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by

K. Koller

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Abstract: The 2-point functions of Euclidean conformal invariant quantum field theory are looked at as intertwining kernels of the conformal group. In this analysis a fundamental role is played by a two-element group W, whose non-identity element $\mathcal{R} = R \cdot I$ consists of the conformal inversion R multiplied by a space-time reflection I. The propagators of conformal invariant quantum field theory are determined by the requirement of $\mathcal{R}$-covariance. The importance of the $\mathcal{R}$-inversion in the theory of Zeta-functions is mentioned.

I. INTRODUCTION

In his article[1] on automorphic functions and the theory of representations I.M. Gel'fand has drawn attention to the fact that >> the Zeta-functions of a homogeneous space are quite analogous to the Heisenberg S-Matrix <<. Some properties relating to this Zeta-function aspect can be studied in conformal invariant quantum field theory (QFT) where the physical space-time is realized as a homogeneous space of the conformal group. The reason that Gel'fand's statement is more transparent in the case of a conformal invariant QFT, lies in the close interrelationship in this theory of group theoretic and quantum field theoretic properties: for example, the 2-point functions are intertwining-kernels, the 3-point functions are Clebsch-Gordan-kernels of the conformal group and the higher n-point functions can be harmonically analysed in such a way that the set of nonlinear integral equations in Lagrangian field theory can be simultaneously diagonalised, with the Wilson expansions in the n-point functions corresponding to the Regge-poles in the SO(2,1) partial wave analysis. These group theoretic aspects have been treated in some detail by Mack[2]. Further discussion of conformal covariant 2-point functions for Minkowski space-time has been given by Rühl[3] and earlier work by a number of other authors on related questions can be found in Ref.(4,5).

In this paper a more refined treatment of the conformal invariant propagators will be given. These are the kernels of the intertwining operators for the conformal group. The central issue discussed in this paper is the significance of the conformal inversion in defining the quantum field propagators. It will be found necessary to change the usual conformal inversion

$$R\, x_\mu = \frac{x_\mu}{x^2} \quad , \; \mu = 1,2,\ldots,D \tag{1.1}$$

slightly in that one must multiply it by a reflection from the orthogonal group in D space-time dimensions $I \varepsilon O(D)$ with $\det I = -1$. This modified conformal inversion

$$\mathcal{R} = R \cdot I \tag{1.2}$$

will be seen to generate the propagators in the conformal quantum field theory.

In this paper conformal invariance will be considered for Euclidean space-time R^D only, but there will be analogous results for the Minkowski case. Besides the physical dimension $D = 4$, other space-time dimensions ($D = 1,2,3$ or 6) may also be of interest. For this reason the discussion of the intertwining operators and the necessary representation theory of the conformal group will be given for all $D = 1,2,..$ There are, however, certain exceptions in the case $D = 1$, which will not be mentioned separately. Mathematically the modified conformal inversion $\mathcal{R}$ will be an element of the so-called Weyl-group[6] W associated to the conformal group

$$W = O(D)/SO(D) \tag{1.3}$$

which will be introduced in the next section. The Weyl-group (of the root space of a semi-simple Lie group) on the other hand plays also the central role in the theory of Zeta-functions on a homogeneous space. Hence, as will be seen, covariance with respect to the Weyl-group W determines the propagators gives rise to the symmetry[7]

$$d \leftrightarrow D-d \tag{1.4}$$

and is also responsible for the Zeta-relation of Zeta-functions. Just as in a Regge-pole theory, the symmetry $d \leftrightarrow D-d$ plays an important role in the partial wave analysis of n-point functions. Thus the conformal invariant QFT(interpreted as the Gell-Mann-Low limit of a massive theory)provides a good illustration of Gel'fand's statement on Zeta-functions. In (1.4) d is the dimension of a physical field. The symmetry (1.4) has been called "shadow-symmetry" by the authors of Ref.(8), who also show its relationship to the conformal inversion R. Furthermore, the conformal inversion R has been used by Schreier[9] to determine the 3-point function. The analysis of the conformal inversion given here explains why this is justified. We have used the mathematical literature on intertwining operators for semi-simple groups. More details, especially on analytic questions and for other groups, may be found in the work of Kunze and Stein[10], Knapp and Stein[11] and Konstant.[12]

II. WEYL GROUP

Now we will give a global parametrization of the identity component of the special orthogonal group $G = SO_e(D+1,1)$, which represents the conformal transformations of the Euclidean space R^D and discuss the Weyl group associated to G. The transformations of $SO_e(D+1,1)$ leave invariant the quadratic form

$$\zeta_o^2 + \zeta_1^2 + \ldots + \zeta_D^2 - \zeta_{D+1}^2 \tag{2.1}$$

There exist two basic decompositions of semi-simple Lie groups, the Iwasawa decomposition and the Bruhat decomposition. These two decompositions lead to a compact and a non-compact realization of space-time, respectively. The Iwasawa decomposition of $G = SO_e(D+1,1)$ is given by

$$G = K\,A\,N \tag{2.2}$$

where the maximal compact subgroup K is the special orthogonal group SO(D+1), the 1-dimensional abelian group A is the dilatation subgroup and the nilpotent group N consists of special conformal transformations. If G is parametrized by (D+2) x (D+2)-dimensional matrices, the abelian group A and the special conformal transformations N are parametrized explicitly by

$$A = \left|\begin{array}{c|c|c} \mathrm{ch}\,s & o\,..\,o & \mathrm{sh}\,s \\ \hline 0 & & 0 \\ \vdots & E & \vdots \\ 0 & & 0 \\ \hline \mathrm{sh}\,s & o\,..\,o & \mathrm{ch}\,s \end{array}\right| \qquad s \in R \tag{2.3}$$

and

$$N = \left|\begin{array}{c|c|c} \frac{1}{2}(2-|\vec{c}|^2) & \vec{c}^{\,t} & \frac{1}{2}|\vec{c}|^2 \\ \hline -\vec{c} & E & \vec{c} \\ \hline -\frac{1}{2}|\vec{c}|^2 & \vec{c}^{\,t} & \frac{1}{2}(2+|\vec{c}|^2) \end{array}\right| \tag{2.4}$$

while the compact subgroup is given by matrices of the following form

$$K = \left|\begin{array}{c|c} SO(D+1) & \begin{matrix} o \\ \vdots \\ o \end{matrix} \\ \hline o \ldots o & 1 \end{array}\right| \qquad (2.5)$$

The translation subgroup X of $SO_e(D+1,1)$, on the other hand, has the parametrization

$$X = \left|\begin{array}{c|c|c} \frac{1}{2}(2-|x|^2) & -\vec{x}^t & -\frac{1}{2}|x|^2 \\ \hline \vec{x} & E & \vec{x} \\ \hline \frac{1}{2}|x|^2 & \vec{x}^t & \frac{1}{2}(2+|x|^2) \end{array}\right| \qquad (2.6)$$

Hereby is E the DxD dimensional identity matrix, $\vec{x} \in R^D$, $\vec{c} \in R^D$ are D-dimensional Euclidean and $\vec{x}^t$, $\vec{c}^t$ their transposed vectors.

