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Abstract

We present a Bethe-Salpeter model, in which phenomenological potentials are used to reproduce the ρ peak in the $\pi\pi$ scattering amplitude. Then we take the off shell amplitude to calculate the pion electromagnetic form factor. We use the Bethe-Salpeter equation in ladder approximation and solve it by iteration and application of the Pade approximation method. Reasonable agreement with the vector dominance model is achieved.

I. Introduction

The concept of the vector dominance model[1] (VDM) has proven to be fruitful in explaining the pion electromagnetic form factor in the timelike region. In this framework one starts (in the limit of zero ρ meson width) with the current field identity

$$j_\mu(x) = -\left[\frac{m_\rho^2}{2\gamma_\rho} \rho_\mu^o(x) + \frac{m_\omega^2}{2\gamma_\omega} \omega_\mu(x) + \frac{m_\phi^2}{2\gamma_\phi} \phi_\mu(x) \right] , \qquad (1)$$

where $j_\mu(x)$ is the electromagnetic current of the hadrons, $V_\mu(x)$ denotes the vector meson fields and m_V their masses $(V = \rho^o, \omega, \phi)$. The constants γ_V determine the strength of the photon vector meson coupling.

Physical conlusions can be drawn from the assumption that

$$< 2\pi \;|K_\mu^\rho \;(0)|0 > = - \frac{2\gamma_\rho}{m_\rho^2} (t - m_\rho^2) < 2\pi \;|\rho_\mu^o \;(x)|0 > \qquad (2)$$

is almost constant for $0 \leq t \leq m_\rho^2$. Here K_μ^ρ is given by

$$K_\mu^\rho(x) = \{\Box - m_\rho^2\} \; \rho_\mu^o(x) \quad ,$$

$t = (p - p')^2$, p and p' being the pion momenta. From the normalization condition

$$F_\pi(0) = 1 \qquad \text{with} \qquad (3)$$

$$< 2\pi \;|j_\mu(0)|0> = \frac{1}{(2\pi)^3} (p + p')_\mu \; F_\pi(t)$$

one obtains

$$F_\pi(t) = \frac{g_{\rho\pi\pi}}{2\gamma_\rho} \frac{m_\rho^2}{m_\rho^2 - t} \qquad \text{and}$$

$$g_{\rho\pi\pi} = 2\gamma_\rho \quad ,$$

$g_{\rho\pi\pi}$ being the $\rho\pi\pi$ coupling constant .

For a finite ρ width, the Omnès formula[2]

$$F_\pi(t) = P_n(t) \exp\left[\frac{t}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\delta(t')dt'}{t'(t-t')}\right] \quad (4)$$

satisfies the normalization condition (3) and the requirement that $F_\pi(t)$ has the same phase $\delta(t)$ as the elastic $\pi\pi$ scattering amplitude. The usual assumption is that $P_n(t)$ is a constant $(P_n(t) = 1)$ and the phase δ is smooth above the resonance. A special model for the elastic $\pi\pi$ scattering phase was given by Gounaris and Sakurai[3] introducing an effective range expansion.

We want to investigate whether the assumption of slow variation of the r.h.s. of Eq.(2) can be understood in a Bethe-Salpeter model of the ρ meson, where the photon couples to the elementary constituents of the ρ. In Feynman graphs the pion form factor is given by the coupling of the photon to the interacting $\pi\pi$ system according to Fig.1, where the blob stands for the $\pi\pi$ scattering amplitude, which we calculate in ladder approximation (Fig.2). Our concept now is to look for a potential which gives a BS scattering amplitude in close agreement with the experimental ρ shape, and then use the corresponding off shell amplitude to calculate the pion form factor. The normalization condition (3) is obtained by the renormalization of the $\gamma\pi\pi$ vertex at $t = 0$. The main difficulty was to find a suitable potential. The single scalar particle exchange with large coupling constants and large exchanged masses give a ρ width about three times larger than the experimental value. Also spin one exchange with a cut off did not work. Therefore we tried scalar particle exchange with derivative coupling to the pions as well as a two channel potential where the ρ is essentially a bound state in the second channel with a threshold at $t = (7\mu)^2$. Both models for the ρ gave reasonable coincidence with the VDM.

