^2}{k4\pi\,\sigma_y\,\sigma_{zo}}$$

must be multiplied without taking into account the exact variation of the beam-to-beam space and the crossing angle. The smaller the ratio $\frac{\beta_o}{\sigma_x}$ and the greater the crossing angle, the less accurately the simple formula describes the actual luminosity value. Fig. 3 shows the dependence of luminosity on the crossing angle for a fixed value of $\frac{\beta_o}{\sigma_x}$ and for varying values of ε, i.e. for various beam heights. $\frac{\beta_o}{\sigma_x} = 1.6$ is the value proposed for the DESY storage ring at 3 GeV. For small angles, i.e. $\sin\phi \approx \phi$ and $\cos\phi \approx 1$ the curves are con. For in this case ϕ and $\frac{\varepsilon}{\sigma_x}$ only occur in the form of $\phi^2 \cdot \frac{\sigma_x}{\varepsilon}$ in the integrand which leads to a shift of the curves in a logarithmic representation of ϕ for changes in $\frac{\varepsilon}{\sigma_x}$.

III. Q-Shift

A charged particle flying in the x direction which crosses a bunch approaching it in the opposite direction experiences a change in momentum along path dx with respect to the z component which is given by

$$dp_z = \frac{1}{c} f_z (x,y,z,t)\, dx$$
$$= \frac{1}{c} \left(f_z(x,y,o,t) + \frac{\partial f_z}{\partial z} \cdot z \right) \, dx$$

where $f_z(x,y,a,t)$ is the z component of the force exerted by the bunch on the individual particle. The first number in the series development leads to a shift of the reference circle while the second member produces a change in the frequency in the vertical betatron oscillation [4]:

$$dQ = - \frac{\beta}{4\pi E} \frac{\partial f_z}{\partial z} dx$$

$$E = \text{particle energy}$$

It is now assumed that the height of the oncoming bunch is small with respect to its length and width. If this condition is not satisfied, the change in the betatron is somewhat smaller. The direction of motion forms the crossing angle δ with respect to the negative x axis. In the relativistic case the electrical and magnetic components of the force acting on the particle are of the same magnitude, but assume different direction. While the magnetic force is parallel to the z axis, the electrical force forms the crossing angle δ with respect to the z axis.

The total z component is then equal to:

$$f_z = f_{magn.} + \cos\delta\, f_{elec.}$$
$$= (1 + \cos\delta)\, f_{elec.}$$
$$= -(1 + \cos\delta)\, e$$

The negative sign stems from the fact that particles with the opposite charges have been assumed. In the rotated coordinate system (x',y',z') the following relationships apply:

$$\frac{\partial \mathcal{E}_{y'}}{\partial y'} << \frac{\partial \mathcal{E}_{z'}}{\partial z'} \quad \text{and} \quad \frac{\partial \mathcal{E}_{x'}}{\partial x'} << \frac{\partial \mathcal{E}_{z'}}{\partial z'}$$

$$\frac{\partial \mathcal{E}_{z'}}{\partial z'} = \frac{\partial \mathcal{E}}{\partial z'} = \frac{\rho}{\varepsilon_o}$$

where ρ = space charge density

ε_o = the dielectric constant

$$\rho(x',y',z',t) = \frac{Ae}{\sigma_z} \exp\left(-\frac{(x' + ct)^2}{2\sigma_x^2} - \frac{y'^2}{2\sigma_y^2} - \frac{z'^2}{2\sigma_z^2}\right)$$

From

$$x' = x\cos\delta - z\sin\delta, \quad y' = y, \quad z' = x\sin\delta + z\cos\delta$$

it follows that

$$\frac{\partial \mathcal{E}}{\partial z} = \cos\delta \frac{\partial \mathcal{E}}{\partial z'}$$

and

$$\frac{\partial f_z}{\partial z} = -\cos\delta\,(1 + \cos\delta)\,\frac{e}{\varepsilon_o}\,\rho$$

At time t = 0 may assume the coordinates (a,b,o) while at the same time the center of gravity of the oncoming bunch may coincide with the coordinate origin. t is then to be substituted by $\frac{1}{c}(x - a)$. The total Q-shift ΔQ results from the integration over the path of the particle through the other bunch.

$$\Delta Q(a,b) = \frac{Ae^2 c\overset{\circ}{o}s\ (1 + \cos\delta)}{4\pi\varepsilon_o E} \int_{-\infty}^{\infty} \frac{\beta}{\sigma_z} \exp\left(- \frac{(x\cos\delta + x - a)^2}{2\sigma_x^2}\right.$$

$$\left. - \frac{b^2}{2\sigma_y^2} - \frac{x^2 \sin^2\delta}{2\sigma_z^2}\right) dx$$

The Q-shift of a particle therefore depends on its position in its own bunch. Its maximum occurs at $b = 0$ and $a = a_m \neq 0$. a_m is then equal to the distance between the center of the bunch and the farthest particles, provided that the crossing angle equals zero, i.e. those particles which lie at the border of the phase-stable range. For an increasing crossing angle, a_m decrease, i.e. the particles with the highest Q-shift lie closer to the center of the bunch.

Since particles whose ΔQ exceeds a critical value ΔQ_{crit}, are lost, the particles with a maximum $\Delta Q\ (a_m, o)$ are those in greatest danger. It must now be taken into account that the position of the particles as a result of the quanta fluctuation, the attenuation and the synchrotron and betatron oscillation changes constantly. For the parameters given in the proposal for the DESY storage ring [5] the periods of the synchrotron and betatron oscillations are 210 times and 0.121 times the time of rotation. The motion produced by the quanta fluctuation averages

0.02 standard deviations per rotation and the motion resulting from the attenuation is 10^{-4} times and $0.5 \cdot 10^{-4}$ times the instantaneous synchrotron and betatron amplitudes respectively.

