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Abstract

An attempt is made to calculate the terms of fourth order in the momenta in $\pi\pi$-scattering from the nonlinear chiral-invariant Lagrangian of pions and nucleons. Two parameters remain arbitrary in standard renormalization theory. Heuristic arguments based on the idea of minimal singularity and the technique of superpropagators lead to values which are in qualitative agreement with empirical information.

I. Introduction

We take the minimal chiral-invariant Lagrangian[1] which contains pion and nucleon fields as a field-theoretic model for pion-pion scattering and we try to calculate from it two parameters, denoted by α_1 and α_2, which occur in a recent low-energy description of this process[2]. As regards the choice of the Lagrangian, we are looking for a model which is simple, i.e. the input parameters are known, and realistic enough so that we may hope to obtain a qualitative or semiquantitative understanding. The simplest possibility, which takes only the self-coupling of pions into account, does not lead to satisfactory results. Therefore, we include the coupling of pions to nucleons. However, we neglect the contributions from other hadrons fields without discussing their significance. We do not consider the linear σ-model[3] since it has an additional parameter and could only give one relation between α_1 and α_2.

The essential parameters of the model are the pion decay constant F_π and the axial coupling constant g_A which is introduced as a phenomenological vertex correction. Alternatively, via the Goldberger-Treiman relation, we may use F_π and G/M_N with G the pion-nucleon coupling constant. Pions are taken as massless. m_π will be chosen non-zero only in kinematical relations to connect momenta correctly with energies. The nucleon mass M_N disappears (except in the ratio G/M_N) since we restrict the pion momenta to $q^2/M_N{}^2 << 1$ (q the c.m. momentum).

The main problem lies in the nonrenormalizability of the Lagrangian. In fact, the parameters α_1 and α_2 are connected with loop diagrams and remain arbitrary if we apply standard renormalization theory. To go beyond it, we propose[4] to choose renormalization constants so that the amplitude is, in a given order of perturbation theory, of minimal growth for large energies. Such a choice can be viewed as a generalization of the principle of minimality used in renormalization theory[5]. It is possible in special cases only. For our problem it leads in the one-loop approximation to a relation between α_1 and α_2 which determines the isospin I = 1 amplitude. If combined with an effective range expansion it gives a value for the ρ mass. To obtain a second relation, needed for the I=0,2 amplitudes, we have to rely on a superpropagator calculation[4] whose significance is less clear.

Both arguments use the one-loop approximation and therefore perturbation theory as an expansion in $f_\pi = \frac{1}{2F_\pi}$. Since the interaction of massless pions gives rise to a logarithmic dependence on momenta[2], the perturbation expansion cannot be a simple power series on dimensional grounds. It contains terms like $f_\pi^4 \log f_\pi^2$ etc which depend logarithmically on the coupling constant. They come in through the superpropagator technique. Concerning the applicability of perturbation theory to our problem, we know from the low-energy theorems of Adler[6] and Weinberg[7] that the lowest order term, which is proportional to $f_\pi^2\, q^2$, is exact in the zero-momentum limit. Therefore, higher order corrections are small for small momenta. However, the relative importance of successive powers of f_π^2 is not known. The parameters α_1 and α_2 (and also the axial vector constant g_A) are functions of $f_\pi^2 M_N^2/4\pi^2 = 0.66$ which appears to be the relevant parameter. We calculate them in lowest order, taking higher order effects partially into account by renormalizing the pion-nucleon vertices. It can be shown that our calculation by means of one-loop diagrams and an effective range expansion is equivalent to a lowest order Padé approximation if terms of sixth and higher orders in q(which are unreliable) are neglected.

The results obtained are satisfactory considering the limitations of the model. They suggest a possible qualitative picture of low-energy pion-pion scattering. Of course, the calculation presented here should be viewed as an attempt to extract results from a nonrenormalizable field theory and further applications are needed before any conclusions can be drawn.

