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Scattering Theory for Strong and Electromagnetic Interactions of Relativistic Two-Particle Systems

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Abstract

Relativistic two-particle scattering is generalized to include electromagnetic interactions. A Hamiltonian consistent with the commutation relations of the inhomogeneous Lorentz group and the principle of minimal electromagnetic coupling is derived and can be defined selfadjoint. The scattering operator is computed and found to admit a representation suitable for calculations due to the invariance property of the Møller operators. Finally, the photodisintegration of bound states is discussed briefly.

1. Introduction

Relativistic quantum mechanics [1-3] provides a general scheme for the dynamical description of relativistic two- and many-particle systems. The interaction is expressed in terms of the 10 hermitian operators H, $\vec{P}$, $\vec{J}$, $\vec{K}$, which are restricted only by the requirement to generate an unitary representation of the inhomogeneous Lorentz group.

For strong interactions, i.e. interactions by short-range potentials, relativistic quantum mechanics is discussed in detail in the literature. The extension to electromagnetic interactions is an outstanding problem and shall be studied in this paper.

Section 2 deals with kinematics. Section 3 and Appendix A give some further description of the dynamics of relativistic two-particle systems. In particular a theorem on invariance of Møller operators and half-off-shell T-matrices is proved. Electrodynamics of relativistic two-particle systems is presented in Section 4 and Appendix B and C. A Hamiltonian is derived consistent with the commutation relations of the inhomogeneous Lorentz group and the general assumed principle of minimal electromagnetic coupling. The Hamiltonian can be extended to a selfadjoint operator, so that electrodynamics of two-particle systems is well defined. In Section 5 Møller operators and scattering operator are discussed and prepared for calculations. Finally, in Section 6 the results are applied to the photo-disintegration of bound states in first order of electromagnetic interaction.

2. Kinematics

The interaction of relativistic two-particle systems with real photons can be described in the Hilbert space

$$\tilde{\mathcal{H}} = \mathcal{H} \otimes \mathcal{H}_{\mathcal{F}} \quad , \tag{2.1}$$

where [3] $\mathcal{H} = \mathcal{H}_{m_1, s_1} \otimes \mathcal{H}_{m_2, s_2}$ and $\mathcal{H}_{\mathcal{F}}$ denotes the photon Fock space. A for our purpose most suitable choice of representation of $\mathcal{H}$ is given by eigenstates* $|\vec{K}\vec{k} j j_3\rangle$ with

$$\vec{P}_0 \, |\vec{K}\vec{k} j j_3\rangle = \vec{K} \, |\vec{K}\vec{k} j j_3\rangle \, ,$$

*The notation of the two-particle system follows Ref.[3].

$$H_0 = (\vec{P}_0^{\,2} + h_0^2)^{\frac{1}{2}} \;, \quad h_0 = r_{10} + r_{20} \;,$$

$$r_{1,2} = ((\vec{r}^{\,2} + m_{1,2}^2)^{\frac{1}{2}} \;, \pm \vec{r}\;) \;,$$

$$\vec{r}\; | \vec{K} \vec{k} j j_3 \rangle = \vec{k}\; | \vec{K} \vec{k} j j_3 \rangle \;,$$

$$\vec{K} = \vec{p}_1 + \vec{p}_2 \;, \quad \vec{k} = \pm \vec{L}(K)\, p_{1,2} \;, \tag{2.2}$$

and the n-photon eigenstates of $\mathcal{H}_F$ are of the form

$$|\vec{q}_1 \sigma_1 , \cdots \vec{q}_n \sigma_n \rangle = \Big(\prod_{i=1}^{n} a^+(\vec{q}_i , \sigma_i) \Big) |0\rangle \;,$$

$$[\, a(\vec{q}, \sigma) \,, \; a^+(\vec{q}\,', \sigma') \,] = 2 q_0 \, \delta^3(\vec{q} - \vec{q}\,') \, \delta_{\sigma\sigma'} \tag{2.3}$$

with

$$(H_0 \,, \vec{P}_0) \, |\vec{q}_1 \sigma_1 , \cdots \vec{q}_n \sigma_n \rangle = (Q_0 \,, \vec{Q}) \, |\vec{q}_1 \sigma_1 , \cdots \vec{q}_n \sigma_n \rangle \;,$$

$$\vec{Q} = \sum_{i=1}^{n} \vec{q}_i \;, \quad Q_0 = \sum_{i=1}^{n} |\vec{q}_i| \;, \quad \sigma_i = \pm 1 \,. \tag{2.4}$$

In $\tilde{\mathcal{H}}$ the generators $\vec{P}_0$ and H_0 can be realized by

$$\vec{P}_0 = \vec{K} + \vec{Q} \;, \quad H_0 = (\vec{P}_0^{\,2} + \hat{h}_0^2)^{\frac{1}{2}} \;, \tag{2.5}$$

where

$$\hat{h}_0 = (\vec{R}^{\,2} + h_0^2)^{\frac{1}{2}} + R_0 \;,$$

$$\vec{R} = - \vec{L}(K+Q)\, Q = \vec{L}(K+Q)\, K \;, \quad R_0 = (L(K+Q)\, Q)_0 \,. \tag{2.6}$$

The corresponding generators $\vec{J}_0$ and $\vec{K}_0$ are given from Ref.[4-6]. In the following, however, we are not interested in their explicit form.