These values must be compared with the number of rotations a particle must cover whose $\Delta Q > \Delta Q_{crit}$, before it gets lost or before its betatron amplitude has become so large that it no longer makes any substantial contribution to luminosity. These times of increase may vary and depend on the type of instability produced by a given ΔQ_{crit}. For example, they are the greater the higher the order of the non-linear resonance leading to the increase of the betatron amplitudes. Two different cases will be examined:

1) The time of increase is greater than the time during which a particle reaches any random position within the bunch. This must then be averaged over the duration of a particle"s stay in the bunch which is given by the Gaussian distribution.

2) The time of increase is greater than a quarter period of the synchrotron oscillation, but smaller than the time during which the particles change their oscillation amplitude substantially as a result of attenuation and quanta fluctuation. It must then be averaged over a quarter period of the synchrotron oscillation since the Q-shift repeats periodically after that.

In the first case, the averaging process yields the integral

$$\Delta Q_1 = \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Delta Q(a,b)\ \exp\left(-\frac{a^2}{2\sigma_x^2} - \frac{b^2}{2\sigma_y^2}\right) da\ db$$

Two of this total of three integrations can be carried out directly and we obtain

$$\Delta Q_1 = \frac{Nr_e\sqrt{\sigma_x}}{2\pi k\gamma\sigma_y\sqrt{\varepsilon}} \cdot N_1\left(\frac{\beta_o}{\sigma_x}\right)$$

where

$$N_1\left(\frac{\beta_o}{\sigma_x}\right) = \sqrt{\frac{2}{\pi}}\,(1 - \tan^2\phi)\,\sqrt{\frac{\sigma_x}{\beta_o}}\,\exp\left(-\frac{F}{D}\right)\int_0^{\infty}\exp\left(-\eta^2 + \frac{F}{D+\eta^2}\right)\sqrt{D+\eta^2}\,d\eta$$

and where

$$D = \cos^4\phi\,\frac{\beta_o^2}{\sigma_x^2}\,, \quad F = 2\sin^2\phi\,\cos^6\phi\,\frac{\beta_o^3}{\varepsilon\sigma_x^2}$$

This last integral was calculated numerically and was entered as a function of $\frac{\beta_o}{\sigma_x}$ in Fig. 4 for various crossing angles. For each crossing angle there is an optimum β_o/σ_x in which the Q-shift reaches a minimum. These minima fall off as the crossing angle increases at a more rapid rate than the maxima of the luminosity in Fig. 1. This behavior becomes evident in Fig.5 in which the quotient of luminosity and the Q-shift has been entered. The quotient

$$L \cdot \frac{2r_e\sigma_x}{f\gamma\ N\ \Delta Q}$$

yields the maximum luminosity that can be attained without exceeding a certain value of ΔQ_{cr}.

The luminosity can also be increased by the crossing angle, namely the more, the smaller the value of β_o / σ_x, i.e. the narrower the beam-to-beam space.

It must be observed here that the beam cross section which was derived from the luminosity normalized for ΔQ becomes smaller as the crossing angle increases. For in order to attain maximum luminosity, the beam cross section must be so selected that the corresponding ΔQ actually occurs, i.e. the beam cross section is given by the curves shown in Fig.4.

In the second case, upon averaging over a quarter synchrtotron oscillation, it is assumed that the dispersion in the interaction zone is equal to zero so that the width of the bunches at this point is determined only by the horizontal betatron oscillations and not by the synchrotron oscillations. The deflection b in the direction of the y-axis is then independent of the phase of the synchrotron oscillation and in the worst case it is equal to zero. The averaging then yields the integral

$$\Delta Q_2 = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \Delta Q(d \cdot \sigma_x \cdot \sin u,\ o)\ du$$

d represents the amplitude of the synchrotron oscillation in units of standard deviations. After some conversions we obtain

$$\Delta Q_2 = \frac{N r_e \sqrt{\sigma_x}}{2\pi\ k\ \gamma\ \sigma_y \sqrt{\varepsilon}}\ N_2 \left(\frac{\beta_o}{\sigma_x}\right)$$

where

$$N_2\left(\frac{\beta_o}{\sigma_x}\right) = \frac{1}{\pi}\sqrt{\frac{2}{\pi}}(1 - \tan^2\phi)\sqrt{\frac{\sigma_x}{\beta_o}}\exp\left(-\frac{F}{D}\right)\int_0^{\frac{\pi}{2}}\int_{-\infty}^{\infty}\exp\left(-\left(\eta - \frac{d}{\sqrt{2}}\sin u\right)^2 + \frac{2F}{2D + \eta^2}\right) \times$$

$$\times \sqrt{2D + \eta^2}\, d\eta\, du$$

The change of this Q-shift as a function of the amplitude is shown in Fig. 6. Here to there exists for each crossing angle a minimum, which, is some 2 to 3 times greater than that for ΔQ_1 and which shifts somewhat more as the crossing angle increases. The break in the curves results from the fact that at this point the maximum Q-shift proceeds from the particles at the edge of the phase-stable range to the particles within the bunch. The oscillation amplitude of the particles at the edge of the phase-stable bunch was assumed to be 8.5 standard deviations of the bunch length. This corresponds to a life of about 8 hrs.

Fig. 7 shows, as did Fig. 5, the luminosity resulting from the elimination of the cross section and at constant ΔQ. Here to, the luminosity can be increased by the crossing angle.

IV. Border Line Cases

For the two border line cases

$$\beta = \text{constant}, \quad \phi \text{ arbitrary}$$

and

$$\beta = \beta_o + \frac{x^2}{\beta_o}, \quad \phi = 0$$

the integrals can be solved in a closed form so that the effect of the parameters becomes more pronounced, particularly the effect of the crossing angle.

a) β = const.