II. The f_π^4-approximation

The classical Lagrangian can be written as[1]

$$L = \frac{1}{16 f_\pi^2} \operatorname{Tr} \{ \partial_\mu e^{2if_\pi \vec{\tau}\vec{\phi}} \partial^\mu e^{-2if_\pi \vec{\tau}\vec{\phi}} \} + \bar{\psi} i\gamma\partial\psi - M_N \bar{\psi} e^{2if_\pi \gamma_5 \vec{\tau}\vec{\phi}} \psi. \qquad (1)$$

The interaction terms are minimal in the number of derivatives. This implies that the deviation of the axial vector constant

$$g_A = g_A\left(\frac{f_\pi^2 M_N^2}{4\pi^2} \right) \approx 1.25$$

from one is to be accounted for by a vertex correction, i.e. $g_A(0) = 1$. In the

Bogoliubov[5] formulation of perturbation theory the quantized interaction Lagrangian is then given by

$$L_I = -\frac{2}{3} f_\pi^{\,2} :(\vec{\phi} \times \partial_\mu \vec{\phi})^2: - 2 f_\pi M_N :\bar{\psi}\, i\gamma_5 \vec{\tau}\vec{\phi}\psi: + 2M_N\, f_\pi^2 : \bar{\psi}\psi\vec{\phi}^2 : + O(f_\pi^{\,3}) \qquad (2)$$

where ϕ and ψ now denote the incoming field operators. In the $f_\pi^{\,4}$ - approximation to $\pi\pi$-scattering the five Feynman diagrams shown in Fig. 1 contribute. The low-energy theorems[6)7)] assure us that the amplitude vanishes for zero momenta and that the terms quadratic in the momenta are exactly given by the tree diagram (a). Therefore, no such terms arise from the subtractions necessary in loop diagrams. Consequently, the nucleon contributions from the diagrams (c), (d) and (e) are well-defined. Only the pion loop (b), which needs three subtractions, leads to two undetermined parameters[4]. We evaluate all diagrams for small momenta discarding terms which are of higher than fourth-order in the pion momenta since for them the $f_\pi^{\,4}$-approximation is certainly inadequate. Using standard notation for the scattering amplitude

$$T = \delta_{i_1 i_2}\,\delta_{i_3 i_4}\,A(s,t,u) + \delta_{i_1 i_3}\,\delta_{i_2 i_4}\,A(t,s,u) + \delta_{i_1 i_4}\,\delta_{i_2 i_3}\,A(u,t,s) \qquad (3)$$

we obtain the following contributions:

$$A_{(a)} = 4\, f\pi^2\, s, \qquad (4)$$

$$A_{(b)} = -\frac{f\pi^4}{6\pi^2}\,\{3\, s^2 \log[\alpha_{1\pi}(-s-i0)] + t(t-u)\log[\alpha_{2\pi}(-t-i0)] + u(u-t)\log[\alpha_{2\pi}(-u-i0)]\}, \qquad (5)$$

$$A_{(c)} = \frac{2}{5}\frac{f\pi^4}{\pi^2}\, s^2, \quad A_{(d)} = -\frac{2}{15}\frac{f\pi^4}{\pi^2}\, s^2, \quad A_{(e)} = -\frac{1}{15}\frac{f\pi^4}{\pi^2}\,[9s^2 - 5(t^2+u^2)]\ . \qquad (6)$$

The total nucleon contribution is therefore

$$A_N = A_{(c)} + A_{(d)} + A_{(e)} = -\frac{1}{3}\frac{f\pi^4}{\pi^2}\,(s^2 - t^2 - u^2)\ . \qquad (7)$$

III. The Renormalization Parameters

We summarize the arguments which lead us to propose definite values for the renormalization parameters $\alpha_{1\pi}$ and $\alpha_{2\pi}$, referring to Lehmann and Trute[4] for a more detailed discussion. As already mentioned, a relation between $\alpha_{1\pi}$ and $\alpha_{2\pi}$ results from the notion of minimal growth for $s \to \infty$. It is $\alpha_{1\pi} = \alpha_{2\pi}$. To show this we take the isospin I = 1 component of $A_{(b)}$.