As mentioned in the introduction a two-element group associated to the conformal group will play a fundamental role. This group is called in the mathematical literature the Weyl group W[6] and is constructed in general for semisimple Lie groups as the quotient group $W = M'/M$ where M is the centralizer of A in K and M' is the normalizer of A in K. Hence in the case of the conformal groups

$$\begin{aligned} M &= \{k \in K \mid k\, a\, k^{-1} \doteq a, \text{ for all } a \in A\} = SO(D) \\ M' &= \{k \in K \mid k\, A\, k^{-1} \subset A\} = O(D) \end{aligned} \qquad (2.7)$$

and it follows that the Weyl group of $SO_e(D+1,1)$ consists only of two cosets

$$W = O(D) / SO(D) = \{\epsilon, \mathcal{R}\} \qquad (2.8)$$

where the identity coset ϵ consists of matrices of the following form

$$\epsilon = \left|\begin{array}{c|c|c} 1 & o..o & o \\ \hline \begin{matrix} o \\ \vdots \\ o \end{matrix} & m & \begin{matrix} o \\ \vdots \\ o \end{matrix} \\ \hline o & o..o & 1 \end{array}\right| \qquad m \in SO(D) \qquad (2.9)$$

and the $\mathcal{R}$-coset is given by

$$\mathcal{R} = \left|\begin{array}{c|c|c} -1 & 0..0 & 0 \\ \hline 0 & & 0 \\ \vdots & I & \vdots \\ 0 & & 0 \\ \hline 0 & 0..0 & 1 \end{array}\right| \tag{2.10}$$

where $I \in O(D)$, det $I = -1$. It can be seen that $\mathcal{R}$ is the product of the usual conformal inversion R with a space-time reflection I, where

$$R\ (\zeta_o, \zeta_1, \ldots, \zeta_{D+1}) = (-\zeta_o, \zeta_1, \ldots, \zeta_{D+1}) \tag{2.11}$$

which is equivalent to (1.1) and as usual $x_\mu = \frac{1}{\kappa}\zeta_\mu$, $\mu = 1,..,D$, $\kappa = \zeta_o + \zeta_{D+1}$, $x^2 = x_1^2 + \ldots + x_D^2$. Hence one obtains with $I \in O(D)$, det $I = -1$

$$\mathcal{R}\, x_\mu = \frac{I\, x_\mu}{x^2} \tag{2.12}$$

Later we will take as representative elements of $\mathcal{R}$

$$\mathcal{R} = R\ I_t \quad \text{if} \quad D \quad \text{is even} \tag{2.13a}$$

$$\mathcal{R} = R\ I_{st} \quad \text{if} \quad D \quad \text{is odd} \tag{2.13b}$$

and the time reflection I_t and space-time reflection I_{st} are simply given by

$$I_t(x_1, x_2, .., x_D) = (-x_1, x_2, .., x_D) \tag{2.14a}$$

$$I_{st}(x_1, x_2, .., x_D) = (-x_1, -x_2, .., -x_D) \tag{2.14b}$$

Note that the usual conformal inversion R is not an element of $SO_e(D+1,1)$ and therefore should frequently be substituted by the $\mathcal{R}$-inversion, which is an element of $SO_e(D+1,1)$. It is the $\mathcal{R}$-inversion, as the non-identity coset of the Weyl group, which fundamentally enters in the construction of the 2-point functions as well as in the analysis of the higher n-point functions in conformal invariant QFT. The symmetry (1.4) of the n-point functions and in Wilson expansions are a consequence of the $\mathcal{R}$-operation.

The conformally compactified Euclidean space-time is the D-dimensional sphere S^D, and is obtained as a homogeneous space of the conformal group with the use of the Iwasawa decomposition. It is the quotient space of $G = SO_e(D+1,1)$ with respect to the stability subgroup P at the zero vector of R^D

$$P = M\,A\,N \tag{2.15}$$

which is called the minimal parabolic subgroup of G.[6] Hence

$$G/P \simeq KAN/MAN \simeq SO(D+1)/SO(D) \simeq S^D \tag{2.16}$$

The representations and intertwining operators will be given later on the non-compact realization of space-time, the D-dimensional Euclidean space R^D. In order to get R^D as a homogeneous space of the conformal group, one can use the Bruhat decomposition of G[6]

$$G = PW\,P \equiv P\,\epsilon\,P + P\,\mathcal{R}\,P \tag{2.17}$$

which involves the Weyl group W. The decomposition (2.17) means that the space of double cosets P g P, $g \in G$, is in 1-1 correspondence with the elements of the Weyl group. Using the relations

$$\mathcal{R}^{-1}\,N\,\mathcal{R} = X \tag{2.18a}$$

$$\mathcal{R}^{-1}\,A\,\mathcal{R} = A \tag{2.18b}$$

$$\mathcal{R}^{-1}\,M\,\mathcal{R} = M \tag{2.18c}$$

and the definition (2.15), then the decomposition (2.17) can be rewritten in an equivalent form

$$G = P\,X + P\,\mathcal{R} \tag{2.19}$$

which shows that the Euclidean space $R^D \simeq X$ is obtained as a homogeneous space of G except that it has to be supplemented by points at infinity. Note that (2.18a) reflects the fact that the special conformal transformations N may be generated by the translations X and the $\mathcal{R}$ operation.

In order to do explicit computations it is more convenient to transform the metric (2.1) by

$$T = \left|\begin{array}{c|c|c} 1/\sqrt{2} & o..o & 1/\sqrt{2} \\ \hline o & & o \\ \vdots & E & \vdots \\ o & & o \\ \hline -1/\sqrt{2} & o..o & 1/\sqrt{2} \end{array}\right| \tag{2.20}$$

to the equivalent form

$$\zeta_1^2 + .. + \zeta_D^2 - 2\,\zeta_o \zeta_{D+1} \tag{2.21}$$

This changes the parametrization of the subgroups to

$$A = \left|\begin{array}{c|c|c} e^s & o..o & o \\ \hline o & & o \\ \vdots & E & \vdots \\ o & & o \\ \hline o & o..o & e^{-s} \end{array}\right| \qquad N = \left|\begin{array}{c|c|c} 1 & \sqrt{2}\ \vec{c}^{\,t} & |c|^2 \\ \hline o & & \\ \vdots & E & \sqrt{2}\ \vec{c} \\ o & & \\ \hline o & o..o & 1 \end{array}\right|$$

$$M = \left|\begin{array}{c|c|c} 1 & o..o & o \\ \hline o & & o \\ \vdots & U & \vdots \\ o & & o \\ \hline o & o..o & 1 \end{array}\right| \qquad X = \left|\begin{array}{c|c|c} 1 & o..o & o \\ \hline \sqrt{2}\ \vec{x} & E & o \\ & & \vdots \\ & & o \\ \hline |x|^2 & \sqrt{2}\vec{x}^{\,t} & 1 \end{array}\right| \tag{2.22}$$