A constant β in the interaction zone reflects a large β_o, i.e.

$$\beta_o >> \sigma_x \, .$$

Thus

$$B >> C >> 1$$

and

$$Z(\frac{\beta_o}{\sigma_x}) = \frac{2}{\sqrt{\pi}}\sqrt{\frac{\beta_o}{\sigma_x}}\frac{1}{\sqrt{C}}\int_0^\infty \exp\left(-\eta^2(1+\frac{B}{C^2})\right) d\eta$$

$$= \sqrt{\frac{\beta_o}{\sigma_x}}\frac{1}{\sqrt{C+\frac{B}{C}}}$$

$$L = \frac{fN^2}{k\, 4\pi\, \sigma_y}\frac{1}{\sqrt{\sigma_z^2\cos^2\phi + \sigma_x^2\sin^2\phi}}$$

where

$$\sigma_z^{\,2} = \varepsilon\, \beta_o$$

The effect of the crossing angle 2ϕ is only noticeable if $\phi \approx \frac{\sigma_z}{\sigma_x}$.

Similarly, it follows from F >> D >> 1 for the Q-shift following the first averaging:

$$N_1(\frac{\beta_o}{\sigma_x}) = \sqrt{\frac{2}{\pi}}(1-\tan^2\phi)\sqrt{\frac{\sigma_x}{\beta_o}}\sqrt{D}\int_0^\infty \exp(-\eta^2(1+\frac{F}{D^2}))\, d\eta$$

$$= \frac{1}{\sqrt{2}}(1-\tan^2\phi)\sqrt{\frac{\sigma_x}{\beta_o}}\frac{D}{\sqrt{D+\frac{E}{D}}}$$

and

$$\Delta Q_1 = \frac{N r_e \beta}{2\pi k \gamma \sigma_y} \frac{1}{\sqrt{2}} \frac{\cos 2\phi}{\sqrt{\sigma_z^2 + 2\sigma_x^2 \tan^2\phi}}$$

Since $\sigma_z << \sigma_x$, the denominator depends much more on ϕ than the numerator so that hereto the effect of the crossing angle becomes noticeable if ϕ reaches the order of magnitude of σ_z/σ_x. The elimination of the beam cross section from the luminosity yields:

$$L = \frac{f\gamma N \Delta Q_1}{2 r_e \beta} \frac{\sqrt{2}\sqrt{\sigma_z^2 + 2\sigma_x^2 \tan^2\phi}}{\cos 2\phi \sqrt{\sigma_z^2 \cos^2\phi + \sigma_x^2 \sin^2\phi}}$$

For small angles, i.e. for $\phi << \frac{\sigma_z}{\sigma_x}$, we obtain

$$L = \frac{f\gamma N \Delta Q_1}{\sqrt{2} r_e \beta} \left(1 + \frac{1}{2} \frac{\sigma_x^2}{\sigma_z^2} \tan^2\phi\right).$$

The maximum attainable luminosity increases therefore for a constant value of β as the crossing angle increases.

For the Q-shift following the second averaging, we obtain

$$\Delta Q_2 = \frac{N r_e \beta}{2\pi k \gamma \sigma_y} \frac{\cos 2\phi}{\sqrt{\sigma_z^2 + \sigma_x^2 \tan^2\phi}}$$

and for the corresponding luminosity

$$L = \frac{f\gamma N \Delta Q_2}{2 r_e \beta} \frac{1}{\cos 2\phi},$$

i.e. it is less dependent on the crossing angle.

b) $\phi = 0$

From $\phi = 0$ it follows that $B = 0$ and thus we obtain the luminosity as a function of the beam-to-beam space:

$$L = \frac{fN^2}{k4\pi\sigma_y\sqrt{\varepsilon\sigma_x}} \frac{1}{\sqrt{\pi}} \sqrt{\frac{\beta_o}{\sigma_x}} \exp\left\{\frac{\beta_o^2}{2\sigma_x^2}\right\} K_o\left(\frac{\beta_o^2}{2\sigma_x^2}\right)$$

Here, K_o represents the modified Hankel function of the zeroeth order [6].

For large and small arguments, we obtain the aymptotic representation

$$\beta_o \gg \sigma_x \qquad L = \frac{fN^2}{k4\pi\sigma_y\sigma_{zo}} \left(1 - \frac{\sigma_x^2}{4\beta_o^2}\right)$$

$$\beta_o \ll \sigma_x \qquad L = \frac{fN^2}{k4\pi\sigma_y\sigma_{zo}} \frac{2}{\sqrt{\pi}} \frac{\beta_o}{\sigma_x} \ln\frac{\sigma_x}{\beta_o}$$

From $F = 0$, there results for ΔQ_1

$$\Delta Q_1 = \frac{Nr_e\sqrt{\sigma_x}}{2\pi k\gamma\sigma_y\sqrt{\varepsilon}} \frac{1}{2\sqrt{2\pi}} \left(\frac{\beta_o}{\sigma_x}\right)^{\frac{3}{2}} \exp\left\{\frac{\beta_o^2}{2\sigma_x^2}\right\}\left(K_o\left(\frac{\beta_o^2}{2\sigma_x^2}\right) + K_1\left(\frac{\beta_o^2}{2\sigma_y^2}\right)\right)$$

with the asymptotic approximations

$$\beta_o \gg \sigma_x \qquad \Delta Q_1 = \frac{Nr_e\beta_o}{2\pi k\gamma\sigma_y\sigma_{zo}} \frac{1}{\sqrt{2}} \left(1 + \frac{\sigma_x^2}{4\beta_o^2}\right)$$

$$\beta_o \ll \sigma_x \qquad \Delta Q_1 = \frac{Nre \quad \beta_o}{2\pi k\gamma\sigma_y\sigma_{zo}} \frac{1}{\sqrt{2\pi}} \frac{\sigma_x}{\beta_o}$$

In the case of $\frac{\sigma_x}{\beta_o} \to 0$ ΔQ_1 differs from the expressions for ΔQ

derived earlier by a factor of $1/\sqrt{2}$, which stems from the averaging over the bunch width.