$$A_{(b)}^{I=1}(s,t,u) = \frac{f\pi^4}{6\pi^2}(u-t)\left[s\log(-s-i0)+t\log(-t-i0)+u\log(-u-i0)-3s\log\frac{\alpha_{1\pi}}{\alpha_{2\pi}}\right]. \tag{8}$$

It has the following behavior for $s \to \infty$, keeping t(or u) fixed:

$$\begin{aligned} &\mathrm{Re}\, A_{(b)}^{I=1} \longrightarrow \mathrm{const} \cdot s^2 \ \text{if}\ \alpha_{1\pi} \neq \alpha_{2\pi} \\ &\mathrm{Re}\, A_{(b)}^{I=1} \longrightarrow \mathrm{const} \cdot s \log s \ \text{if}\ \alpha_{1\pi} = \alpha_{2\pi} . \end{aligned} \tag{9}$$

The other diagrams of order $f\pi^4$ do not destroy this behavior since they increase at most as s log s. We take this property as a heuristic basis for choosing $\alpha_{1\pi} = \alpha_{2\pi}$. An alternative formulation follows by calculating the I = 1 amplitude from its imaginary part by an ordinary dispersion relation. This leads to

$$A_{(b)}^{I=1} = \frac{f_\pi^4}{6\pi^2}(t-u)\int_0^\infty d\alpha\left[\frac{s}{\alpha-s}+\frac{t}{\alpha-t}+\frac{u}{\alpha-u}\right]. \tag{10}$$

The integral converges on the energy-shell. This dispersion integral gives the minimal definition. We could, as always, add to it a polynomial whose lowest order term, due to the low-energy theorems and the antisymmetry in t,u, is proportional to s(t-u) corresponding to $\alpha_{1\pi} \neq \alpha_{2\pi}$ [8].

The idea of making a least singular definition of renormalization parameters has been discussed some time ago in the context of nonpolynomial Lagrangians[9]. Here we use it in a more restrictive manner, i.e. in a definite order of perturbation theory. This is possible only in special cases. For example, the leading term of the I = 0,2 amplitudes is proportional to s^2logs for $s \to \infty$ independent of $\alpha_{1\pi}$ and $\alpha_{2\pi}$. Therefore, an analogous argument for the remaining parameter $\alpha_\pi = \alpha_{1\pi} = \alpha_{2\pi}$ cannot be given. A value for α_π follows from the superpropagator method[4] with the result

$$\alpha_\pi = \frac{f\pi^2}{\pi^2} \exp \ [3\gamma - \frac{91}{20}] \qquad (11)$$

where γ is Euler's constant. Unfortunately the structure of chiral-invariant Lagrangians is such that it is not known whether this result can be connected with the concept of minimal singularity. It is obtained as the limit of a regularization procedure applied to a set of diagrams which is not chiral-invariant.

IV. Renormalization of Pion-Nucleon Vertices

The coupling constants which occur at the pion-nucleon vertices have to be renormalized. In the case of the three-vertex it follows from the Goldberger-Treiman relation, which is exact for massless pions, that f_π has to be replaced by $f_\pi g_A$. For the four-vertex the situation is more complicated. Here too we make the substitution $f_\pi \to f_\pi g_A$ which implies that we neglect a momentum-dependent term.

The exact form of the renormalized on-shell vertices is known from the low-energy theorems on pion-nucleon scattering[6)7)]. However, these theorems refer to the derivative coupling between pions and nucleons which follows from our Lagrangian (2) by a transformation on the nucleon field[1)]. If we were to use this derivative coupling for the loop diagrams (c), (d), (e) the nucleon contribution would become, for $g_A \neq 1$, more singular and lead to a new undetermined subtraction constant. To avoid singularities which enter through an incomplete treatment of higher order effects we continue to use the Lagrangian(2) with momentum-independent vertices. With this restriction the correct normalization of pion-nucleon vertices is obtained by replacing f_π by $f_\pi g_A$ as seen from the effective Lagrangian which gives the low-energy theorems for πN-scattering in the renormalized tree-approximation. In the derivative form this Lagrangian is

$$L_{tree} = -\frac{2}{3} f_\pi^2 :(\vec{\phi}\times\partial_\mu\vec{\phi})^2: - f_\pi g_A :\bar{\psi}\gamma_5\gamma_\mu\vec{\tau}\partial^\mu\vec{\phi}\psi : - f_\pi^2 :\bar{\psi}\gamma_\mu\vec{\tau}\cdot(\vec{\phi}\times\partial^\mu\vec{\phi})\psi: . \qquad (12)$$