This parametrization yields for the products

$$MAN = \left|\begin{array}{c|c|c} e^s & \sqrt{2}\ e^s\ \vec{c}^{\,t} & e^s\ |x|^2 \\ \hline o & & \\ \vdots & U & \sqrt{2}\ U\ \vec{c} \\ o & & \\ \hline o & o..o & e^{-s} \end{array}\right| \tag{2.23}$$

$$MAN\ X = \left|\begin{array}{c|c|c} e^s(1+2\ c\cdot x+|c|^2|x|^2) & \sqrt{2}\ e^s(\vec{c}^{\,t}+|c|^2\ \vec{x}^{\,t}) & e^s|c|^2 \\ \hline \sqrt{2}(U\vec{x}+|x|^2\ U\vec{c}) & U+2(U\vec{c})\cdot\vec{x}^{\,t} & \sqrt{2}\ U\ \vec{c} \\ \hline e^{-s}|x|^2 & \sqrt{2}\ e^{-s}\ \vec{x}^{\,t} & e^{-s} \end{array}\right| \tag{2.24}$$

and the two elements of the Weyl group are now given by

$$\epsilon = \left|\begin{array}{c|c|c} 1 & 0..0 & 0 \\ \hline 0 & & 0 \\ \vdots & m & \vdots \\ 0 & & 0 \\ \hline 0 & 0..0 & 1 \end{array}\right| \qquad \mathcal{R} = \left|\begin{array}{c|c|c} 0 & 0..0 & 1 \\ \hline 0 & & 0 \\ \vdots & I & \vdots \\ 0 & & 0 \\ \hline 1 & 0..0 & 0 \end{array}\right| \tag{2.25}$$

with $m \in SO(D)$, $I \in O(D)$, $\det I = -1$. The matrix representations (2.22)-(2.25) will be needed later for the computation of the conformal invariant 2-point functions.

III. REPRESENTATION THEORY OF THE CONFORMAL GROUP.

In this section the irreducible representations of $G = SO_e(D+1,1)$ will be constructed. They will be realized on certain function spaces over space-time R^D.

The intertwining operators for the conformal group G will also be given in the R^D formulation, which allows a direct comparison with the expression for the 2-point function used in QFT. On the whole, it has to be said, that the non-compact formulation of conformal invariance is a completely legitimate procedure. One should bear in mind, however, that the action of G on R^D will be undefined at certain points, and if this happens, the compactified version of conformal invariance allows for a more systematic analysis in this case. According to the general theory of induced representations, the representations of G can be induced from those of the stability subgroup MAN at the origin of R^D. Hence, let

$$\begin{aligned} m &\to \mathcal{D}^{\ell}(m) \equiv \mathcal{D}(m),\ m \in M = SO(D) \\ a &\to \mathcal{L}^{d}(a) \equiv \mathcal{L}(a),\ a \in A \end{aligned} \tag{3.1}$$

be irreducible representations of SO(D) and A respectively.

The tensor representations of SO(D) are then labeled by the weight vector ℓ

$$\begin{aligned} \ell &= (\ell_{\frac{1}{2}D}, \ldots, \ell_2, \ell_1) \text{ in the case D even} \\ \ell &= (\ell_{\frac{1}{2}(D-1)}, \ldots, \ell_2, \ell_1) \text{ in the case D odd} \end{aligned} \tag{3.2}$$

where the ℓ_i are integers restricted by

$$\begin{aligned} 0 < \ell_{\frac{1}{2}D} < \ldots < \ell_2 < \ell_1 \quad \text{for D even} \\ |\ell_{\frac{1}{2}(D-1)}| < \ldots < \ell_2 < \ell_1 \text{ for D odd} \end{aligned} \tag{3.3}$$

and the 1-dimensional representations of the dilatation group A are given by

$$\mathcal{L}^{d}(a) = e^{(d-D/2)\alpha \ln a} \equiv e^{(d-D/2)s} \tag{3.4}$$

where α maps the generator of a onto the real number 1 and d is a complex number. The representations of $G = SO_e(D+1,1)$ may be induced from representations of MAN in a standard way where N is represented by the identity representation. The unitary characters of A lie on the line Im d = D/2 parallel to the imaginary axis and induce the unitary principal series of G. Denoting the Hilbert space for the finite dimensional representations of M by H^{ℓ}, the principal series of representations of G may be represented in the Hilbert space $L^2(X,H^{\ell})$ of square-integrable functions over X with values in H^{ℓ}, and this Hilbert space may be identified with $L^2(R^D,H^{\ell})$. The matrix elements for other representations can be obtained by analytic continuation in the Casimir variable d.

Let the modular function $\delta_P \equiv \delta$ of P = MAN be defined by

$$\int_P dp f(pq^{-1}) = \delta(q) \int_P dp f(p) \tag{3.5}$$

where it can be shown[13] that $\delta(man)$ does not depend on $m \epsilon M$ and $n \epsilon N$ and that

$$\delta(man) = \delta(a) = e^{-D\alpha \ln a} \equiv e^{-Ds} \tag{3.6}$$

According to the construction procedure of induced representations the functions $f \in L^2(X,H^{\ell})$ obey the covariance property

$$f(man\ x) = \delta^{-1/2}(a)\ \mathcal{L}^d(a)\mathcal{D}^{\ell}(m)\ f(x) \tag{3.7}$$

$x \in X$, man $\in$ MAN; in particular they are homogeneous of degree d

$$f(\rho x) = \rho^d\ f(x) \qquad \rho > 0$$

and the representations $U^{\ell,d}$ of G induced from the representations $\mathcal{L}^d(a)\cdot \mathcal{D}^{\ell}(m)$ of MAN are then given by right multiplication

$$U^{\ell,d}(g)\ f(x) = \delta^{-1/2}(\Pi_P(xg))\ \mathcal{L}^d(\Pi_A(xg))\ \mathcal{D}^{\ell}(\Pi_M(xg))\ f(\Pi_X(xg)) \tag{3.8}$$

where $g \in G$, $x \in X$, $xg \in$ MAN X , $f \in L^2(X,H^{\ell})$ and the projections

$$\Pi_A : \text{MAN X} \rightarrow A$$
$$\Pi_X : \text{MAN X} \rightarrow X \qquad \text{etc.} \tag{3.9}$$

of xg ε MAN X will be computed below. In (3.8) the Bruhat decomposition (2.19) of g ε G will be used, i.e. almost all g ε G (except a set of Haar measure zero) may be uniquely decomposed into the product MAN X and for such a g ε G fixed $\Pi_X(xg)$ again belongs to X for all but one x ε X. Hence using for g ε G the parametrization (2.24) it follows that x•g has the following general form

$$x \cdot g = \left|\begin{array}{c|c|c} e^s \sigma(c,b) & \sqrt{2}\, e^s (\vec{c}^{\,t} + |c|^2 b^t) & e^s |c|^2 \\ \hline \begin{array}{l}\sqrt{2}\, e^s \sigma(c,b)\vec{x} \\ +\sqrt{2}(U\vec{b} + |b|^2 U\vec{c})\end{array} & \begin{array}{l} 2e^s\, \vec{x}(\vec{c}^{\,t} + |c|^2 \vec{b}^{\,t}) \\ +U+2(U\vec{c})\, \vec{b}^{\,t}\end{array} & \begin{array}{l}\sqrt{2}\, e^s |c|^2 \vec{x} \\ +\sqrt{2}\, U \vec{c}\end{array} \\ \hline \begin{array}{l} e^s |x|^2 \sigma(c,b) \\ +2x^t (U\vec{b} + |b|^2 U\vec{c}) \\ +e^{-s} |b|^2 \end{array} & \begin{array}{l} \sqrt{2}\, e^s |x|^2 (\vec{c}^{\,t} + |c|^2 \vec{b}^{\,t}) \\ +\sqrt{2}\, x^t (U + 2(U\vec{c})\, \vec{b}^{\,t}) \\ +\sqrt{2}\, e^{-s} \vec{b}^{\,t} \end{array} & e^{-s} \sigma(e^s x^t U, c) \end{array}\right| \tag{3.10}$$