For ΔQ_2, no simple analytical expression for the dependence on the beam-to-beam space can be found. For both border line cases we obtain

$\beta_o \gg \sigma_x$

$$\Delta Q_2 = \frac{N r_e \beta_o}{2\pi k \gamma \sigma_y \sigma_{zo}} \left(1 + \frac{1}{8} \frac{\sigma_x^2}{\beta_o^2} \quad (1 + \frac{d^2}{2})\right)$$

$\beta_o \ll \sigma_x$

$$\Delta Q_2 = \frac{N r_e \beta_o}{2\pi k \gamma \sigma_y \sigma_{zo}} \frac{1}{\sqrt{2\pi}} \frac{\sigma_x}{\beta_o} f(d)$$

where

$$f(d) = \exp(-\frac{d^2}{4}) I_o(\frac{d^2}{4}) - \frac{2}{\sqrt{\pi}} \sum_{k=o}^{\infty} \frac{(-1)^k d^{2k} \Gamma(k + \frac{1}{2})}{2^k k! (k - 1)! \ (2k - 1)}$$

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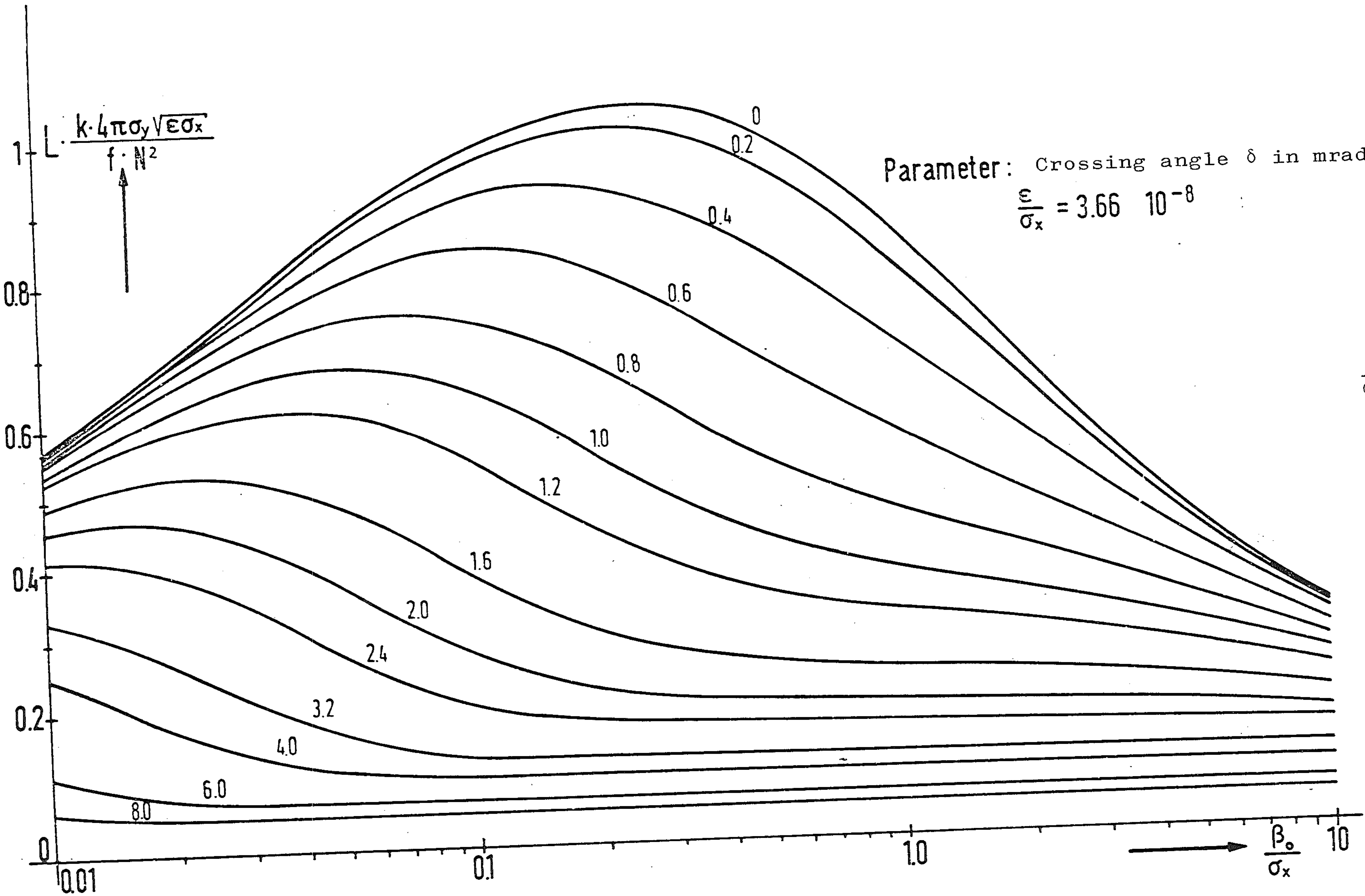