This is unitarily equivalent to

$$L_{tree} = -\frac{2}{3} f_\pi^2 :(\vec{\phi}\times\partial_\mu\vec{\phi})^2: - 2f_\pi g_A M_N :\bar{\psi} i\gamma_5 \vec{\tau}\vec{\phi}\psi: + 2M_N f_\pi^2 g_A^2 :\bar{\psi}\psi\vec{\phi}^2: + \tag{13}$$

$$+ f_\pi^2 (g_A^2 - 1) :\bar{\psi}\gamma_\mu\vec{\tau}\cdot(\vec{\phi}\times\partial^\mu\vec{\phi})\psi:.$$

Our normalization of vertices corresponds to neglecting the last term in (13).

V. Results

The results for the parameters α_1 and α_2 can now be read off by comparing their definition[2)] with the equations (5), (7) multiplied by g_A^4, and (11). With the notation

$$\log \alpha_i = \log \alpha_{i\pi} + \log \alpha_{iN}, \tag{14}$$

where $\alpha_{i\pi}$ is the contribution from the pion loop diagram (b), α_{iN} the nucleon contribution, we obtain

$$\log \alpha_{1N} = \frac{1}{3} g_A^4, \quad \log \alpha_{2N} = - g_A^4 \tag{15}$$

while $\alpha_{1\pi} = \alpha_{2\pi}$ is given by (11). For the parameters ξ and η defined in terms of α_1, α_2 this implies

$$\xi = \frac{2}{3} g_A^4 - 1/18$$

$$\eta = \frac{25}{18} \left(\beta_\pi + \frac{11}{150} + \frac{3}{25} g_A^4\right) \tag{16}$$

where

$$\beta_\pi = \log\left(\frac{f_\pi^2}{4\pi^2\alpha_\pi}\right) = \frac{91}{20} - 3\gamma - \log 4.$$

The parameter ξ which determines the P wave depends only on the condition $\alpha_{1\pi} = \alpha_{2\pi}$. As seen from Eq.(16) it is a consequence of this condition that the contribution of the pion loop to the P wave phase is extremely small and the ρ resonance does not follow from the $\pi\pi$ self-coupling. However, the nucleon diagrams provide, in the effective range approximation[2], an attraction of the correct order of magnitude. ξ is related to the ρ mass by

$$\xi = \frac{16\pi^2 F_\pi^2}{m_\rho^2 (1 - \frac{4m_\pi^2}{m_\rho^2})} \tag{17}$$

Therefore, with $m_\pi = 0$ and neglecting the pion loop contribution (- 1/18) we obtain approximately

$$m_\rho \approx \sqrt{\frac{3}{2}}\, \frac{4\pi F_\pi}{g_A^2} \approx \sqrt{\frac{3}{2}}\; \frac{1}{G^2/4\pi}\; \frac{M_N^2}{F_\pi} \quad , \tag{18}$$

the two expressions being connected by the Goldberger-Treiman relation $F_\pi G \approx M_N g_A$. (18) gives $m_\rho \approx 925$ MeV with $g_A = 1{,}25$ or $m_\rho \approx 815$ MeV with $G^2/4\pi = 14.7$, the difference being due to the rather large corrections[10] to the Goldberger-Treiman relation. Use of the complete expressions (16) and (17) raises these values for m_ρ by about 50 MeV. Therefore, in this approach the behavior of the P wave is governed by the scattering length given by the chiral-invariant four-pion vertex and the range due to the nucleon diagrams, the main contribution coming from the box diagram (e). Indeed of only the diagrams (a) and (e) are taken into account (the imaginary part of the amplitude which is determined by unitarity is of course related to the pion loop) satisfatory values for the ρ mass and width follow, the latter being given by the Brown-Goble relation[11]. For the S waves, which depend on the superpropagator value (11) for α_π, evaluation of η from Eq.(16) shows that the $\pi\pi$ self-coupling is essential. Numerically the I = 0 phase shift comes out as $\delta_o^o\,(q_\rho^2) \approx 110^o$ at $\sqrt{s} = m_\rho$.