where $\vec{x} \in R^D$ parametrises x $\in$ X, $\vec{b} \in R^D$ parametrises b $\in$ X $\subset$ MAN X and as usual

$$\sigma(b^t, c) = 1 + 2b \cdot c + |b|^2 \, |c|^2 \tag{3.11}$$

Comparing matrix elements of (3.10) with those of (2.24) yields

$$\Pi_A(xg) = \left|\begin{array}{c|c|c} e^s [\sigma(e^s x^t U, c)]^{-1} & o..o & o \\ \hline \begin{array}{c} o \\ \vdots \\ o \end{array} & E & \begin{array}{c} o \\ \vdots \\ o \end{array} \\ \hline o & o..o & e^{-s} \sigma(e^s x^t U, c) \end{array}\right| \tag{3.12}$$

and by the same method the transformed vector $\vec{x}' \varepsilon R^D$ of $x' = \Pi_X(x \cdot g)$ ε X is derived to be

$$\vec{x}' = \frac{e^{2s} |x|^2 (\vec{c} + |c|^2 \vec{b}) + e^s U^t \vec{x} + 2e^s (\vec{x}^{\,t} \cdot U\vec{c}) \cdot \vec{b} + \vec{b}}{\sigma(e^s x^t U, c)} \tag{3.13}$$

from which follows the conformal space-time transformation law.

$$\vec{x}' = \vec{b} + \frac{e^{s}\, U^{t}\vec{x} + e^{2s}|x|^{2}\,\vec{c}}{\sigma(e^{s}x^{t}U,c)} \tag{3.14}$$

The rotation $U' = \Pi_M(x\cdot g) \in SO(D)$, finally, is again computed by equating matrix elements (2.24) and (3.10)

$$\begin{aligned} U' + 2(U\vec{c})' \cdot (\vec{x}')^{t} &= 2e^{s}\vec{x}\,(\vec{c}^{\,t} + |c|^{2}\,\vec{b}^{\,t}) + U + 2(U\vec{c}) \cdot \vec{b}^{\,t} \\ (U\vec{c})' &= e^{s}|c|^{2}\vec{x} + U\vec{c} \end{aligned} \tag{3.15}$$

which determine the rotation

$$U'_{ij} = U_{ij} + 2e^{s}\, x_i c_j + 2[(Uc)_i + |c|^{2}\, e^{s}x_i]\,(b_j - x'_j) \tag{3.16}$$

where x'_j is given in (3.14) and $U'_{ij} \in SO(D)$ $i,j = 1,2,\ldots,D$ is a generalized Wigner rotation. Hence, identifying $L^2(X,H^{\ell})$ with $L^2(R^D,H^{\ell})$ the representations $U^{\ell,d}(g)$ of $SO_e(D+1,1)$ are given by

$$U^{\ell,d}(g)\, f(x) = e^{sd}\,[\sigma(e^{s}x^{t}U,c)]^{-d}\,\mathcal{D}^{\ell}(U')\, f(x') \tag{3.17}$$

where $\mathcal{D}^{\ell}(U')$ is a representation of $SO(D)$ and the transformed elements $U' \in SO(D)$, $\vec{x}' \in R^D$ may be taken from (3.14) and (3.16).

IV. INTERTWINING OPERATORS

Let us define representations $\mathcal{R}\chi = (\mathcal{R}\ell, \mathcal{R}d)$ by

$$\mathcal{D}^{\mathcal{R}\ell}(m) \equiv \mathcal{D}^{\ell}(\mathcal{R}^{-1} m \mathcal{R}) \qquad m \in M = SO(D)$$
$$\mathcal{L}^{\mathcal{R}d}(a) \equiv \mathcal{L}^{d}(\mathcal{R}^{-1} a \mathcal{R}) \qquad a \in A \tag{4.1}$$

Choosing for $\mathcal{R}$ the representative elements (2.13) one obtains if D is an odd integer for all $m \in M$,

$$\mathcal{R}^{-1} m \mathcal{R} = m \tag{4.2}$$

and this means that the representation $\mathcal{R}\ell$ is always equivalent to ℓ for D odd. If D is an even integer, on the other hand, it follows that

$$\mathcal{R}\ell = (-\ell_{\frac{1}{2}D}, +\ell_{\frac{1}{2}(D-2)}, \ldots, \ell_2, \ell_1) \tag{4.3}$$

and this implies that $\mathcal{R}\ell$ is equivalent to ℓ if and only of $\ell_{\frac{1}{2}D} = 0$. Secondly, observe that

$$\mathcal{R}^{-1} a \mathcal{R} = a^{-1} \tag{4.4}$$

which yields for the representation $\mathcal{L}^{\mathcal{R}d}$ of A

$$\mathcal{L}^{\mathcal{R}d}(a) = e^{-(d-D/2)s} = \mathcal{L}^{D-d}(a) \tag{4.5}$$

and hence

$$\mathcal{R}d = D-d \tag{4.6}$$

is the shadow dimension of d in the language of Ref. (8). Next, we will write down the intertwining operators Δ^{χ}, which are maps[11]

$$\Delta^{\chi}: C_c^{\infty}(R^D, H^{\ell}) \rightarrow C^{\infty}(R^D, H^{\ell}) \tag{4.7}$$

and which intertwine the two representations $U^{\ell,d}(g)$ and $U^{\ell,\mathcal{R}d}(g)$, e.g. they satisfy

$$\Delta^{\chi}\, U^{\ell,d}(g) \;=\; U^{\ell,\mathcal{R}d}(g)\; \Delta^{\chi} \tag{4.8}$$

Writing down the integral representation for

$$\Delta^{\chi}\; f(z) = \int_{R^D} d^D x\; \Delta^{\chi}(x)\; f(x\cdot z) \tag{4.9}$$

it will be demonstrated below that the kernel $\Delta^{\chi}(x)$ is the conformal invariant two-point function. In (4.9) $x\cdot z$ means multiplication in X, which is equivalent to $x+z$ in R^D. In the construction of the intertwining kernel $\Delta^{\chi}(x)$, $x \in R^D$ or $x \in X$ essential use is made of the $\mathcal{R}$-operation. The final expression for $\Delta^{\chi}(x)$, however, will not depend on any particular choice of the inversion I, where $\mathcal{R} = R\cdot I$, but only on the entire coset $\mathcal{R}$ of the Weyl group $\{\epsilon, R\}$. Hence the $\mathcal{R}$-dependence of $\Delta^{\chi}(x)$ has been dropped. The resulting expression for the intertwining kernel looks as follows

$$\Delta^{\chi}(x) = a(\chi)\;\; \delta^{-1/2}(\Pi_P(x\mathcal{R}))\mathcal{L}^{-1}(\Pi_A(x\mathcal{R}))\,\mathcal{D}(\mathcal{R})\mathcal{D}^{-1}(\Pi_M(x\mathcal{R})) \tag{4.10}$$

and will be computed explicitly with the parametrization introduced in (2.22)-(2.25). The normalization factor $a(\chi)$ is very important, it is related to the Plancherel measure of $SO_e(D+1,1)$. This will be discussed in Section V. Apart from the representations $\mathcal{L}^{-1} \equiv [\mathcal{L}^d]^{-1}$ of the dilatation group A and the representation $\mathcal{D}^{-1} \equiv [\mathcal{D}^{\ell}]^{-1}$ of SO(D) the factor $\mathcal{D}(\mathcal{R})$ also occurs in (4.10).