3. Further Dynamics of Relativistic Two-Particle Systems

Relativistic two-particle dynamics is determined by the mass operator $h = h_0 + v$, or in case there are no bound states equivalently by the Møller operators [3]

$$\Omega_{\pm}(h,h_o) = 1 - \int \delta_{\pm}(h_o - E_k)\, T_{\mp}(k)\, d\mathcal{P}_o(k) . \qquad (3.1)$$

The underlying Hilbert space is $\mathcal{H}$.

It can be shown and comes out to be important that the Mϕller operators (3.1) are invariant under $h \to \phi\,(h)$, $h_o \to \phi\,(h_o)$ for a certain class of real valued functions* ϕ . The general proof, however, is beyond the scope of this paper. In place of that we prove the following

Theorem**: Let one of the Mϕller operators

$$\Omega_{\pm}(h,h_o)\ ,\ \Omega_{\pm}(h^2,h_o^2),\ \Omega_{\pm}(\vec{P}^2+h^2,\vec{P}^2+h_o^2),\ \Omega_{\pm}((\vec{P}^2+h^2)^{\frac{1}{2}},(\vec{P}^2+h_o^2)^{\frac{1}{2}})$$

exist $(\vec{P} = \vec{P}_o)$. Then

$$\Omega_{\pm}(h,h_o) = \Omega_{\pm}(h^2,h_o^2)$$

$$= \Omega_{\pm}(\vec{P}^2+h^2,\ \vec{P}^2+h_o^2) = \Omega_{\pm}((\vec{P}^2+h^2)^{\frac{1}{2}},(\vec{P}^2+h_o^2)^{\frac{1}{2}}) . \qquad (3.2)$$

The invariance of the Mϕller operators (3.2) is a property of the T-matrices defined by (3.1) and [Appendix A]

$$\Omega_{\pm}(h^2,h_o^2) = \Omega_{\pm}(\vec{P}^2+h^2,\ \vec{P}^2+h_o^2) = 1 - \int \delta_{\pm}(h_o^2-E_k^2)\, \tilde{T}_{\mp}(k)\, d\mathcal{P}_o(k) ,$$

$$\Omega_{\pm}((\vec{P}^2+h^2)^{\frac{1}{2}},(\vec{P}^2+h_o^2)^{\frac{1}{2}}) = 1 - \int \delta_{\pm}((\vec{P}^2+h_o^2)^{\frac{1}{2}}-(\vec{P}^2+E_k^2)^{\frac{1}{2}})\, \hat{T}_{\mp}(k)\, d\mathcal{P}_o(k) . \qquad (3.3)$$

In Appendix A it is shown that

$$T_{\pm}(k)\, d\mathcal{P}_o(k) = (h_o+E_k)^{-1}\, \tilde{T}_{\pm}(k)\, d\mathcal{P}_o(k)$$

$$= (h_o+E_k)^{-1}((\vec{P}^2+h_o^2)^{\frac{1}{2}}+(\vec{P}^2+E_k^2)^{\frac{1}{2}})\, \hat{T}_{\pm}(k)\, d\mathcal{P}_o(k) , \qquad (3.4)$$

which proves the theorem.

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*In time-dependent scattering theory this fact holds under the strong condition $\| v \|_1 < \infty$ [7,8]. In time-independent scattering theory the theorem can be proved under more general and weaker conditions. In addition, the proof is in closer contact with physical quantities.

**Finally, the theorem gives a consistent justification of the Mϕller operators (3.1). See Ref. [3].

The bound states of mass m_α and spin s_α are given by

$$h\, |\vec{p}\, m_\alpha s_\alpha s_3\rangle = m_\alpha\, |\vec{p}\, m_\alpha s_\alpha s_3\rangle . \tag{3.5}$$

They can be normalized according to

$$\langle \vec{p}\,' m_\beta s_\beta s_3' | \vec{p}\, m_\alpha s_\alpha s_3\rangle = 2 p_0^\alpha\, \delta^3(\vec{p}\,'-\vec{p})\, \delta_{\beta\alpha}\, \delta_{s_3' s_3} , \tag{3.6}$$

so that

$$\langle \vec{K}\vec{k}\, j j_3 | \vec{p}\, m_\alpha s_\alpha s_3\rangle = 2 K_0^{\frac{1}{2}} p_0^{\alpha\,\frac{1}{2}}\, \delta^3(\vec{K}-\vec{p})\, \langle \vec{k}\, j j_3 | m_\alpha s_\alpha s_3\rangle . \tag{3.7}$$

The matrix elements $\langle \vec{k}\, j j_3 | m_\alpha s_\alpha s_3\rangle$ are the reduced wave functions of the bound states and fulfil the integral equation

$$(m_\alpha - E_k)\, \langle \vec{k}\, j j_3 | m_\alpha s_\alpha s_3\rangle = \sum_{j', j_3'} \int d^3\vec{k}\,'\, v_{j j_3, j' j_3'}(\vec{k},\vec{k}\,')\, \langle \vec{k}\,' j' j_3' | m_\alpha s_\alpha s_3\rangle . \tag{3.8}$$

In $\tilde{\mathcal{H}}$ the two-particle interaction determined by h leads to

$$H = (\vec{P}_0^{\,2} + \hat{h}^2)^{\frac{1}{2}} , \quad \hat{h} = (\vec{K}^2 + h^2)^{\frac{1}{2}} + R_0 . \tag{3.9}$$

The Hamilton operator H implies a set of generators $H, \vec{P}, \vec{J}, \vec{K}$ with $\vec{P} = \vec{P}_0$ and $\vec{J} = \vec{J}_0$. The corresponding Møller operators $\Omega_\pm\, (H, H_0)$ are given by

$$\Omega_\pm (H, H_0) = \Omega_\pm (h, h_0) \otimes 1_F \tag{3.10}$$

as a consequence of the theorem.