In view of the uncertainties inherent in the model and the various approximations we have made, we can only remark that this attempt to obtain information from a nonrenormalizable field theory leads to reasonable results which may encourage further applications.

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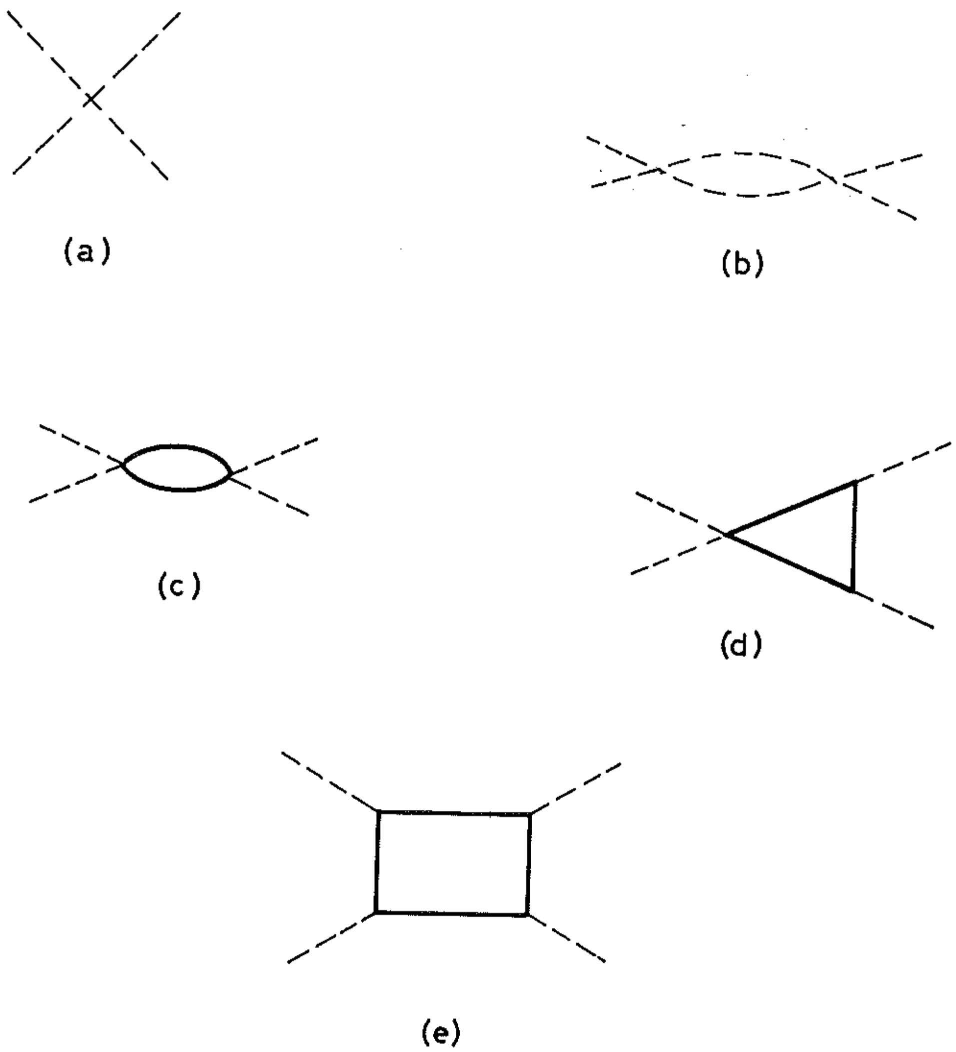

Fig.1

Diagrams which contribute to the f_{π}^{2}-approximation