For arbitrary representations χ $\mathcal{D}(\mathcal{R})$ is not well defined, because $\mathcal{D} \equiv \mathcal{D}^{\ell}$ is a representation of SO(D) and $\mathcal{R} \in O(D)$ only. It is possible, however, to extend the representation $\mathcal{D}^{\ell}$ of SO(D) to a representation $\bar{\mathcal{D}}^{\ell}$ of O(D) on the same Hilbert space H^{ℓ} if and only if[14]

$$\mathcal{R}\,\ell \equiv \ell \tag{4.11}$$

If D is an even integer this condition restricts the representations χ which can be used for the intertwining kernel (4.10). In accordance with the remarks following Equ.(4.3), this implies that $\Delta^{\chi}(x)$ is defined only for representations

$$\chi = (\ell;d) = (\ell_1,\dots,\ell_{1/2D};\ d)\ \text{with}\ \ell_{1/2D} = 0,\ D\ \text{even} \tag{4.12}$$

and resembles the fact that unitary representations of the complementary series exist only for $\ell_{1/2D} = 0$. For example the complementary series of $SO_e(3,1)$ in our parametrization of the Casimir labels is given by

$$\chi = (o;d) \quad \text{with } 0 < d < 2 \tag{4.13a}$$

and the complementary series of $SO_e(5,1)$ lies in the intervals

$$\begin{aligned} &\chi = (o,o;d) \text{ with } o < d < 4 \\ &\chi = (\ell_1,o;d) \text{ with } 1 < d < 3,\ \ell_1 \neq 0 \end{aligned} \tag{4.13b}$$

etc., here d is a real number and there are two equivalent sets of unitary representations divided by the point $d = D/2$. For these representations of the complementary series the intertwining kernel (4.10) gives rise to a scalar product

$$(f,g)_\chi = \int_{R^D} d^Dx\ d^Dy\ \bar{f}(x)\ \Delta^\chi(x-y)\ g(y) \tag{4.14}$$

On the other hand there is no restriction on the representation χ in (4.10) for the groups $SO_e(D+1,1)$ with D an odd integer. In addition, there is a $\pm$ sign ambiguity (a signature factor) when extending a representation $\mathfrak{D}$ of SO(D) to a representation $\bar{\mathfrak{D}}$ of O(D),[14] we will take here the +sign by requiring positivity for the 2-point function.

It will now be demonstrated that the intertwining kernel (4.10) reduces to the usual 2-point function in conformal invariant QFT. In order to do this we will now compute $\Pi_A(x\mathcal{R})$ and $\Pi_M(x\mathcal{R})$. Using the matrix representations for $x \in X$ and $\mathcal{R}$ (2.22) and (2.25) one obtains for their product

$$x\,\mathcal{R} = \left|\begin{array}{c|c|c} o & o..o & 1 \\ \hline o & & \\ \vdots & I & \sqrt{2}\vec{x} \\ o & & \\ \hline 1 & \sqrt{2}\vec{x}^{\,t} I^t & |x|^2 \end{array}\right| \quad I \in O(D) \tag{4.15}$$

Recalling that

$$x\,\mathcal{R} \in P\,X \equiv MAN\,X$$

and the definition of the projections Π_A, Π_M

$$\Pi_A : MAN\,X \rightarrow A$$
$$\Pi_M : MAN\,X \rightarrow M \tag{4.16}$$

one obtains the relation

$$e^{-s} = |x|^2 \tag{4.17}$$

by comparing matrix elements between $x\,\mathcal{R}$ and MAN X in the parametrization (2.24). Hence

$$\Pi_A(x\,\mathcal{R}) = \left|\begin{array}{c|c|c} \frac{1}{|x|^2} & o..o & o \\ \hline o & & o \\ \vdots & E & \vdots \\ o & & o \\ \hline o & o..o & |x|^2 \end{array}\right| \tag{4.18}$$

and using the expression of the modular function (3.6) it follows that

$$\delta^{-1/2}(\Pi_P(x\mathcal{R})) = \delta^{-1/2}(\Pi_A(x\mathcal{R})) = |x|^{-D} \tag{4.19}$$

and similarly for the representation $\mathcal{L}$ of A Equ.(3.4)

$$\mathcal{L}^{-1}(\Pi_A(x\mathcal{R})) = |x|^{2(D/2-d)} \tag{4.20}$$

This shows that the propagator for scalar particles

$$\Delta^{\chi}(x) = a(\chi)(|x|^2)^{d-D} \qquad \chi = (\vec{o};d) \tag{4.21}$$

is obtained as the $\delta^{-1/2}\mathcal{L}^{-1}$ matrix element of $\Pi_A(x\mathcal{R})$, e.g. it arises by right multiplication with the conformal inversion $\mathcal{R}$ on $x \in X$. In the more general case when the representation ℓ of SO(D) is not the scalar representation

$\ell = \vec{0}$ one has to evaluate in addition the factor $\bar{D}(\mathcal{R})D^{-1}(\pi_M(x\mathcal{R}))$. Hence, comparing three matrix elements of (4.15) with those of MANX in (2.24)

$$\begin{aligned} U + 2(U\vec{c})\vec{z}^t &= I \\ (U\vec{c}) &= \vec{x} \\ \vec{z}^t &= \frac{1}{|x|^2}\vec{x}^t I^t \end{aligned} \tag{4.22}$$

where $x \in X$ and $\vec{z} \in R^D$ parametrizes X in MANX, $U = \pi_M(x\mathcal{R})$ can be computed by solving the equations (4.21)

$$U^{-1} = I^{-1}(E - \frac{2\vec{x}\,\vec{x}^t}{|x|^2}) \quad . \tag{4.23}$$

Hence

$$D^{-1}(\pi_M(x\mathcal{R})) = D(I^{-1}(E - \frac{2\vec{x}\,\vec{x}^t}{|x|^2})) \quad . \tag{4.24}$$

Note that

$$\det(E - 2\frac{\vec{x}\,\vec{x}^t}{|x|^2}) = -1$$

and therefore the expression $E - 2\vec{x}\vec{x}^t/|x|^2$ is only in SO(D) when multiplied by the reflection I^{-1}. Combining (4.21) with (4.24) then allows one to write the intertwining kernel (4.10) as

$$\begin{aligned} \Delta^{\chi}(x) &= a(\chi)\,\frac{\bar{\mathcal{D}}^{\ell}(\mathcal{R})\,\mathcal{D}^{\ell}(I^{-1}(E - \frac{2\vec{x}\,\vec{x}^t}{|x|^2}))}{|x|^{2(D-d)}} \\ &= a(\chi)\,\frac{\bar{\mathcal{D}}^{\ell}(E - \frac{2\vec{x}\,\vec{x}^t}{|x|^2})}{|x|^{2(D-d)}} \end{aligned} \tag{4.25}$$

where $\chi = (\ell;d)$ is a representation of SO(D+1,1) with the requirement that ℓ is equivalent to $\mathcal{R}\ell$ and $\bar{\mathcal{D}}^{\ell}$ is a representation of O(D) extended from the representation $\mathcal{D}^{\ell}$ of SO(D). This shows that the intertwining kernel (4.10) between the representations $U^{\ell,d}(g)$ and $U^{\ell,\mathcal{R}d}(g)$ of SO(5.1) is the analytically regularized propagator of Euclidean QFT and is postulated to be the 2-point function for interacting fields in conformally invariant QFT. In this case $\chi = (\ell_1,0;d)$ is a representation of $SO_e(5,1)$ and the spin label ℓ_1 takes on the values $\ell_1 = 0,1,2\ \ldots$, for example in the spin 1 case one has

$$\Delta^{\chi}(x) = a(\chi) \frac{\delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{|x|^2}}{|x|^{2(D-d)}} \tag{4.26}$$

where $\mu,\nu = 1,2,3,4$, $\ell_1 = 1$, and the higher spin two-point functions may be obtained by taking tensor products of $\delta_{\mu\nu} - 2\, x_\mu x_\nu / |x|^2$.