II. The $\pi\pi$ Scattering Amplitude

The $\pi\pi$ scattering amplitude is calculated from the BSE in ladder approximation, which in graphical form is represented in Fig.2. The kinematics are chosen so that in the CMS $p = (W/2,\vec{0})$ is half the total four-momentum of the incoming particles and the q's are relative four-momenta. The partial wave yields for the partial wave amplitude the following integral equation:

$$M_\ell(q,q'') = M_\ell^B(q,\ q'')$$

$$- \frac{i}{\pi^2} \int_{-\infty}^{\infty} dq_o' \int_0^{\infty} |\vec{q}\,'|^2 \, d\,|\vec{q}\,'| \quad \frac{M_\ell^B(q,q')\ M_\ell(q',q'')}{[(q_o' - \frac{W}{2})^2 - |\vec{q}\,'|^2 - \mu^2][q_o' + \frac{W}{2})^2 - |\vec{q}\,'| - \mu^2]} \tag{5}$$

$W = \sqrt{s}$ and μ is the pion mass. In the case of scalar particle exchange the Born term reads

$$M_\ell^B(q,q') = \frac{g^2/4\pi}{2|\vec{q}||\vec{q}\,'|} \; Q_\ell\big(\mathcal{H}(q,q')\big) \quad . \tag{6}$$

$Q_\ell(z)$ is the Legendre function of second kind and its argument is:

$$\mathcal{H}\,(q,q') = -\,[(q_o - q_o')^2 - |\vec{q}|^2 - |\vec{q}\,'|^2 - m^2]/2|\vec{q}||\vec{q}\,'| \quad , \tag{7}$$

where m is the mass of the exchanged particle. g is the coupling constant of the exchanged particle to the pions.

Our technique to solve the BSE was to iterate it up to the order g^{16} utmost and form the associated Padè-approximants[4] for the T-matrix. Since we work below inelastic threshold, the singularities in the BSE are simple poles in both variables and are subtracted numerically and added analytically according to standard methods.

Searching for a suitable potential we began with spin zero exchange (6). Our experience was, that this potential can produce a resonance in the ρ region, but varying g and m we were not able to bring its width below 400 MeV. As the bootstrap idea suggests a ρ exchange, the next potential we investigated was a spin one particle exchange. The propagator of a massive spin one particle is

$$\Delta_{\mu\nu}(k^2) = (g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2})/(k^2 - m^2) \;, \qquad k = q - q' \qquad \text{(Fig.2)}.$$

Even dropping the longitudinal term $k_\mu k_\nu/m^2$ in the nominator, a cut off has to be introduced, which we did by the substitution $(m_C = 5\mu)$:

$$1/(k^2 - m^2) \rightarrow 1/(k^2 - m^2)(1 - \frac{k^2 - m^2}{m_c^2}) \quad .$$

This potential did not give much narrower resonances than the scalar particle exchange. Thus we examined the longitudinal term separately. Introducing a cut off, the Born term which we finally used is (s. Fig.2)

$$- \frac{g^2}{4\pi} \frac{\left((p - q')^2 - (p + q')^2\right)\left((p - q)^2 - (p + q)^2\right)}{\left((p - q')^2 + (p + q')^2 - M^2\right)\left((p - q)^2 - (p + q^2) - M^2\right)} / (k^2 - m^2),$$

M being a cut off mass. Even with a cut off of the order of 5 GeV the width was larger than 400 MeV.

The next trial was a derivative coupling (DC) of the exchanged scalar particle to the pions. This results in the substitution (s.Fig.2)

$$- \frac{g^2}{4\pi} / (k^2 - m^2) \rightarrow f(q,p)\{- \frac{g^2}{4\pi} / (k^2 - m^2)\} \; f(q',p')$$

where we chose

$$f(q,p) = \frac{m_2^4}{m_1^4} \frac{\left((p - q)^2 - \mu^2 - m_1^2\right)\left((p + q)^2 - \mu^2 - m_1^2\right)}{\left((p - q)^2 - \mu^2 - m_2^2\right)\left((p + q)^2 - \mu^2 - m_2^2\right)} \quad .$$

This can be looked at as introducing a form factor at the vertex between the exchanged particle and the pions. Such strong off shell effects can only be understood if multiparticle intermediate states are important at the vertices. This potential gave reasonable results.

As it turned out to be impossible to produce a resonance of the mass and the width of the ρ meson in the $\pi\pi$ channel via one particle exchange without derivative coupling, we also examined a two channel potential[5] (TCH).* In addition to the $\pi\pi$ channel we introduced a second channel of scalar particles. This second channel is chosen in such a way that the ρ meson can accur mainly as bound state in this channel. Thus its elastic threshold has to be above the ρ meson mass and the coupling has to be strong compared to the coupling as the $K\bar{K}$ channel because we see no reason for a stronger coupling in the $\pi\pi$ channel. We don't like to interprete this second channel as the $K\bar{K}$ channel because we see no reason for a stronger coupling in the $K\bar{K}$ than in the $\pi\pi$ channel. A more realistic picture could be provided by the $N\bar{N}$ channel, the

*A dispersion model for the pion form factor involving several channels has been set up by S. Serio[6] spin complications of which prevent us from treating it properly. For definiteness we call the second channel the $N\bar{N}$ channel in what follows.

The two channel scattering is described by a system of two coupled integral equations, which in graphical form are represented in Fig.3. In the direct channel the exchanged particle has to be a meson, in the coupling of both channels the exchanged particle has to be a nucleon. We have taken the mass of the exchanged particles equal in both Born terms, simply in order to save computation time. According to the mechanism explained above, the TCH potential can produce very narrow resonances.