4. Electrodynamics of Relativistic Two-Particle Systems

The generalization of relativistic two-particle dynamics to electromagnetic interactions consists first in the construction of a Hamilton operator, which again results in a complete set of generators $H, \vec{P}, \vec{J}, \vec{K}$ of the inhomogeneous Lorentz group [1] and, in addition, corresponds to the usual assumption of minimal electromagnetic coupling.

As in the instant form of dynamics [9] we set

$$\vec{P} = \vec{P}_0 , \quad \vec{J} = \vec{J}_0 . \tag{4.1}$$

Eq.(4.1) means that linear and angular momentum are conserved and implies that the commutation relations, which contain H, $\vec{P}$ and $\vec{J}$, are automatically satisfied. Apart from spin interactions, the most general ansatz for a Hamiltonian H consistent with the above mentioned conditions is given from (3.9) by the substitution

$$\vec{r}_{1,2} \rightarrow \vec{r}_{1,2} - e_{1,2}\, \vec{a}_{1,2} \tag{4.2}$$

and

$$\vec{R} \rightarrow \vec{R} - e\vec{A} \quad , \quad e = e_1 + e_2 \quad . \tag{4.3}$$

Therein $\vec{a}_1$ ($\vec{a}_2$) denotes the electromagnetic potential in the center-of-momentum (c.m.) system of the cluster of the connected two-particle and photon states at the position of particle 1 (2), and $\vec{A}$ means the c.m. electromagnetic potential at the position of the center of mass of the two-particle system. The quantities e_1 , e_2 are the charges of the corresponding particles. The electromagnetic potentials are understood to be in the Coulomb gauge. In case (4.2) the Coulomb interaction v_c of the two-particle system must be introduced by the further substitution

$$r_{10} + r_{20} \rightarrow r_{10} + r_{20} + v_c \quad . \tag{4.4}$$

Let us now discuss (4.2). Later we shall see that (4.3) is ruled out by physical arguments. In the following we omit the Coulomb potential v_c . On the one hand Coulomb contributions can be neglected in comparison to the strong part of the interaction. On the other hand usual scattering theory does not work for singular potentials.

The local electromagnetic potential $\vec{A}_{1,2}$ at the position of particle 1, 2 is given in Appendix B. Transformation of expression (B.3) to the instantaneous c.m. system of the connected two-particle and photon states then yields [6]

$$\begin{aligned}\vec{a}^{(+)}_{1,2} = (2\pi)^{-\frac{3}{2}} \sum_{j, j_3, j', j_3', \bar{j}, \sigma} \int \frac{d^3\vec{q}}{2q_0} \int d^3\vec{k} \int \frac{d^3\vec{K}}{(2K_0\, 2K_0^{1,2})^{\frac{1}{2}}}\, \Delta^{1,2}\, \tilde{d}^{\,1,2}_{j'j_3', jj_3} \\ \times\, \mathcal{D}^{j'}_{\bar{j}_3 j_3}(\mathcal{R}_{1,2})\, \exp\{-i\sigma\,\Theta(\vec{q}, L(K^{1,2}))\}\, \vec{e}\,(\vec{L}(K^{1,2})q, \sigma) \\ \times\, a^{+}(\vec{q}, \sigma)\, |\vec{K} - \vec{q}\;\mathcal{R}_{1,2}[\vec{k}]^{1,2}\, j'\bar{j}_3 \rangle \langle \vec{K}\,\vec{k}\, j j_3 | \;,\end{aligned} \tag{4.5}$$

where [Appendix B]

$$K^{1,2} = [K-q]^{1,2} + q \;, \quad (M^{1,2})^2 = (K^{1,2})^2 \;, \quad \mathcal{R}_{1,2} = \mathcal{R}([K-q]^{1,2}, K^{1,2}) \;,$$

$$\Delta^{1,2} = \frac{q_0}{(L(K^{1,2})q)_0} \frac{\partial(\vec{L}(K^{1,2})q)}{\partial(\vec{q})} \left\{ \frac{M^{1,2}}{(L(K^{1,2})[K-q]^{1,2})_0} \frac{\partial(\mathcal{R}_{1,2}[\vec{k}]^{1,2})}{\partial(\vec{k})} \right\}^{\frac{1}{2}} \;,$$

$$\tilde{d}^{1,2}_{j'j_3', jj_3} = \langle j' j_3' | s_1 s_1^3, s_2 s_2^3 \rangle \, \mathcal{D}^{s_1}_{s_1^3 \bar{s}_1^3}(\mathcal{R}^{-1}(k_1^{1,2}, k_1^{1,2}+k_2^{1,2})) \, \mathcal{D}^{s_2}_{s_2^3 \bar{s}_2^3}(\mathcal{R}^{-1}(k_2^{1,2}, k_1^{1,2}+k_2^{1,2}))$$

$$\times \langle s_1 \bar{s}_1^3, s_2 \bar{s}_2^3 | j j_3 \rangle \;,$$

$$k^{1,2}_{1,2} = [k_{1,2} - L(K^{1,2})q] \;, \quad k^{1,2}_{2,1} = k_{1,2} \;. \qquad (4.6)$$

The negative frequency part $\vec{a}^{(-)}_{1,2}$ is the adjoint of (4.5).