Fig. 1: Luminosity as a function of amplitude

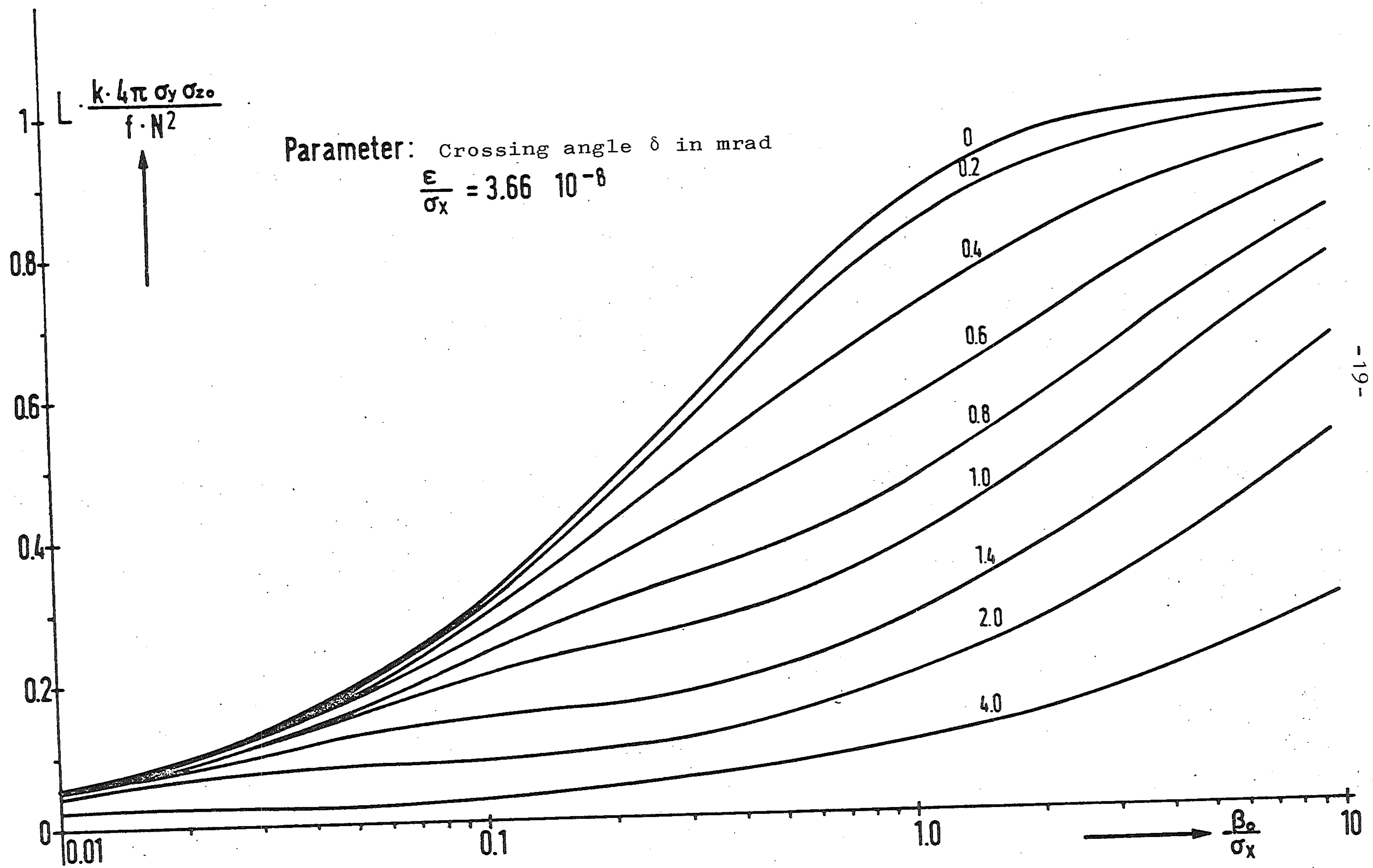


Fig. 2: Luminosity