The results we represent here, are calculated with the [4,4] Padè-approximants and dependent on what we have calculated, we used an integration of about 14 Gaussian points in each variable. As this takes quite a bit of computation time even on the IBM 360/75 computer, we find it more convenient only to reproduce roughly the ρ peak in the $\pi\pi$ scattering amplitude and then compare the resulting electromagnetic form factor with the GS form factor in which the width has been matched* to our phases with an $m = 767.7$ MeV.

For the DC-potential we report our results for the following two parameter sets (Γ_ρ parametrizes the width in the GS-model and is determined by the fit to our phases):

1) $g^2/4\pi = 92$, $m = 5.5\mu$, $m_1 = 4.8\ \mu$, $m_2 = 9\mu$ $\quad$ ($\Gamma_\rho = 162$ MeV)

2) $g^2/4\pi = 174$, $m = 7\ \mu$ $\quad m_1 = 5.15\mu$, $m_2 = 9\mu$ $\quad$ ($\Gamma_\rho = 196$ MeV)

and for the TCH potential they are according to Fig.3:

$$g^2_{\pi\pi}/4\pi \cong 0 \ , \ g^2_{N\bar{N}}/4\pi = 8960 \ , \ g^2_{\pi N}/4\pi = 1180 \ , \ m = 7\mu \ (\Gamma_\rho = 123 \text{ MeV}).$$

Both of these potentials we used to fit the $\pi\pi$ scattering amplitude involve four parameters. This is actually a large number to reproduce roughly the resonance shape of the ρ meson, but m and m_2 may vary in a wide range without changing the results significantly. For the DC potential we cannot really give a physical interpretation and accept it just as phenomenological potential that gives us the desired result. Likewise the large coupling constant in the direct nucleon channel of the TCH case seems to be somehow unphysical. This may be due to our neglection of spin.

*The phases at the lower half width and the resonance mass have been made to coincide.

III. The Pion Form Factor

The integral equation for the renormalized vertex function $\tilde{\Gamma}_\mu$ of the $\gamma\pi\pi$ vertex reads[7]

$$\tilde{\Gamma}_\mu(p + q, p - q) = Z_1 \cdot 2q_\mu - \int \frac{d^4q'}{(2\pi)^4} \tilde{\Delta}'_F(p + q') \times$$

$$\times \tilde{\Gamma}_\mu(p + q', p - q') \tilde{\Delta}'_F(p - q')\tilde{K}(q, q') . \tag{8}$$

Here $\tilde{K}$ is the kernel of the integral equation, given by the sum of all two particle irreducible graphs for $\pi\pi$ scattering, for which we take the same approximation as in Part II. $\tilde{\Delta}'_F$ is the meson propagator including self energy contributions due to strong interactions, Z_1 being the vertex renormalization constant. If we neglect self energy effects by taking

$$\tilde{\Delta}'_F(k) = \frac{1}{k^2 - \mu^2} ,$$

where μ is the π meson mass, we violate the Ward identity

$$2p^\mu \tilde{\Gamma}_\mu = \tilde{\Delta}'_F(p + q) - \tilde{\Delta}'_F(p - q)$$

since we do not couple the photon to all charged lines in our diagrams.

Iterating (8) shows that the vertex function can be written in the form

$$\tilde{\Gamma}_\mu(p + q, p - q) =$$

$$= Z_1\{2q_\mu - i \int \frac{d^4q'}{4\pi^3} M(s, q, q') \frac{2q'_\mu}{[(p - q')^2 - \mu^2][(p + q')^2 - \mu^2]} \}$$

where M is the $\pi\pi$ scattering amplitude and CM - coordinates have been chosen according to Fig.2. From Lorentz invariance we conclude that $\tilde{\Gamma}_\mu$ can be written as:

$$\tilde{\Gamma}_\mu = Z_1\{2q_\mu F_o(p,q) + 2p_\mu G_o(p,q)\} \tag{9}$$

F_o and G_o being unrenormalized form factors.

We observe that the two particle scattering amplitude (even off shell) is symmetric under simultaneous change of sign of:
$q_o, q_o' \leftrightarrow -q_o, -q_o'$ (CP conjugation). Using this symmetry we get for $G_o(W^2)$ by contraction of (9) with p_μ an expression which is antisymmetric in q_o and which for q_o on shell $(q_o = 0)$ vanishes. This means that on shell our model is gauge invariant, although we have selected only a small class of diagrams for the photon coupling.