In relativistic quantum mechanics, however, local potentials are no longer distinguished in comparison with nonlocal ones. On the contrary, local potentials result in divergence difficulties, which can only be avoided by introduction of certain cut-off form factors. Therefore, in the following we set

$$\vec{a}^{(+)}_{1,2} = (2\pi)^{-\frac{3}{2}} \sum_{j, j_3, j', j_3', \bar{j}_3, \sigma} \int \frac{d^3\vec{q}}{2q_0} \int d^3\vec{k} \int \frac{d^3\vec{K}}{(2K_0 \, 2K_0^{1,2})^{\frac{1}{2}}} \Delta^{1,2} \, \tilde{d}^{1,2}_{j'j_3', jj_3}$$

$$\times F((M^{1,2} - E_k)^2) \, \mathcal{D}^{j'}_{\bar{j}_3 j_3'}(\mathcal{R}_{1,2}) \exp\{-i\sigma\,\theta(\vec{q}, L(K^{1,2}))\} \; \vec{e}(\vec{L}(K^{1,2})q, \sigma)$$

$$\times \; a^+(\vec{q},\sigma) \, |\vec{K}-\vec{q} \; \mathcal{R}_{1,2}[\vec{k}]^{1,2} j \bar{j}_3 \rangle \langle \vec{K}\vec{k} j j_3 | \;,$$

$$(4.7)$$

where $F((M^{1,2} - E_k)^2)$ is a real function to be specified later.

Now we compute the Hamilton operator coming out of (4.2). For simplicity the substitution is only done in h_0, and it is assumed that $m = m_1 = m_2$. The general case of unequal masses is discussed in

Appendix C. In general, the potential v will be affected too. But since v is not specified further this case is left open. Those expressions, which contain square roots of electromagnetic potentials can be expanded according to

$$((\vec{r}_1 - e_1\vec{a}_1)^2 + m^2)^{\frac{1}{2}} = (\vec{r}_1^{\,2} + m^2)^{\frac{1}{2}}$$

$$+ \int ((\vec{r}_1^{\,2} + m^2)^{\frac{1}{2}} + (\vec{k}^2 + m^2)^{\frac{1}{2}})^{-1} \{ -e_1(\vec{r}_1\vec{a}_1 + \vec{a}_1\vec{r}_1) + e_1^2\vec{a}_1^{\,2} \} \, d\tilde{P}_0(\cdots, \vec{k}, \cdots)$$

$$+ \; - \; - \; - \; - \; - \quad , \tag{4.8}$$

where $d\tilde{P}_0$ denotes the spectral projection of H_0. The single terms of expansion (4.8) are finite, provided that the form factor F decreases sufficiently strong for large energy transfer. Then, if we consider only first order contributions and write for instance

$$[(\vec{r}_1^{\,2} + m^2)^{\frac{1}{2}} / -e_1(\vec{r}_1\vec{a}_1 + \vec{a}_1\vec{r}_1) + e_1^2\vec{a}_1^{\,2}]_0$$

$$= \int ((\vec{r}_1^{\,2} + m^2)^{\frac{1}{2}} + (\vec{k}^2 + m^2)^{\frac{1}{2}})^{-1} \{ -e_1(\vec{r}_1\vec{a}_1 + \vec{a}_1\vec{r}_1) + e_1^2\vec{a}_1^{\,2} \} \, d\tilde{P}_0(\cdots, \vec{k}, \cdots) \,, \tag{4.9}$$

and in the same manner for the spectrum of the Hamiltonian (3.9), denoted by brackets [ ], we get

$$H = (\vec{P}^2 + \mathcal{B} + [h_0/\mathcal{B}]_0 v + v[h_0/\mathcal{B}]_0 + [h_0/\mathcal{B}]_0^2$$

$$+ [(\vec{R}^2 + h^2)^{\frac{1}{2}} / \mathcal{B} + [h_0/\mathcal{B}]_0 v + v[h_0/\mathcal{B}]_0 + [h_0/\mathcal{B}]_0^2] R_0$$

$$+ R_0 [(\vec{R}^2 + h^2)^{\frac{1}{2}} / \mathcal{B} + [h_0/\mathcal{B}]_0 v + v[h_0/\mathcal{B}]_0 + [h_0/\mathcal{B}]_0^2] + \hat{h}^2)^{\frac{1}{2}} \tag{4.10}$$

with

$$\mathcal{B} = 2\,\mathcal{B}_1 + 2\,\mathcal{B}_2 \,,$$

$$\mathcal{B}_{1,2} = - e_{1,2} (\vec{\pi}_{1,2} \vec{a}_{1,2} + \vec{a}_{1,2} \vec{\pi}_{1,2}) + e^2_{1,2} \vec{a}^{\,2}_{1,2} \quad . \tag{4.11}$$

The Hamilton operator (4.10) is a priori undefined. If the form factor F is chosen properly, for instance has compact support, it can be shown that $D(H^2)$ is dense in $\tilde{\mathcal{H}}$, provided that v satisfies the usual assumptions [3] and $h > 0$. In consequence of (3.9), (4.2) and (4.10) and the fact that the square of a symmetric operator is positive, H^2 comes out to be positive. Therefore, according to Friedrichs theorem [10], H^2 can be extended to a selfadjoint operator, which admits a spectral representation. In the following we define H to be the positive square root of that selfadjoint operator. That is always possible, since every positive symmetric operator has at least one positive selfadjoint extension.

The Hamilton operator H is by construction of generalized Bakamjian-Thomas type [11]. The corresponding mass operator, if taken in an appropriate representation, commutes with $\vec{P}$, $\vec{J}$ and depends only on c.m. variables for the single cluster contributions do have this property. Hence it is a possible candidate for a relativistic Hamiltonian.