as a function of amplitude

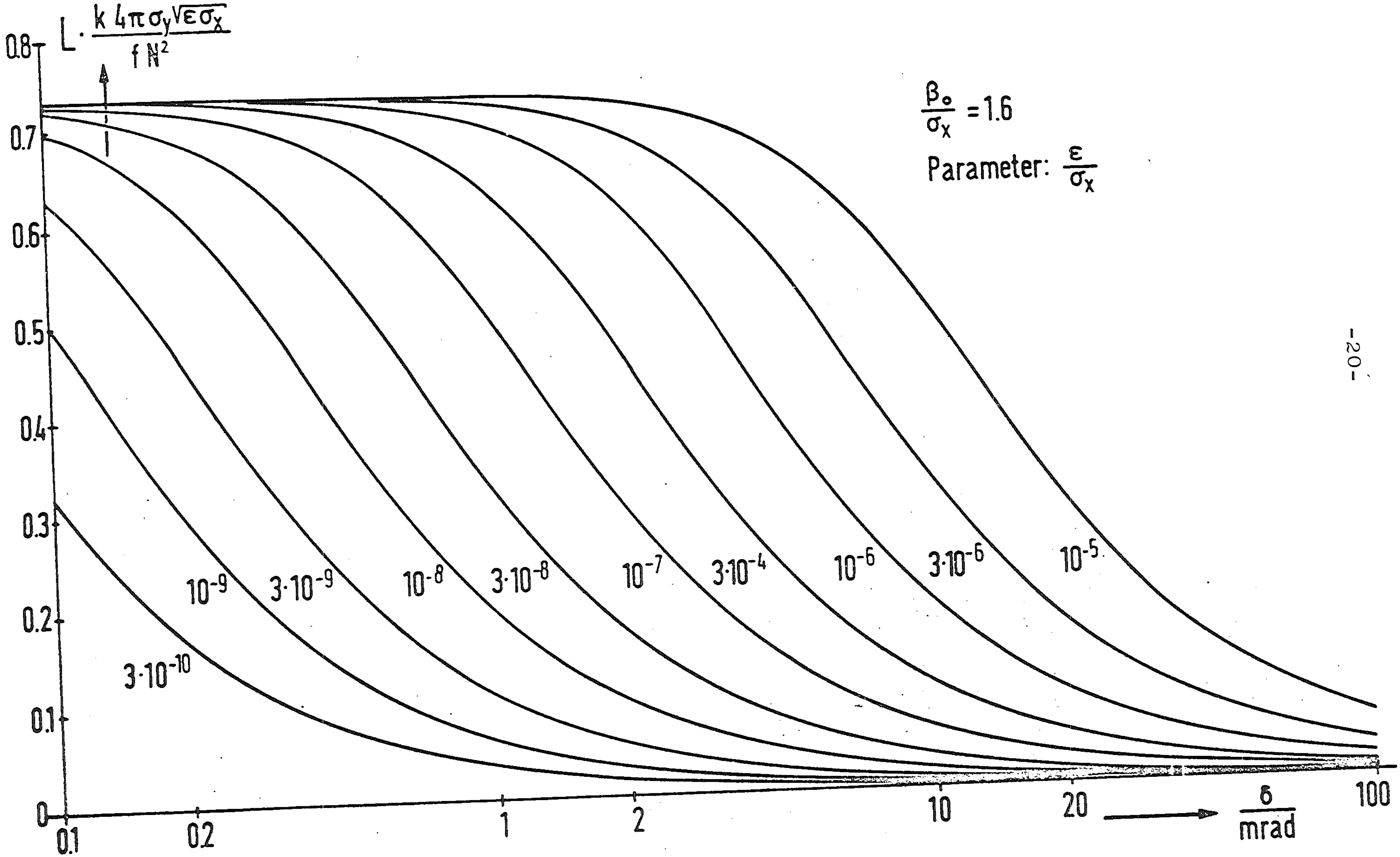
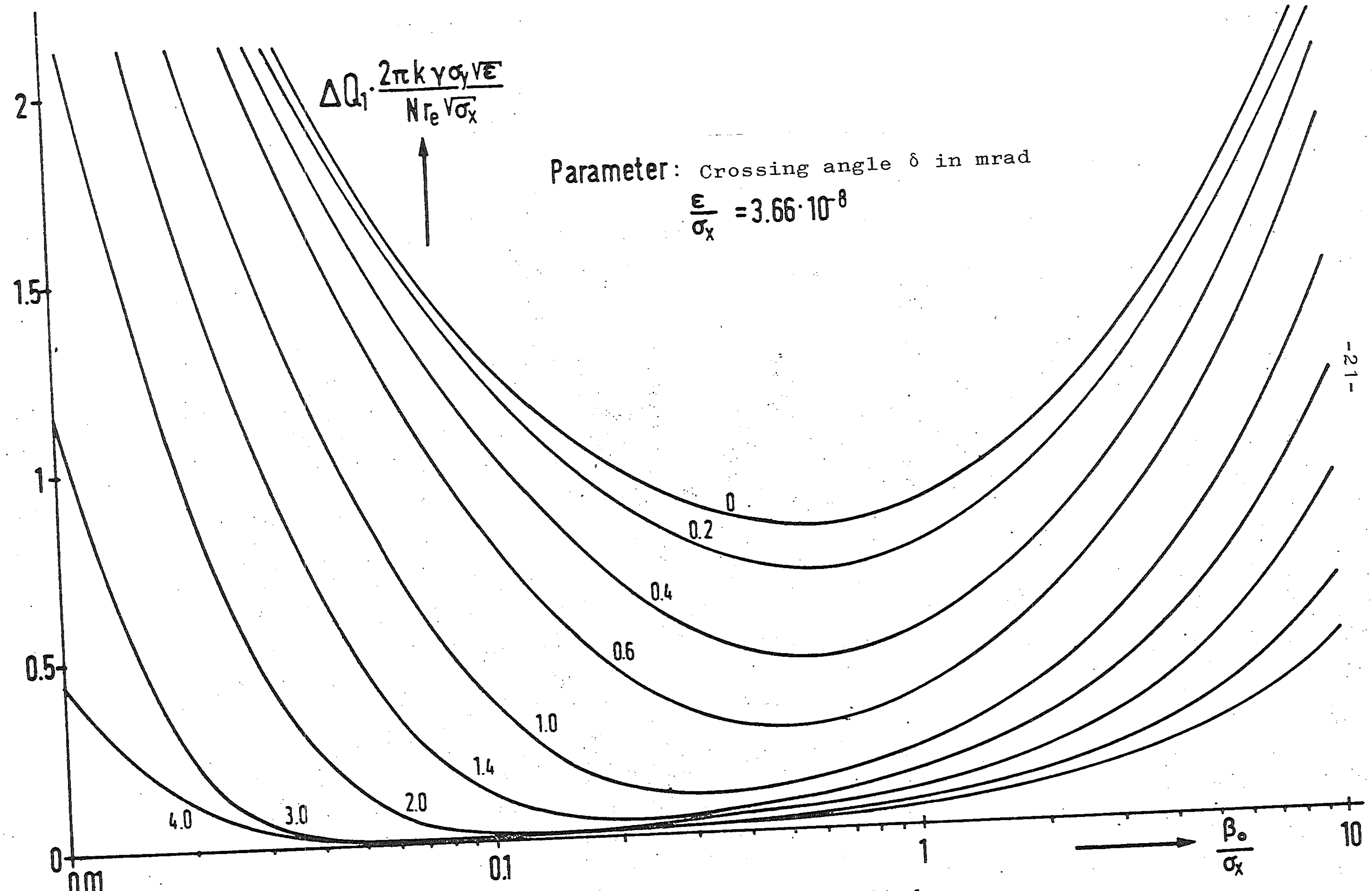