Evaluating $F_o(t)$ again by contraction with p_μ we see after angular integration, that only the p-wave contribution of the amplitude $M(q,q')$ remains (the s-wave part drops out because of $q_o = 0$). So we finally get for the form factor:

$$F_o(s) = 1 - \frac{i}{\pi^2} \int_{-\infty}^{\infty} dq_o' \int_0^{\infty} |\vec{q}\,'|^2 \, d|\vec{q}\,'| \, \frac{|\vec{q}\,'|}{|\vec{q}\,|} \times$$

$$\times \frac{M_1(s, q', q)}{[(q_o' - \frac{W}{2})^2 - |\vec{q}|^2 - \mu^2][(q_o' + \frac{W}{2})^2 - |\vec{q}\,'|^2 - \mu^2]} \tag{10}$$

Our model is superrenormalizable and so Z_1 is finite and is given by $1/F_o(0)$.

These observations are also valid for the two channel potential. In this case the photon couples to the pions as well as to the nucleons in a way analogues to Fig.1, where the Clebsch-Gordan coefficient for the nucleon channel is $1/\sqrt{2}$.

IV. Numerical Results for the Form Factor

We present the results for the pion electromagnetic form factor according to (9) and (10), where M_1 is calculated by the proceedure described in II and compare them to the form factor of the GS-model with a similar phase as explained.

The results for the DC potential are represented in Fig.4 for the two parameter sets given above. The dashed curves show the GS-form factor for $\Gamma_\rho = 62$ MeV and $\Gamma_\rho = 96$ MeV while the crosses and circles represent four values of $|F_\pi(s)|^2$ resp., which we obtained from our model. We observe that our ρ peak is slightly higher. The renormalization constant is $Z_1 = 0.064$ for both parameter sets.

In Fig.5 $|F_\pi(s)|^2$ is plotted for the TCH potential and the comparison with GS in this case shows that the peak is too low compared to the VDM, the areas under the curves differing by 25%. The normalization constant in this case is $Z_1 = 0.15$. For this potential we also calculated $F_\pi(t)$ in the spacelike region, the results being plotted in Fig.6 in comparison with the ρ pole and the GS model. Deviations can be understood by the observation, that our ρ meson does not exhaust the dispersion integral for the form factor. If we calculate the scattering amplitude

$$f(s) = \frac{e^{2i\delta} \sin\delta}{2p} ,$$

we observe that the width of the ρ in the scattering process is essentially the same as for the form factor, the resonance peak being shifted to higher mass values for about 10 MeV.

V. Conclusions

In our two simple BS models for the ρ meson the slow variation of the matrix element (2) seems to be not automatically fulfilled but depends on the type of the interaction. If one takes a simple one particle exchange interaction of the Yukawa type (and in that case only by the TCH potential the ρ can be reproduced) the renormalization constant is not very small compared to 1, and in that case we observe deviation from the VDM and from experiment.[8] If we neglect the coupling of the photon to the "nucleon", the discrepancy is still larger. Probably the inclusion of spins with the corresponding contribution of higher momentum components in the BS wave function would improve the situation.

For the DC potential the renormalization constant is smaller and agreement with the VDM is better. Since the physical interpretation of this potential is dubious, we cannot claim to have achieved a good understanding of the ρ meson as a pure $\pi\pi$ resonance.

Acknowledgement

One of us (J.F.) wants to thank Professor H. Joos for the kind hospitality extended to him at DESY as well the Deutsche Forschungsgemeinschaft for financial support for a stay at DESY.

Figure Captions

1) Diagram responsible for the vector meson - photon coupling. The broken lines indicate structureless pions.
2) Kinematics of the Bethe-Salpeter equation
3) The coupled system of integral equations representing the two channel BSE. The second amplitude ($N\bar{N}\rightarrow N\bar{N}$) decouples from the two others and is not calculated. $g_{\pi\pi}$, $g_{\pi N}$ and $g_{N\bar{N}}$ are the coupling constants at the corresponding vertices and m is the mass of the exchanged scalar particle.
4) $|F_\pi(s)|^2$ calculated with the DC-potential at four energies with two parameter sets as quoted in the text, is compared with the GS-formfactor, in which the width has been matched to our phases. The dashed curves 1) and 2) correspond to m = 767.7 MeV and Γ_ρ = 162 MeV resp. Γ_ρ = 196 MeV. The corresponding values from our calculation are indicated by crosses resp. circles.
5) $|F_\pi(s)|^2$ calculated with the TCH-potential (full curve) is compared with the GS-formfactor (dashed curve), the latter corresponding to m = 767.7 MeV and Γ_ρ = 123 MeV.
6) $F_\pi(t)$ in the spacelike region calculated with the TCH-potential is represented in comparison with the pion pole (from VDM) and the GS-model again with m = 767.7 MeV and Γ_ρ = 123 MeV.