The exact choice of the form factor F is left open. For mathematical purposes it is sufficient to assume that F has compact support. A generalization to spin interactions is given in Appendix C.

The interpretation of the Hamiltonian (4.10) is quite clear. The photons are coupled directly to particles 1 and 2 as is usually assumed. On the contrary, the Hamiltonian that would come out of (4.3) has a different meaning. In that case the two-particle system is no longer treated as a composite system as far as electromagnetic interactions are concerned. This is in contrast to the physical situation discussed here. Therefore we need not go into details any further for that case.

5. The Scattering Operator

Electrodynamics of relativistic two-particle systems comes out to be a multi-channel problem if bound states are present or can be produced. In absence of electromagnetic interactions the various bound states behave like elementary particles and, in addition to continuum states, constitute the different channels. Let the bound states be labelled by α. Then the Hilbert space $\tilde{\mathcal{H}}$ can be decomposed according to

$$\tilde{\mathcal{H}} = \bigoplus_{\varkappa} \tilde{\mathcal{H}}^{\varkappa} \quad , \qquad \tilde{\mathcal{H}}^{\varkappa} = \Lambda_{\varkappa} \tilde{\mathcal{H}} \quad , \qquad \varkappa \in \{0, \alpha\} \quad , \tag{5.1}$$

where Λ_α are the projection operators onto the bound states and $\Lambda_0 = \Omega_{\pm}(k, k_0)\, \Omega_{\pm}^{+}(k, k_0) \otimes \mathbb{1}_{\mp}$. In view of this the Hamilton operator H_0 must be changed to

$$H_0 = \sum_{\varkappa} H_{0\varkappa} \quad , \tag{5.2}$$

where

$$H_{00} = (\vec{P}^2 + \hat{k}_0^2)^{\frac{1}{2}} \Lambda_0 \tag{5.3}$$

and

$$H_{0\alpha} = (\vec{P}^2 + \hat{k}^2)^{\frac{1}{2}} \Lambda_\alpha \quad , \tag{5.4}$$

which accordingly yields

$$d\,\tilde{P}_0 = \sum_{\varkappa} d\,\tilde{P}_{0\varkappa} \quad . \tag{5.5}$$

The Møller operators are formally defined by

$$\Omega_{\pm}(H, H_0) = 1 - \int \delta_{\pm}(H_0 - E)\, T_{\mp}(E)\, d\,\tilde{P}_0(\cdots, E, \cdots) \, ,$$

$$T_{\pm}(E) = \operatorname*{s-lim}_{\epsilon \to +0} T_{\pm}^{\epsilon}(E) \, , \quad T_{\pm}^{\epsilon}(E) = H - H_0 + (H - H_0)(E - H \pm i\epsilon)^{-1}(H - H_0) \, , \tag{5.6}$$

and lead to the scattering operator

$$S = 1 - 2\pi i \int \delta(H_0 - E)\, \mathcal{T}_+(E)\, d\tilde{P}_0(\cdots, E, \cdots) \,. \tag{5.7}$$

In consequence of our previous results $\mathcal{T}^{\epsilon}_{\pm}(E)$ exists for $\varepsilon > 0$, and there should be no trouble in doing the limit $\varepsilon \to 0$. The infrared divergences can be avoided by setting $F(0) = 0$ during calculation. In this paper, however, we do not treat the limit $\varepsilon \to 0$ any further but leave it as well as renormalization questions to a future paper.* In place of that we prepare (5.7) for perturbation-theoretical calculations.

In the expansion of $\mathcal{T}_{\pm}(E)$ the single terms are meaningless from the physical point of view if the square roots implied by H cannot be expanded. In principle, this is possible according to (4.8). But there is a better way already known from Section 3. Let

$$\tilde{\mathcal{T}}_{\pm}(E) = \operatorname*{s-lim}_{\epsilon \to +0} \tilde{\mathcal{T}}^{\epsilon}_{\pm}(E) \,,$$

$$\tilde{\mathcal{T}}^{\epsilon}_{\pm}(E) = H^2 - H_0^2 + (H^2 - H_0^2)\left((E \pm i\epsilon)^2 - H^2\right)^{-1}(H^2 - H_0^2) \,. \tag{5.8}$$

Then

$$\mathcal{T}^{\epsilon}_{\pm}(E) = (E + H_0 \pm i\epsilon)^{-1}\, \tilde{\mathcal{T}}^{\epsilon}_{\pm}(E)\, (E + H_0 \pm i\epsilon)^{-1}\, 2(E \pm i\epsilon)$$

$$+ (E - H_0 \pm i\epsilon)(E + H_0 \pm i\epsilon)^{-1}\, \mathcal{T}^{\epsilon}_{\mp}(-E)\, (E + H_0 \pm i\epsilon)^{-1}(E - H_0 \pm i\epsilon) \,. \tag{5.9}$$

The second term on the right hand side of Eq.(5.9) converges for $\varepsilon \to 0$ and therefore does not contribute to $\Omega_{\pm}(H, H_0)$ and S because of the factors $(E - H_0)$. Formally, or order by order this yields

$$\Omega_{\pm}(H, H_0) = 1 - \int \delta_{\pm}(H_0^2 - E^2)\, \tilde{\mathcal{T}}_{\mp}(E)\, d P_0(\cdots, E, \cdots) \tag{5.10}$$

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*The renormalization procedure is outlined in Ref.[12].

and

$$\mathcal{S} = 1 - 2\pi i \int \delta(H_0^2 - E^2)\, \tilde{\mathcal{T}}_+(E)\, d\tilde{\mathcal{P}}_0(\cdots, E, \cdots) . \tag{5.11}$$

The scattering operator (5.11) does not contain any root of electromagnetic potentials and therefore can directly be calculated term by term.