More generally, it is possible to construct intertwining operators between the representations $U^{\ell,d}(g)$ $U^{\mathcal{R}\ell,\mathcal{R}d}(g)$. Define

$$\Delta^{\chi} = \bar{\mathcal{D}}(R)\bar{\Delta}^{\chi} , \tag{4.27}$$

then it can be seen from (4.8) and (4.1) that

$$\begin{aligned} \bar{\Delta}^{\chi}\, U^{\ell,d}(g) &= \bar{\mathcal{D}}^{-1}(\mathcal{R})U^{\ell,\mathcal{R}d}(g)\,\bar{\mathcal{D}}\,(\mathcal{R})\,\bar{\Delta}^{\chi} \\ &= U^{\mathcal{R}\ell,\mathcal{R}d}(g)\bar{\Delta}^{\chi} \end{aligned} \tag{4.28}$$

hence $\bar{\Delta}^{\chi}$ intertwines $U^{\ell,d}(g)$ with $U^{\mathcal{R}\ell,\mathcal{R}d}(g)$, however in this case the $\bar{\Delta}^{\chi}$ operators are left with the I^{-1} dependence from (4.24).

It has been shown by Kunze and Stein[10] that an equivalent definition of the intertwining operators can be given by

$$\bar{\Delta}^{\chi} f(x) = a(\chi) \int_X f(z\, \mathcal{R}^{-1} x)dz \tag{4.29}$$

where f obeys the covariance condition (3.7) and

$$g(x) = \int_X f(z\, \mathcal{R}^{-1}x)dz \tag{4.30}$$

has the property

$$g(\,man\ x) = \delta^{-1/2}(a)\,\mathcal{L}^{\mathcal{R}d}(a)\mathcal{D}^{\mathcal{R}\ell}(m)\ \ g\,(x) \tag{4.31}$$

$man \in MAN$, and using (4.31) it can be immediately proved that $\bar{\Delta}^{\chi}$ obeys the intertwining relation (4.28). We sketch a proof that the integral representation (4.29) is equivalent to the representation (4.9) (4.10) in the appendix. The significance of the modified conformal inversion $\mathcal{R}$ for the definition of the propagators in quantum field theory is demonstrated very clearly in (4.29).

V. NORMALIZATION

It has been shown by Knapp and Stein[11] that there is a close relationship between the pole structure in d of the intertwining operators and the zeros of the Plancherel measure. Dropping for a moment the normalization factor $a(\chi)$ of Δ^{χ} then it can be seen from the defintion of the (unnormalized) intertwining operators $\Delta^{\chi} = a(\chi)\tilde{\Delta}^{\chi}$ that

$$\tilde{\Delta}^{\tilde{\chi}}\,\tilde{\Delta}^{\chi} = \frac{1}{c(\chi)}\, I \qquad (5.1)$$

in the domain of the unitary complementary series of representations. For example in the case of scalar particles $c^{-1}(\chi)$ will have (according to the analytic structure of $|x|^{2d}$ as a distribution in x) a double pole at $d = \frac{D}{2}$ and simple poles at $d = \frac{D}{2} \pm k > 0$ $k = 1,2,3\ldots$. In (5.1) I is the identity operator and $\Delta^{\mathcal{R}\chi} \equiv \Delta^{\mathcal{R}\chi}(\mathcal{R}^{-1})$, $\mathcal{R}\chi = (\mathcal{R}\ell, \mathcal{R}d)$. In order to have non-singular intertwining operators for all representations of the complementary series one can identify $c(\chi)$ with the Plancherel measure of $SO_e(D+1,1)$ and then one obtains the normalizing condition

$$a(\mathcal{R}\chi)a(\chi) = c(\chi) \qquad (5.2)$$

for the intertwining operators Δ^{χ} , which in turn satisfy

$$\Delta^{\mathcal{R}\chi}\,\Delta^{\chi} = I \quad . \qquad (5.3)$$

For even dimensional space-time D the Plancherel measure $c(\chi)$ of $SO_e(D+1,1)$ is given by (15)

$$c(\chi) = c^2 \prod_{1\leq s<r\leq \frac{D}{2}+1} (\rho_s^2 - \rho_r^2) \qquad (5.4)$$

where

$$\rho_n = \frac{D}{2} - n + \ell_n \quad , \quad n = 1,2,\ldots \frac{D}{2}$$

$$\rho_{\frac{D}{2}+1} = d - \frac{D}{2} \quad , \qquad (5.5)$$

$\chi = (\ell_1,\ldots, \ell_{\frac{D}{2}};d)$ is a representation of $SO_e(D+1,1)$ and c is a constant.

It can be seen that the zeros of $c(\chi)$ are distributed symmetrically around the axis $\mathrm{Re}\, d = \frac{D}{2}$, with a double zero at $d = \frac{D}{2}$ if and only if $\ell_{\frac{D}{2}} = 0$. The normlization factor $a(\chi)$ is then determined by the requirement that $a(\chi)$ contains exactly all the zeros which lie left to the $\mathrm{Re}\, d = \frac{D}{2}$ axis. This gives in the case $\ell_{\frac{D}{2}} = 0$

$$a(\chi) = c\ \{ \prod_{s=1}^{D/2} (\rho_s + \rho_{\frac{D}{2}+1})\}\{ \prod_{1\le s<r\le\frac{D}{2}} (\rho_s^2 - \rho_r^2)\ \}^{\frac{1}{2}} \tag{5.6}$$

and if in the case $\ell_{\frac{D}{2}} \neq 0$ $a(\chi)$ is required to be independent of the sign of $\ell_{\frac{D}{2}}$,

then this condition changes the factor $(\rho_{\frac{D}{2}} + \rho_{\frac{D}{2}+1})$ in (5.6) into

$$\{\rho^2_{\frac{D}{2}+1} - \rho^2_{\frac{D}{2}}\}^{1/2} = +\{(\ell_{\frac{D}{2}} + d - \tfrac{D}{2})(-\ell_{\frac{D}{2}} + d - \tfrac{D}{2})\}^{1/2} \tag{5.7}$$