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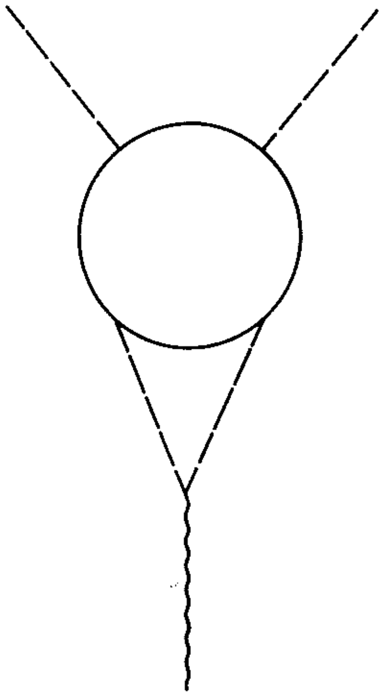

Fig. 1

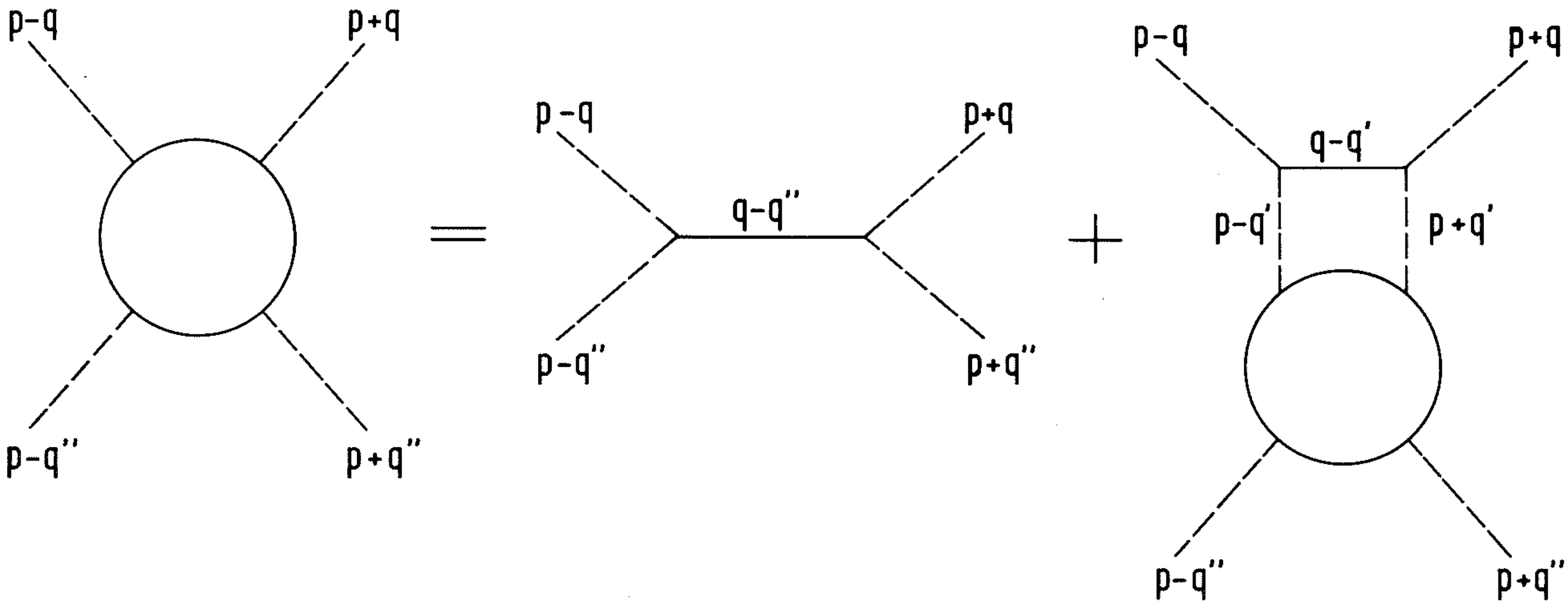

Fig.2

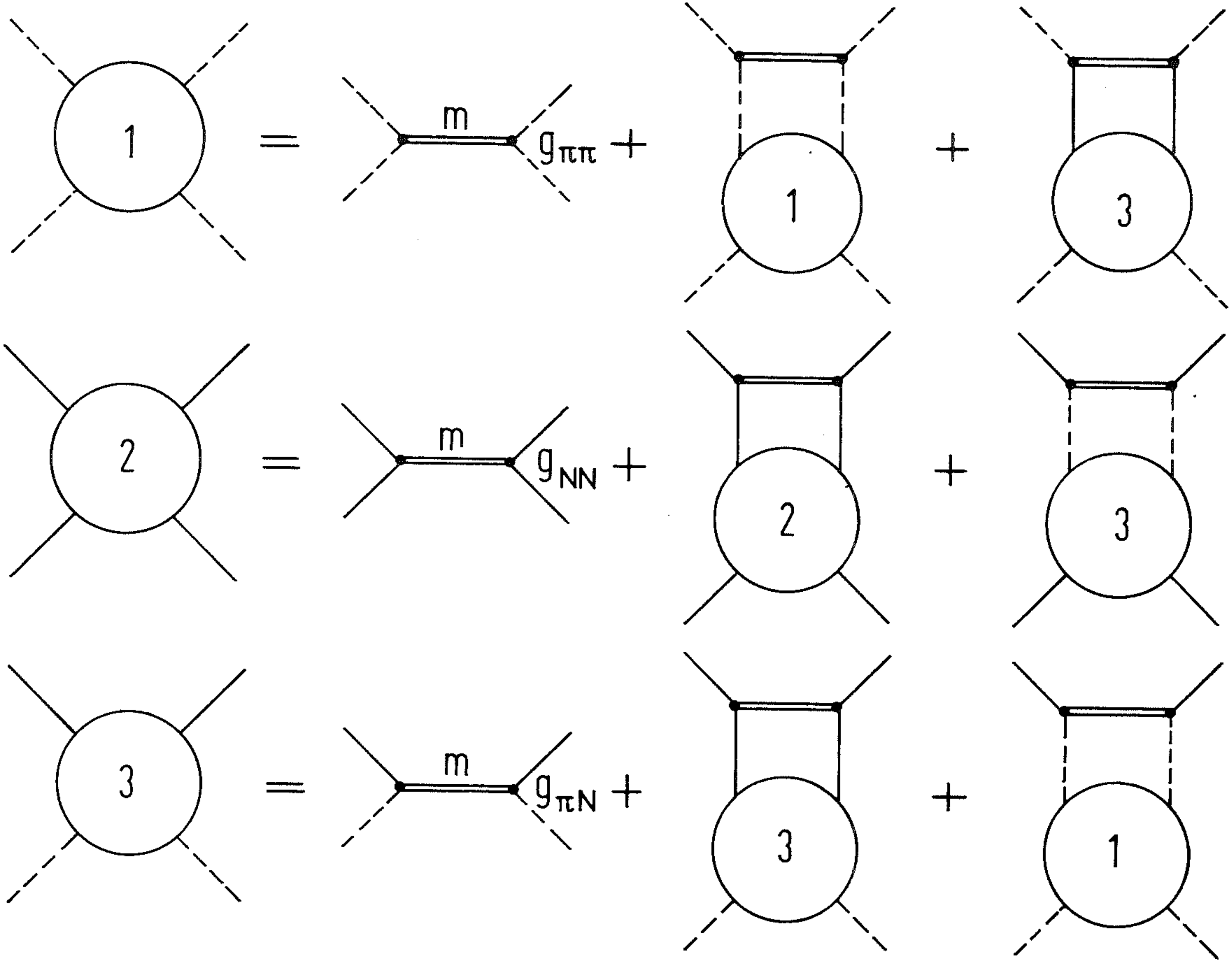

Fig. 3

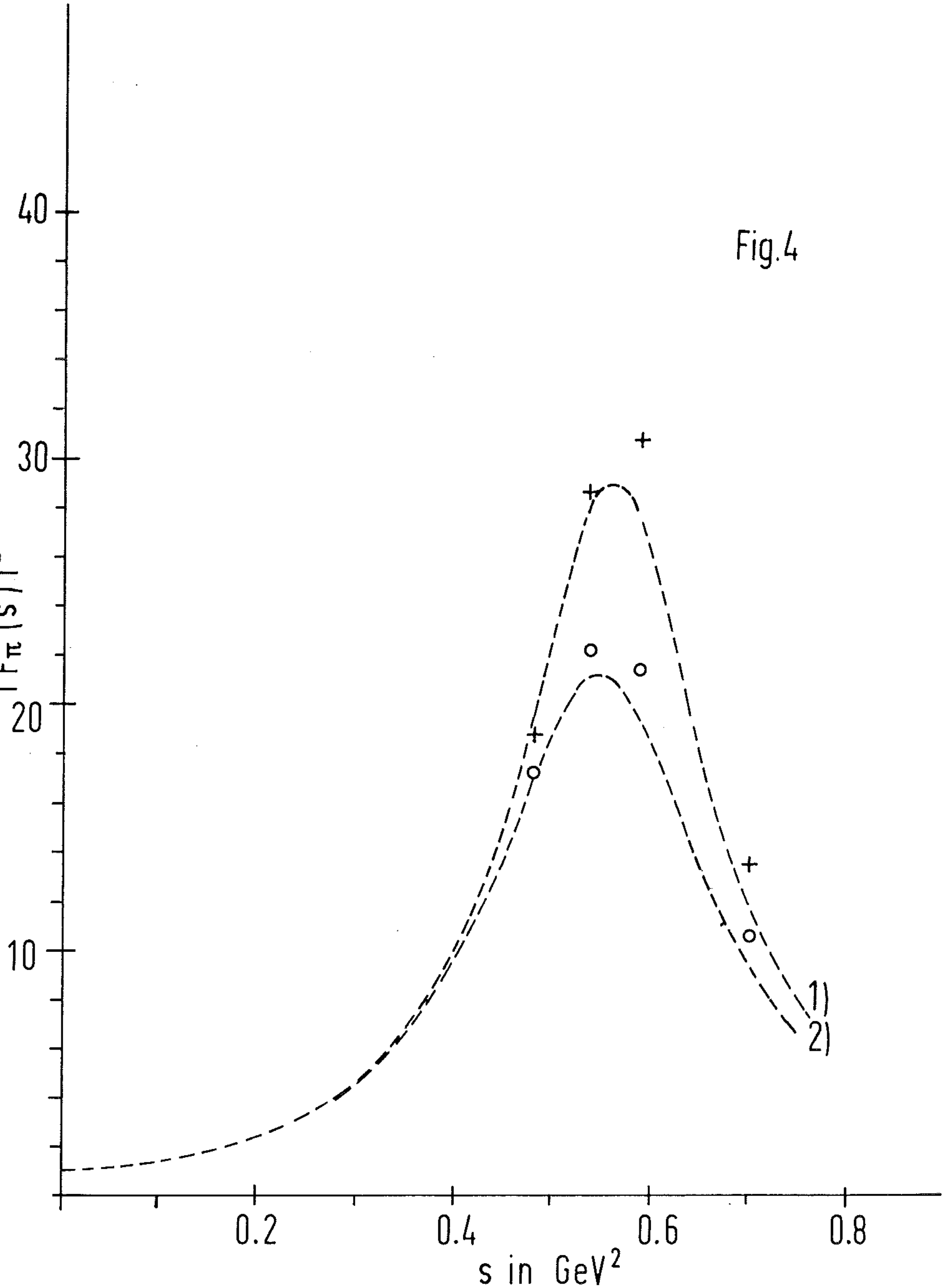