6. Application: Photodisintegration of Bound States

The photodisintegration of bound states is one of the problems we are mainly interested in. In first order of electromagnetic interaction* we get from (3.3), (A.1), (4.10) and (5.8)

$$\begin{aligned} &\langle \vec{K}\vec{k}\, j j_3 \,|\, \tilde{\mathcal{T}}_+(E) \,|\, \vec{p}\, m_\alpha s_\alpha s_3 ; \vec{q}\sigma \rangle \\ &= \langle \vec{K}\vec{k}\, j j_3 \,|\, \{ 1 + \tilde{T}_+(\tilde{k})\, \delta_+(E^2 - H_0^2) \} \{ B + [h_0/B]_0\, v \\ &+ v\,[h_0/B]_0 + [(\vec{K}^2 + h^2)^{\frac{1}{2}} / B + [h_0/B]_0\, v + v\,[h_0/B]_0]\, R_0 \\ &+ R_0\, [(\vec{K}^2 + h^2)^{\frac{1}{2}} / B + [h_0/B]_0\, v + v\,[h_0/B]_0] \} \,|\, \vec{p}\, m_\alpha s_\alpha s_3 ; \vec{q}\sigma \rangle \end{aligned} \tag{6.1}$$

with

$$E = p_0^\alpha + q_0 \quad , \quad E_{\tilde{k}}^2 = (p^\alpha + q)^2 . \tag{6.2}$$

However, Eq.(6.1) is not very useful because $\tilde{T}_+(\tilde{k})$ is generally unknown and cannot be computed as simple as $T_+(\tilde{k})$. In consequence of the theorem stated in Section 3, especially of Eq.(3.4), Eq.(6.1) can be simplified to

$$\begin{aligned} &\langle \vec{K}\vec{k}\, j j_3 \,|\, \tilde{\mathcal{T}}_+(E) \,|\, \vec{p}\, m_\alpha s_\alpha s_3 ; \vec{q}\sigma \rangle \\ &= \langle \vec{K}\vec{k}\, j j_3 \,|\, \{ 1 + T_+(\tilde{k})\, \delta_+(E_{\tilde{k}} - h_0) \} \{ B + [h_0/B]_0\, v \\ &+ v\,[h_0/B]_0 + [(\vec{K}^2 + h^2)^{\frac{1}{2}} / B + [h_0/B]_0\, v + v\,[h_0/B]_0]\, R_0 \\ &+ R_0\, [(\vec{K}^2 + h^2)^{\frac{1}{2}} / B + [h_0/B]_0\, v + v\,[h_0/B]_0] \} \,|\, \vec{p}\, m_\alpha s_\alpha s_3 ; \vec{q}\sigma \rangle , \end{aligned} \tag{6.3}$$

*For the general case of unequal masses see Appendix C.

provided that it is taken on shell, i.e. $E_{k} = E_{\tilde{k}}$. The latter holds always for physical amplitudes.

In Eq.(6.3) the strong interaction is calculated exactly. The matrix elements of $T_{+}(\tilde{k})$ can be computed from the integral equations already discussed in Ref.[3], and the bound states are given by Eq.(3.8). Terms like $[k_0/B]_0 v_1$, $[(\vec{K}^2+k^2)^{\frac{1}{2}}/B]\, K_0$ etc. belong to relativistic corrections and are important for strongly bound composite systems. The form factor F can be chosen equal to one on a compact support and zero elsewhere. That is consistent with our general assumptions. Therefore, if the support is taken to be large enough, the matrix elements (6.3) are not effected by F .

I would like to thank Professor G. Kramer for useful discussions concerning this subject. I am also indebted to thank Professor H. Joos for critical remarks.

Appendix A

The T-matrices defined by (3.1) and (3.3) are of the form

$$T_{\pm}(k) = v + \underset{\epsilon \to +0}{s\text{-}\lim}\; v\,(E_k - h \pm i\epsilon)^{-1} v,$$

$$\tilde{T}_{\pm}(k) = h_0 v + v h_0 + v^2$$

$$+ \underset{\epsilon \to +0}{s\text{-}\lim}\; (h_0 v + v h_0 + v^2)\left((E_k \pm i\epsilon)^2 - h^2\right)^{-1}(h_0 v + v h_0 + v^2),$$

$$\hat{T}_{\pm}(k) = (\vec{P}'^2 + h^2)^{\frac{1}{2}} - (\vec{P}'^2 + h_0^2)^{\frac{1}{2}}$$

$$+ \underset{\epsilon \to +0}{s\text{-}\lim}\; \left((\vec{P}'^2 + h^2)^{\frac{1}{2}} - (\vec{P}'^2 + h_0^2)^{\frac{1}{2}}\right)\left(w - (\vec{P}'^2 + h^2)^{\frac{1}{2}} \pm i\epsilon\right)^{-1}\left((\vec{P}'^2 + h^2)^{\frac{1}{2}} - (\vec{P}'^2 + h_0^2)^{\frac{1}{2}}\right),$$

$$w = (\vec{P}'^2 + E_k^2)^{\frac{1}{2}}. \qquad \text{(A.1)}$$

Without restriction we assume that $T_{\pm}(k)$ exists. Then from (A.1) we get

$$\tilde{T}_{\pm}(k) = h_0 T_{\pm}(k) + T_{\pm}(k) h_0 + \frac{1}{2E_k}(E_k - h_0)\, T_{\pm}(k)\,(E_k - h_0)$$