Fig. 3: Luminosity as a function of crossing angle δ



... ΔQ_1 as a function of amplitude

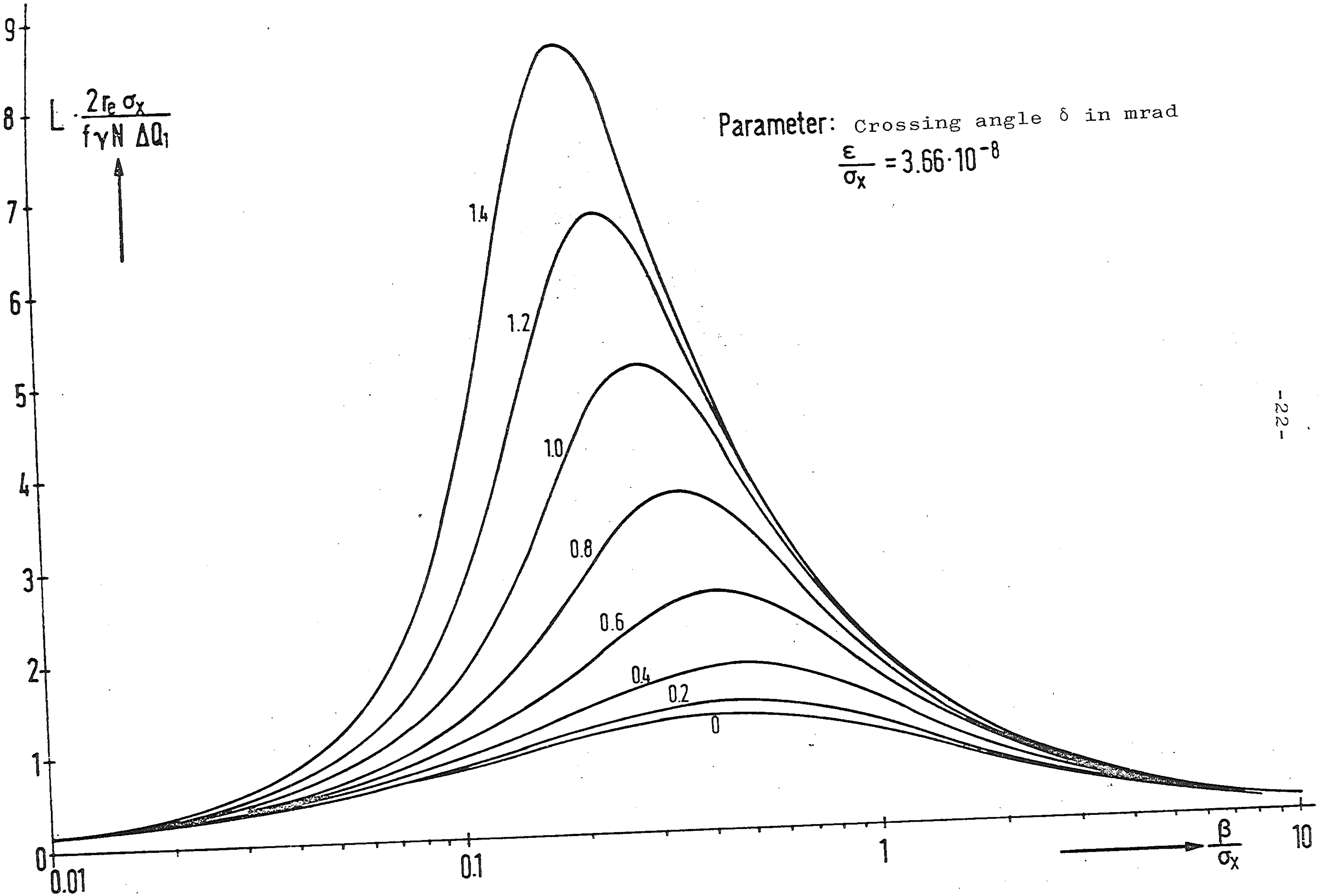
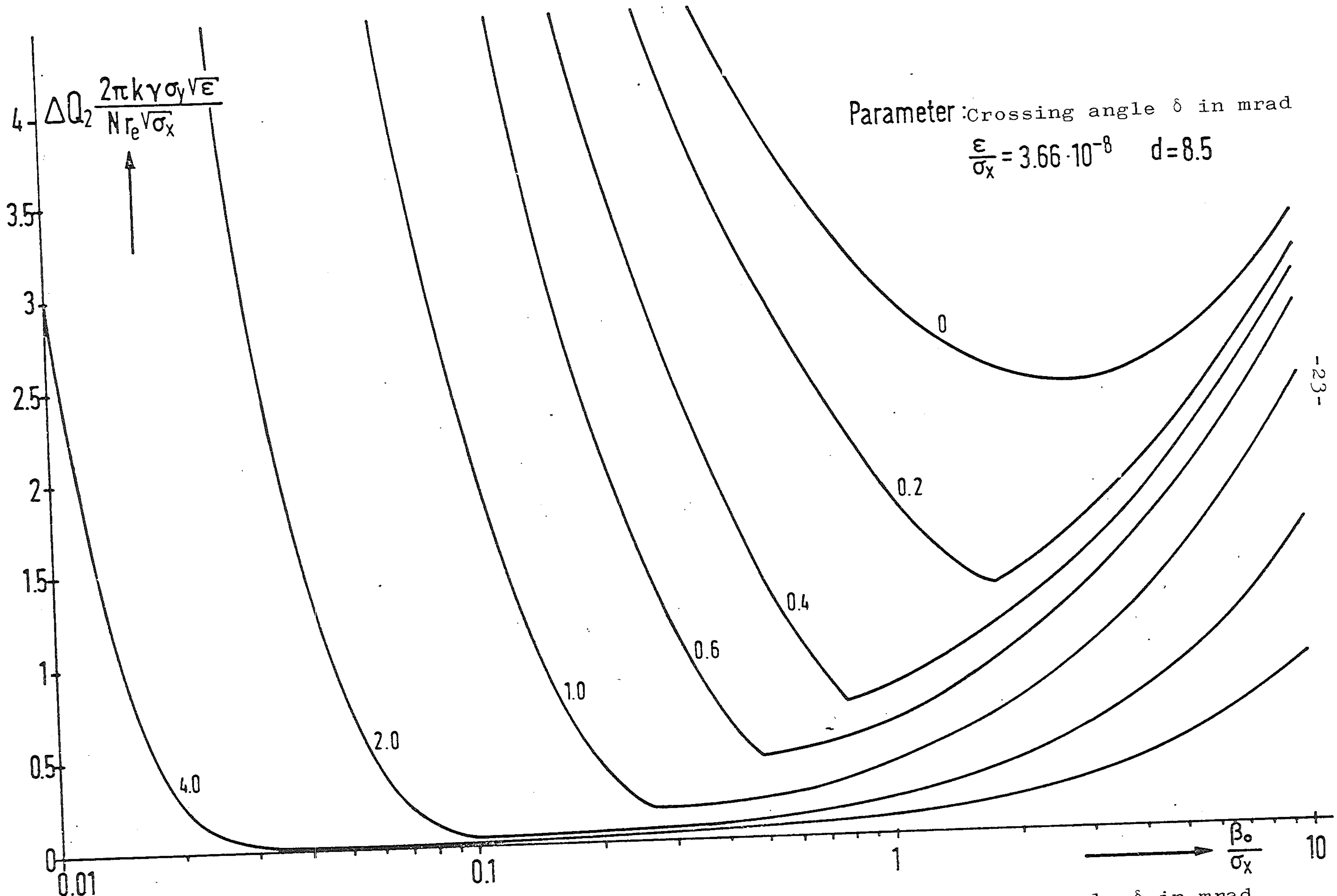