which again reduces with the above choice of the sign of the square root to $(\rho_{\frac{D}{2}} + \rho_{\frac{D}{2}+1}) = d - \frac{D}{2}$ in the limit $\ell_{\frac{D}{2}} = 0$. Hence the normalization factor $a(\chi)$ for $SO_e(5,1)$ and $\chi = (\ell,\ 0;d)$ reads

$$a(\chi) = c\cdot(d - 1 + \ell)(d - 2)(\ell+1) = c\cdot(\ell+1)\ \frac{\Gamma(d+\ell)\ \ \Gamma(d-1)}{\Gamma(d-1+\ell)\Gamma(d-2)} \tag{5.8}$$

and the normalized inverse propagator for scalar fields is determined to be

$$\Delta^{\mathcal{R}\chi}(x) = c\ \frac{\Gamma(4-d)}{\Gamma(2-d)}\ \frac{1}{|x|^{2d}} \tag{5.9}$$

which is regular in the strip of the complementary series $0 < d < D$ and has singularities at the integer points $d = 4,5,\ldots$. The use of the normalizing factor $a(\chi)$ and its relation to the Plancherel measure (5.2) prevent singularities occurring at $d = 2$ and $d = 3$ in $\Delta^{\mathcal{R}\chi}(x)$. Choosing the constant $c = \frac{8^d}{(8\pi)^2}$, determines the value of $\Delta^{\mathcal{R}\chi}(x)$ to be

$$\Delta^{\mathcal{R}\chi}(x) = \frac{1}{4\pi^2|x|^2} \qquad d = 1$$

$$\Delta^{\mathcal{R}\chi}(x) = \delta^4(x) \qquad d = 2$$

$$\Delta^{\mathcal{R}\chi}(x) = -\Delta\ \delta^4(x) \qquad d = 3 \tag{5.10}$$

where Δ is the Laplace operator. Hence in contrast to the normalization used by Gel'fand and Shilov[16] $\Delta^{\mathcal{R}\chi}(x)$ is not singular at $d = 2,3$ and reduces to $\delta^4(x)$ at $d = 2$ as is required by group theory. It can be shown that Δ^χ is positive definite in the "critical strip" of the complementary series (4.13b), i.e. it maps positive functions $f \in C_c^\infty(R^D, H^\ell)$ into positive functions. At the limit points $d = 4$ for $\ell_1=\ell=0$ and $d = 3$ for $\ell_1 \neq 0$ Δ^χ is a positive semi-definite operator, i.e. it maps non-vanishing functions into the zero function. The positivity condition of Euclidean quantum field theory for the 2-point function in the massless case is here identical to the requirement that the corresponding quantum fields transform as representations of the complementary series of $SO(5,1)$, limit points are meant to be included. Furthermore, the length of the strip $d = 0$ to $d = 4$ seems to relate the renormalizability of Euclidean quantum fields $\phi^d(x)$, $d = 2,3,4$ to the existence of unitary representations of the conformal group $SO_e(5,1)$ at these points. It has been suggested in[17] that one can use non integer powers of fields in order to obtain summation formulas for Feynman graphs in perturbation theory. It can be shown that at the integer points $d = 4,5,\ldots$ there exist in the language of Gelfand, Graev and Vilenkin[18] operator irreducible representations which have however two invariant subspaces, one finite-dimensional the other infinite-dimensional. Moreover at these integer points the usual equivalence of the representations

$$\chi = (\ell,0;d) \quad \text{with} \quad \mathcal{R}\chi = (\ell,0;D-d) \tag{5.10}$$

breaks down. For all integers d, except those in the critical strip $d = 1,2,3$, the representations χ are only partially equivalent to $\mathcal{R}\chi$ and one has to form quotient spaces in order to recover equivalent representations. The representations on the boundary of the complementary series have to be treated as limiting cases, see (4.13b). The proof of these statements for the Lorentz group $SO_e(3,1)$ can be found in Ref.(18). From this group theoretical point of view the ultraviolet divergences for d outside the complementary series are a manifestation of the appearance of inequivalent representations at the integer points.

VI. ZETA RELATIONS

In this last section some remarks concerning the Zeta-function aspect will be given. As has been stated in the introduction, the conformal invariant n-point functions in Euclidean invariant QFT will possess the symmetry

$$d \leftrightarrow D - d \tag{6.1}$$

that is the partial wave amplitudes are symmetric with respect to the symmetry axis, which lies at

$$\mathrm{Re}\, d = \frac{D}{2} \tag{6.2}$$

e.g. half of the dimension of space-time. This symmetry is a Zeta-function symmetry in the sense that the Riemann Zeta-function for the real line R

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{6.3}$$

obeys the functional equation

$$\zeta(s) = \rho(s)\, \zeta(1 - s) \tag{6.4}$$

with

$$\rho(s) = \pi^{s-1/2} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma(\frac{s}{2})} \tag{6.5}$$

Hence the value of $\zeta(s)$ is "symmetric" with respect to the line $\mathrm{Re}\, s = \frac{1}{2}$, e.g. again given by (6.2) with $D = 1$. In order to see the analogy in the $D = 1$ dimensional case more clearly one has to go over to the corresponding "local" Zeta-functions, as will be done now. Consider the local Zeta-function on the real line

$$Z(f,s) = \int_{R^x} f(x) \frac{1}{|x|^{s-1}} \frac{dx}{|x|} \quad . \tag{6.6}$$

Here R^x is the multiplicative group of the real line $R^x = GL(1,R) = R \setminus \{o\}$ and $dx/|x|$ is the Haar measure on R^x. It can be seen that $Z(f,s)$ essentially coincides with the intertwining integral (4.9) and kernel (4.25) for $D = 1$ and if f is evaluated with z at the origin (e.g. $f(z\cdot x) = f(x)$) and furthermore $a(\chi) = 1$ and $s = 2-2d$ is taken. $Z(f,1-s)$ is the Mellin transform of f and we may suppose that f has compact support $f \in C_o^\infty(R^x)$.

Introducing the Fourier transform of f

$$\mathcal{F}(f(y)) \equiv \tilde{f}(y) = \int_{R^x} f(x)\, e^{2\pi i\, x\cdot y} dx \tag{6.7}$$

one may consider

$$Z(\tilde{f},\ 1-s) = \int_{R^x} \tilde{f}(x)\ |x|^s \frac{dx}{|x|} \tag{6.8}$$

and establish that the ratio $Z(f,s)/Z(\tilde{f},\ 1-s)$ does not depend on $f \in C_o^\infty(R^x)$ but only on s . Because of this independence the local Zeta-function may be computed for the special function

$$f(x) = e^{-\pi\, x^2} \tag{6.9a}$$

which is an eigenfunction of the Fourier transform and gives therefore

$$f(x) = \tilde{f}(x) \tag{6.9b}$$

Introducing the Gaußian (6.9) into the Zeta integral (6.6) and (6.8) allows one to establish the local Zeta relation for $f \in C_o^\infty(R^x)$

$$Z(f,s) = \rho(s)\ Z(\hat{f},\ 1-s) \tag{6.10}$$

where

$$\rho(s) = \left.\frac{Z(f,s)}{Z(\hat{f},\ 1-s)}\right| \text{ for } f(x) = e^{-\pi x^2} \tag{6.11}$$

is again given by the expression (6.5).