$$- \frac{1}{2E_k}(E_k - h_0)\left(v - v(E_k + h)^{-1} v\right)(E_k - h_0),$$

$$\hat{T}_{\pm}(k) = 2w\left(w + (\vec{P}'^2 + h_0^2)^{\frac{1}{2}}\right)^{-1} \tilde{T}_{\pm}(k) \left(w + (\vec{P}'^2 + h_0^2)^{\frac{1}{2}}\right)^{-1}$$

$$+ \left(w - (\vec{P}'^2 + h_0^2)^{\frac{1}{2}}\right)\left\{\left(w + (\vec{P}'^2 + h_0^2)^{\frac{1}{2}}\right)^{-1} - \left(w + (\vec{P}'^2 + h^2)^{\frac{1}{2}}\right)^{-1}\right\}\left(w - (\vec{P}'^2 + h_0^2)^{\frac{1}{2}}\right), \qquad \text{(A.2)}$$

which finally yields

$$T_{\pm}(k)\, d\mathcal{P}_o(k) = (h_o + E_k)^{-1}\, \tilde{T}_{\pm}(k)\, d\mathcal{P}_o(k)$$

$$= (h_o + E_k)^{-1} \left((\vec{P}^2 + h_o^2)^{\frac{1}{2}} + (\vec{P}^2 + E_k^2)^{\frac{1}{2}} \right) \hat{T}_{\pm}(k)\, d\mathcal{P}_o(k). \tag{A.3}$$

Appendix B

In $\tilde{\mathcal{H}}$ the local electromagnetic potentials $\vec{A}_{1,2}$ at the position of particle 1, 2 and instant $t = 0$ are given by

$$\vec{A}_{1,2} = \vec{A}^{(+)}_{1,2} + \vec{A}^{(-)}_{1,2}\,, \quad \vec{A}^{(+)}_{1,2} = \vec{A}^{(-)+}_{1,2}\,,$$

$$\vec{A}^{(+)}_{1,2} = (2\pi)^{-\frac{3}{2}} \sum_{s_1^3, s_2^3, \sigma} \int \frac{d^3\vec{q}}{2q_o} \int \frac{d^3\vec{p}_{2,1}}{2p^0_{2,1}} \int \frac{d^3\vec{p}_{1,2}}{(2p^0_{1,2}\, 2p^{1,2\,0}_{1,2})^{\frac{1}{2}}}$$

$$\times\ \vec{e}(\vec{q},\sigma)\ a^{+}(\vec{q},\sigma)\ |\vec{p}_1^{\,1,2} s_1^3 ; \vec{p}_2^{\,1,2} s_2^3\rangle \langle \vec{p}_1 s_1^3 ; \vec{p}_2 s_2^3| \tag{B.1}$$

with [6]

$$p^{1,2}_{1,2} = [p_{1,2} - q]\,, \quad p^{2,1}_{1,2} = p_{1,2}\,,$$

$$[p_{1,2} - q] = \left(((\vec{p}_{1,2} - \vec{q})^2 + m^2_{1,2})^{\frac{1}{2}},\ \vec{p}_{1,2} - \vec{q} \right),$$

$$\vec{e}(\vec{q},\sigma) = \frac{1}{\sqrt{2}}\ \mathcal{L}(\vec{q})\ (0, 1, -i\sigma, 0)\,, \tag{B.2}$$

if $\mathcal{H}$ is spanned by eigenstates $|\vec{p}_1 s_1^3 ; \vec{p}_2 s_2^3\rangle$. The transition to representation states $|\vec{K}\vec{k} J J_3\rangle$ can be performed according to Ref.[3] and yields

$$\vec{A}^{(+)}_{1,2} = (2\pi)^{-\frac{3}{2}} \sum_{j'_1 j'_3, j, j_3, \sigma} \int \frac{d^3\vec{q}}{2q_0} \int d^3\vec{k} \int \frac{d^3\vec{K}}{(2K_0\, 2[K-q]_0^{1,2})^{\frac{1}{2}}} \left(\frac{\partial([\vec{k}]^{1,2})}{\partial(\vec{k})} \right)^{\frac{1}{2}}$$

$$\times\, d^{1,2}_{jj'_3, jj_3}\ \vec{e}(\vec{q},\sigma)\ a^+(\vec{q},\sigma)\ |\vec{K}-\vec{q}\ [\vec{k}]^{1,2} j' j'_3\rangle \langle \vec{K}\vec{k} j j_3|\ , \tag{B.3}$$

where

$$[K-q]^{1,2} = \left(((\vec{L}^{-1}(K)k_{1,2} - \vec{q})^2 + m_{1,2}^2)^{\frac{1}{2}} + ((\vec{L}^{-1}(K)k_{2,1})^2 + m_{2,1}^2)^{\frac{1}{2}},\ \vec{K}-\vec{q} \right),$$

$$[\vec{k}]^{1,2} = \mp\, \vec{L}([K-q]^{1,2})\, L^{-1}(K)\, k_{2,1}\ , \quad k_{1,2} = ((\vec{k}^2 + m_{1,2}^2)^{\frac{1}{2}}, \pm\vec{k})\ ,$$

$$d^{1,2}_{jj'_3, jj_3} = \langle j j'_3 | s_1 s'^3_1, s_2 s'^3_2 \rangle\ \mathcal{D}^{s_1}_{s'^3_1 \bar{s}^3_1}(\mathcal{R}^{-1}(p_1^{1,2}, [K-q]^{1,2}))\ \mathcal{D}^{s_2}_{s'^3_2 \bar{s}^3_2}(\mathcal{R}^{-1}(p_2^{1,2}, [K-q]^{1,2}))$$

$$\times\ \mathcal{D}^{s_1}_{\bar{s}^3_1 \tilde{s}^3_1}(\mathcal{R}(p_1, K))\ \mathcal{D}^{s_2}_{\bar{s}^3_2 \tilde{s}^3_2}(\mathcal{R}(p_2, K))\ \langle s_1 \tilde{s}^3_1, s_2 \tilde{s}^3_2 | j j_3 \rangle\ . \tag{B.4}$$