Fig.4

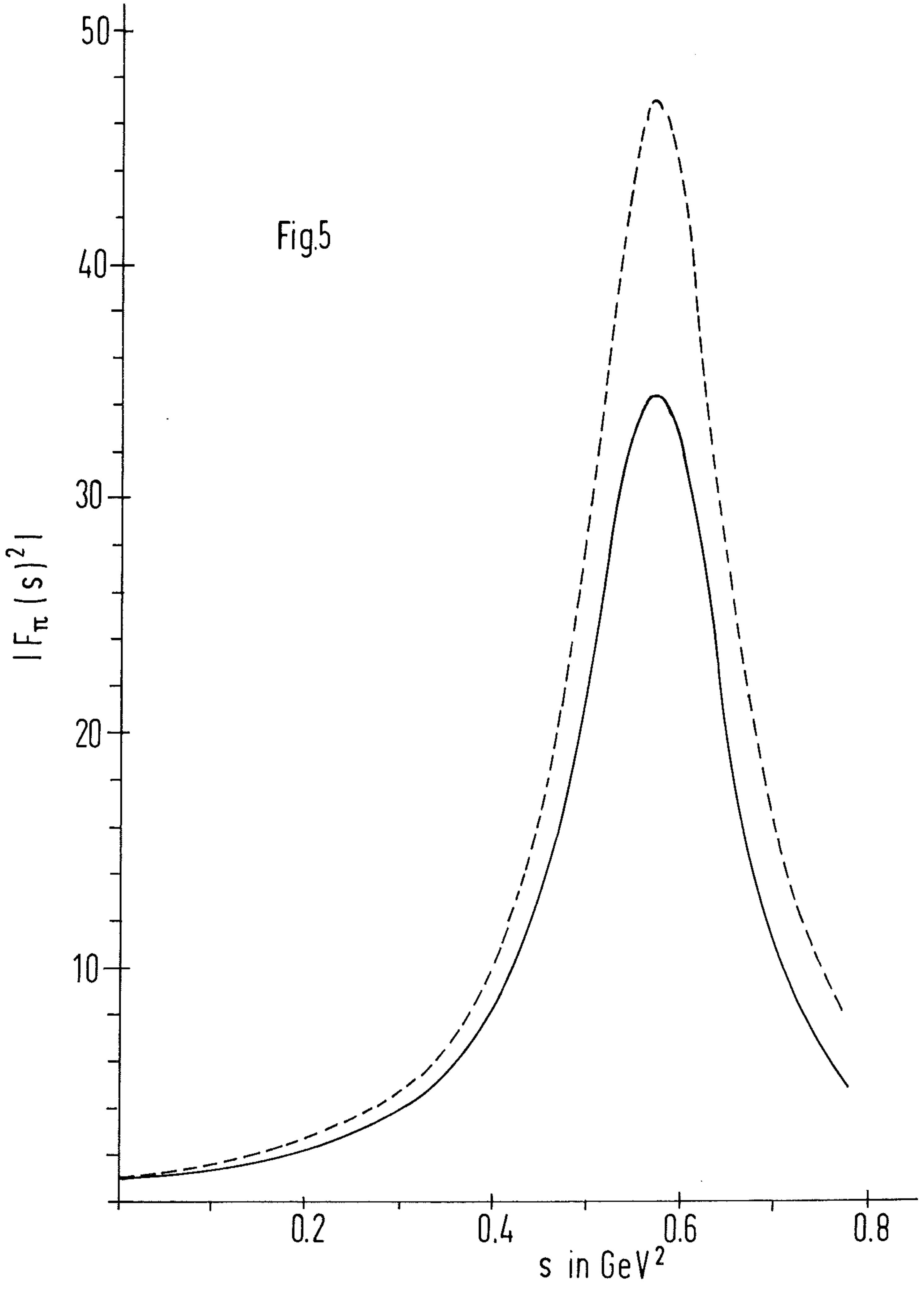
Fig.5
50
40
30
20
10
$|F_\pi(s)^2|$
0.2
0.4
0.6
0.8
s in GeV^2