Fig. 5: Luminosity as a function of amplitude



amplitude crossing angle δ in mrad

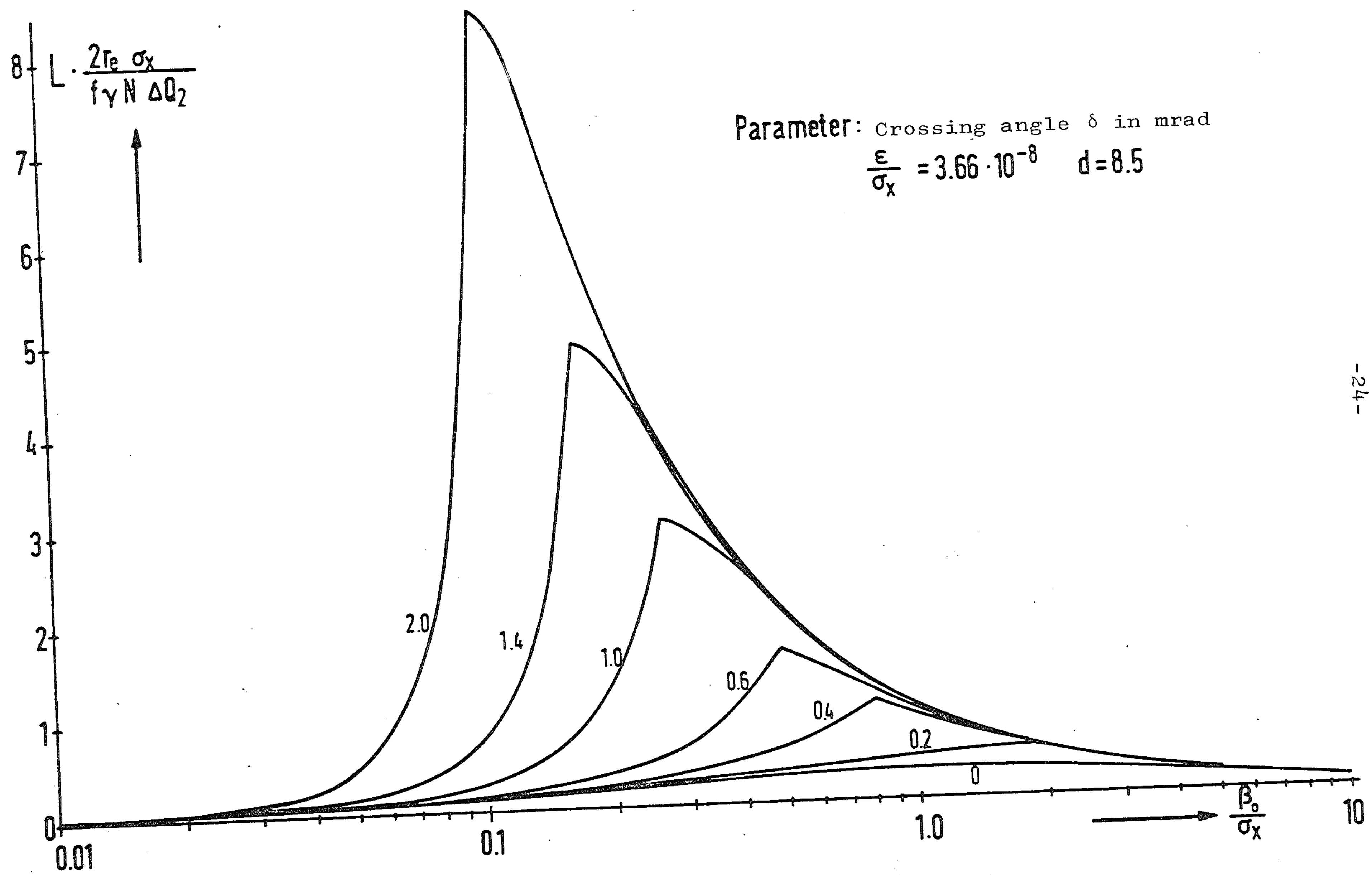


Fig. 7: Luminosity as a function of amplitude crossing angle δ in mrad