Appendix C

The generalization of the Hamilton operator (4.10) to the case of unequal masses can be done along the lines of Section 4. We get

$$H = \left(\vec{P}^2 + \Sigma + [(\vec{R}^2 + \ell^2)^{\frac{1}{2}} | \Sigma] R_0 + R_0 [(\vec{R}^2 + \ell^2)^{\frac{1}{2}} | \Sigma] + \hat{\ell}^2 \right)^{\frac{1}{2}} \tag{C.1}$$

with

$$\Sigma = B_1 + B_2 + (\vec{r}_1^2 + m_1^2)^{\frac{1}{2}} [(\vec{r}_2^2 + m_2^2)^{\frac{1}{2}} | B_2]_0 + [(\vec{r}_2^2 + m_2^2)^{\frac{1}{2}} | B_2]_0 (\vec{r}_1^2 + m_1^2)^{\frac{1}{2}}$$

$$+ (\vec{r}_2^2 + m_2^2)^{\frac{1}{2}} [(\vec{r}_1^2 + m_1^2)^{\frac{1}{2}} | B_1]_0 + [(\vec{r}_1^2 + m_1^2)^{\frac{1}{2}} | B_1]_0 (\vec{r}_2^2 + m_2^2)^{\frac{1}{2}}$$

$$+ \left([(\vec{r}_1^2 + m_1^2)^{\frac{1}{2}} | B_1]_0 + [(\vec{r}_2^2 + m_2^2)^{\frac{1}{2}} | B_2]_0 \right)^2$$

$$+ \left\{ \left[(\vec{p}_1^{\,2}+m_1^2)^{\frac{1}{2}} / B_1\right]_0 + \left[(\vec{p}_2^{\,2}+m_2^2)^{\frac{1}{2}} / B_2\right]_0 \right\} v$$

$$+ v \left\{ \left[(\vec{p}_1^{\,2}+m_1^2)^{\frac{1}{2}} / B_1\right]_0 + \left[(\vec{p}_2^{\,2}+m_2^2)^{\frac{1}{2}} / B_2\right]_0 \right\} . \qquad \text{(C.2)}$$

In presence of spin interactions the Hamilton operator must be generalized further. We restrict ourself to spin-1/2 particles. Let μ_1 and μ_2 be the magnetic moments of particle 1 and 2. Then, in order to be in agreement with the Dirac theory we are led to set

$$((\vec{p}_1 - e_1 \vec{a}_1)^2 + m_1^2)^{\frac{1}{2}} \longrightarrow \Big((\vec{p}_1 - e_1 \vec{a}_1)^2 + m_1^2$$

$$- \mu_1 m_1 \epsilon_{ikl} \, [s_{1i} , f_{1kl}]_+ + \text{higher terms} \Big)^{\frac{1}{2}} \qquad \text{(C.3)}$$

with

$$f_{1\,kl}^{(+)} = i \, (2\pi)^{-\frac{3}{2}} \sum_{j, j_3, j', j_3', \bar{j}_3, \sigma} \int \frac{d^3\vec{q}}{2 q_0} \int d^3\vec{k} \int \frac{d^3\vec{K}}{(2K_0 \, 2K_0^1)^{\frac{1}{2}}} \, \Delta^1 \tilde{d}^{\,1}_{j j_3', j j_3}$$

$$\times \, F((M^{12}-E_k)^2) \, \mathcal{D}^{j'}_{\bar{j}_3 j_3'}(R_1) \, \exp\{-i\sigma\Theta(\vec{q}, L(K^1))\}$$

$$\times \, (L(K^1)q)_k \, e_l(\bar{L}(K^1)q, \sigma) \, a^+(\vec{q},\sigma) \, |\vec{K}-\vec{q}\, R_1 [\vec{k}]^1 j' \bar{j}_3 \rangle \langle \vec{K} \vec{k} j j_3 |$$

$$- \, (k \longleftrightarrow l) , \qquad \text{(C.4)}$$

and similarly for particle 2. Therein $[\ \ ]_+$ denotes the anti-commutator, and the individual spin operators $\vec{s}_{1,2}$ are defined by [2,5] $\vec{s}_1 + \vec{s}_2 = \vec{j}$. In (C.3) higher terms are needed in order to keep the radicand positive. We do not specify them any further for they are ambiguous to some extent. Finally, this leads to

$$\mathcal{B}_{1,2} = - \quad e_{1,2} \, (\vec{r}_{1,2} \, \vec{a}_{1,2} + \vec{a}_{1,2} \, \vec{r}_{1,2}) + e_{1,2}^2 \, \vec{a}_{1,2}^{\,2}$$

$$- \mu_{1,2} \, m_{1,2} \, \epsilon_{ikl} \, [s_{1,2_i} , f_{1,2_{kl}}]_+ + \text{higher terms} \quad .$$

(C.5)

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