The relationship of the global Zeta-function $\zeta(s)$ to the local Zeta-function $Z(f,s)$ is explained by introducing the more general global Zeta-function $\zeta(f,s)$ and writing it as an infinite product of integrals

$$\zeta(f,s) = \prod_p \int_{Q_p^x} f_p(\lambda_p)|\lambda_p|_p^s\, d^x\lambda_p \tag{6.12}$$

where the product runs over all prime numbers $p = 2,3,5\ \ldots$ including $p = \infty$ and Q_p^x is the multiplicative group of the p- adic numbers with the measure $d^x\lambda_p$ and the p-adic norm $|\lambda_p|_p$. Formula (6.12) is a generalized Mellin transform and for more details the reader may consult the references[19].

Taking for f_p $p = 2,3,...$ generalized Gaußian's which are eigenfunctions of the Fourier operator, then expression (6.12) reduces to the Riemann Zeta-function $\zeta(s)$ in the product representation

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1} \tag{6.13}$$

where the product is taken over all prime numbers $p = 2,3,...$ On the other hand the Zeta-function $Z(f,s')$, $s' = 1-s$, is contained as a local factor of $\zeta(f,s)$; namely, in the limit $p \to \infty$ the p-adic numbers will go over into the real numbers

$$\lim_{p\to\infty} Q_p^x = R^x \tag{6.14}$$

hence $Z(f, 1-s)$ is just the local factor of (6.12) at $p = \infty$. This explains the relationship of the local Zeta-function $Z(f,s)$ to the global function $\zeta(s)$. The above discussion concerns the real line $R^x = GL(1,R)$ and its resulting Zeta relation (6.4) with symmetry axis $\text{Re } s = \frac{1}{2}$. This Zeta-function analysis can be generalized to higher dimensional groups.[19]

In the partial wave analysis of n-point functions of Euclidean conformal invariant QFT the symmetry axis $\text{Re } d = 2$ again occurs when the dimension of space-time is 4. This symmetry plays a very important role in the analysis of the short distance behaviour of amplitudes and in Wilson's operator product expensions. It is always needed in the transition from functions of the first kind to functions of the second kind (as used in Regge-pole theory). As explained here, this symmetry is generated by the conformal inversion $\mathcal{R}$, it manifests itself in the functional equations of Zeta-functions, and plays a fundamental role in defining the 2-point function of QFT.

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VIII. APPENDIX:

We sketch here a proof that the two expressions (4.10) and (4.29) for the intertwining operators of $SO_e(D+1,1)$ are equivalent. A more detailed proof can be found in Ref.(10). Extending the representations $\mathcal{L}(a)\mathcal{D}(m) \equiv V(p)$ of $P = MAN$ $p \in P$ to function on $P \cdot X$ in such a way that

$$V(p \cdot x) = V(p) \tag{A.1}$$

for all $x \in X$, choosing a positive function $h \in C_o^\infty(G)$ such that

$$\int_P h(pg)dp = 1 \tag{A.2}$$

for all $g \in G$, and the using the covariance property (3.7) and the integral identity

$$\int_G dg\ f(g) = \int_X dx \int_P dp\ f(pz)\ \delta(p) \tag{A.3}$$

yields the desired result

$$\overset{\gamma}{\Delta}{}^X\ f(z) = \int_X dx\ f(x\,\mathcal{R}^{-1} z)$$

$$= \int_X dx \int_P dp\ h(px)\ f(x\,\mathcal{R}^{-1}\ z)$$

$$= \int_X dx \int_P dp\ h(px)\ \delta^{1/2}(p) V^{-1}(p) f(px\,\mathcal{R}^{-1} z)$$

$$= \int_X dx \int_P dp\ h(px)\ \delta(p)\ \delta^{-1/2}(px) V^{-1}(px) f(px\,\mathcal{R}^{-1} z)$$

$$= \int_G dg\ h(g)\delta^{-1/2}(g) V^{-1}(g) f(g\,\mathcal{R}^{-1} z)$$

$$= \int_G dg\ h(g\mathcal{R})\delta^{-1/2}(g\mathcal{R}) V^{-1}(g\mathcal{R}) f(gz)$$

$$= \int_X dx \int_P dp\ h(px\mathcal{R})\ \delta(p)\ \delta^{-1/2}(px\mathcal{R}) V^{-1}(px\mathcal{R}) f(pxz)$$

$$= \int_X dx \int_P dp\ h(px\mathcal{R})\delta(p)\delta^{-1/2}(px\mathcal{R})V^{-1}(px\mathcal{R})\delta^{-1/2}(p)V(p)f(xz)$$

$$= \int_X dx \int_P dp\ h(px\mathcal{R})\delta^{-1/2}(x\mathcal{R})V^{-1}(x\mathcal{R})f(xz)$$

$$= \int_X dx\ \delta^{-1/2}(x\mathcal{R})V^{-1}(x\mathcal{R})f(xz) \qquad \text{(A.4)}$$

IX. REFERENCES AND FOOTNOTES:

(1) I. M. Gel'fand; Proc. Int. Congr. Math., Stockholm, p. 74 (1962)

(2) G. Mack; Renormalization and Invariance in Quantum Field Theory, Plenum Press, New York (to appear)

(3) W. Rühl; Commun, math. Phys. 30, 287 (1973) and 34, 149 (1973)

(4) S. Ferrara, R. Gatto and A. Grillo; Springer Tracts in Modern Phys., Vol. 67, Springer Verlag, Berlin (1973); Ann. Phys. (N.Y.) 76, 161 (1973)

(5) I. T. Todorov; Conformal Invariant Quantum Field Theory with Anomalous Dimensions, Cargése Lectures (1973)

(6) See for example G. Warner; Harmonic Analysis on Semi-Simple Lie Groups, Vol.I, II, Springer (1972)

(7) This symmetry is, of course, The Regge pole symmetry for D = 1 and the Lorentz pole symmetry for D = 2

(8) S. Ferrara, R. Gatto, A. F. Grillo and G. Parisi; Lett. Nuovo Cimento 4, 115 (1972)

(9) E. Schreier; Phys. Rev. D3, 980 (1971)

(10) R. A. Kunze and E. M. Stein; Amer. J. Math., 89, 385 (1967)

(11) A. W. Knapp and E. M. Stein; Ann. of Math., 93, 489 (1971)

(12) B. Kostant; Bull. Amer. Math. Soc., 75, 627 (1969)

(13) See ref.(6), Vol.I, p.446

(14) Weyl, H. The Classical groups, Princeton Univ. Press, Princeton (1939)

(15) T. Hirai, Proc. Japan Acad., 42, 323 (1965)

(16) I. M. Gelfand and G. E. Shilov, Generalized Functions, Vol.1, Academic Press (1964)

(17) K. Koller; Homogeneous Interactions and a Summation Method in Perturbation Theory, Karpacz Lectures (1973)

(18) I. M. Gelfand, M. I. Graev and N. Ya. Vilenkin; Generalized Functions Vol.5, Academic Press (1966)

(19) See for example: I. M. Gel'fand, M. I. Graev and I. I. Pyatetskii-Shapiro; Representation Theory and Automorphic Functions, W. B. Saunders Co. (1969); H. Jacquet and R. P. Langlands; Automorphic Forms on GL(2), Lecture notes in Math. Vol. 114, Springer Verlag (1970); A. Weil, Dirichlet Series and Automorphic Forms, Lecture notes in Math.Vol.189, Springer Verlag (1971).