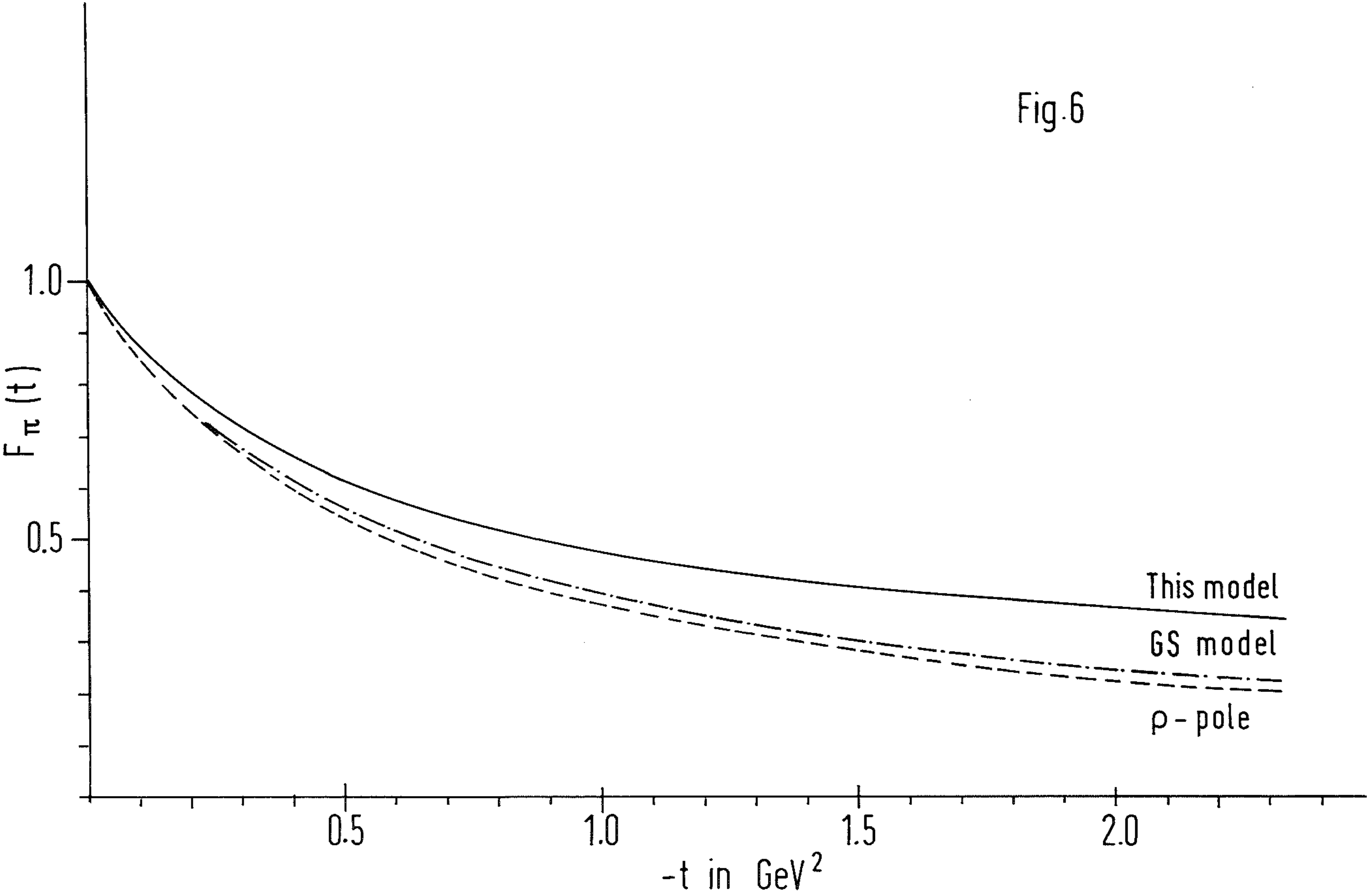
Fig.6
1.0
0.5
F_π (t)
This model
GS model
ρ - pole
0.5
1.0
1.5
2.0
-t in GeV^2