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Definitions and Selected Applications of Feynman-Type Integrals

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Definitions and selected applications
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1. Examples and definitions

This appears to be a convenient time for a review of the work done on Feynman-type integrals, both of the rigorous work dealing with the foundations, and of the applications to specific problems. One may note that the rigorous work comprises so far only a few articles, and seems to be far away from what even the simpler of applications require. Still, the available results can furnish a first orientation into problems which may arise.

The integrals we have in mind are in particular those characterized by the weight factor e^{iA}, where A is the action of the physical system. This includes the path integrals for the (quantum) particles and the history integrals for fields. There are, besides, some problems which lead to integrals not of this form, for instance to a Gaussian integral with a complex variance. However, the subsequent discussion can be readily adapted to such Gaussian integrals.

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We will present several alternative definitions for Feynman-type integrals. None of the definitions can at present be regarded as superior over the others, and all may produce some useful tools. We ourselves, however, tend to favour the definition based on generalized measures. Before reviewing these definitions, we will give several forms of Feynman-type integrals, and will comment on the history of the subject.

A basic formula for the path integral is the following (where the $\underset{\sim}{x}_k \in R^n$, usually $n = 3$, the t_k refer to the time, and m below is the mass):

$$G(t_1, \underset{\sim}{x}_1 ; t_0, \underset{\sim}{x}_0) = N \int_{\eta(t_k) = \underset{\sim}{x}_k} \mathcal{D}(\eta) e^{iA(\eta)} . \tag{1}$$

Here G is the Green's function for the corresponding Schrödinger equation and A is the classical action, *e.g.*

$$A(\eta) = \int_{t_0}^{t_1} d\tau \left\{ \frac{1}{2} m \sum_{j=1}^{n} [\dot{\eta}^j(\tau)]^2 - V(\eta(\tau)) \right\}. \tag{2}$$

The functions η are real, the entity $\mathcal{D}(\cdot)$ is, heuristically, an infinite product of Lebesque measures, and is invariant under translation,

$$\mathcal{D}(\cdot) = \mathcal{D}(\cdot + \alpha) . \tag{3}$$

Finally, N in (1) is a normalizing factor that will depend on the definition of the integral.

We will write $\mathcal{D}(\cdot)$ quite generally in functional integrals, and two such entities occurring in different integrals need not be related in a simple way. If need be, one can write $\mathcal{D}_1$, $\mathcal{D}_2$, etc.

In (1) we may set $\underset{\sim}{x}_0 = 0$, $t_0 = 0$, and $m = 1$, and consider more general integrands, so as to have

$$\int_{\eta(0)=0} \mathcal{D}(\eta)\, e^{\frac{1}{2}i\langle\dot{\eta},\dot{\eta}\rangle} f_0(\eta). \tag{4}$$

The restriction $\eta(t_1)=\underset{\sim}{x}_1$ can come through a factor $\delta(\eta(t_1)-\underset{\sim}{x}_1)$ in f_0. A change of the variable function brings (4) to the form

$$\int \mathcal{D}(\xi)\, e^{\frac{1}{2}i\langle\xi,\xi\rangle} f(\xi). \tag{5}$$

Furthermore, several works recently discussed the following variant of (1),

$$G(t_1, \underset{\sim}{x}_1; t_0, \underset{\sim}{x}_0) = N \int_{\underset{\sim}{q}(t_k)=\underset{\sim}{x}_k} \mathcal{D}(\underset{\sim}{q})\mathcal{D}(\underset{\sim}{p})\, e^{iA(\underset{\sim}{p},\underset{\sim}{q})}, \tag{6a}$$

$$A(\underset{\sim}{p},\underset{\sim}{q}) = \int_{t_0}^{t_1} d\tau \left\{ \sum_{j=1}^{n} p^j \dot{q}^j - \left[\frac{1}{2m} \sum_{j=1}^{n} (p^j)^2 + V(\underset{\sim}{q}(\tau)) \right] \right\}. \tag{6b}$$

An integration over $\underset{\sim}{p}$ in (6a), carried out in accordance with the usual rules for Gaussian integrals, brings the integral to the form (6a). The variable $\underset{\sim}{q} = (q^j)$ in (6a) is analogous to η in (4) or in (1), while $\underset{\sim}{p}$ is analogous to ξ in (5).

In the case of quantized fields, one has the following formula for time-ordered functions,

$$\langle (F(\varphi))_+ \rangle_0 = \int \mathcal{D}(\gamma) e^{iA(\gamma)} F(\gamma) , \tag{7}$$

where we assumed a single scalar field with action A, and integration is over the histories of the field. The time interval is $(-\infty, \infty)$. The propagator of the theory can be obtained by taking $F(\varphi) = \varphi(x)\varphi(y)$. One can give, moreover, an expression which is intermediate between (1) and (7), namely a Green's functional,

$$G(t_1, \zeta_1; t_0, \zeta_0) = N \int_{\gamma(t_k, \underset{\sim}{x}) = \zeta_k(\underset{\sim}{x})} \mathcal{D}(\gamma) e^{iA(\gamma)} . \tag{8}$$

Here the time interval for integration is $[t_0 , t_1]$, and N is a normalizing factor.

With regard to the history, it seems fair to say that the path integral had its roots in a paper of Dirac of 1933 [1] who elaborated on the theme that "... it would seem desirable to take up the question of what corresponds in the quantum theory to the Lagrangian method of the classical theory."

Feynman took up the subject afterwards in his thesis, and in his well-known article of 1948 [2] . Shortly thereafter he applied these ideas in his basic works on quantum electrodynamics [3] . Around 1954 several other authors independently exploited such integrals for fields and constructed solutions, necessarily heuristic, to more general quantized fields with interaction [5 - 9] . These solutions have the general form of eq. (7).

The first attempt at a rigorous construction of the path integral was apparently that of Cameron of 1960 [10]. He observed that there is no measure (having the usual properties) for the path integral, even if the exponent in e^{iA} is modified so as to have a real part. He adapted therefore the approximation scheme which is familiar in case of the Wiener integral (cf. [38]), where the time interval $[0, t]$ is first broken into n parts by the separating points

$$0 = t_0 < t_1 < \dots < t_n = t. \tag{9}$$

The paths η are then assumed to be polygonal, with the successive corners at $(t_j, \eta(t_j))$. The functionals e^{iA} are evaluated for such polygonal paths, one integrates over the values $\eta(t_j)$, and passes to the limit $\max (t_j - t_{j-1}) \downarrow 0$. For path integral, the integrals over $\eta(t_j)$ must be defined with the aid of a convergence factor, e.g. by letting $m \to m + i\delta$, where $\delta \downarrow 0$ subsequently.

Cameron showed that the integrals evaluated as above converge to a path integral satisfying (1), if the potential V satisfies some rather strong conditions, which include in particular some analyticity.

We can point out two drawbacks to this approach. First, the method appears awkward in case of more intricate systems like fields. Second, the method does not seem to tie in with any convenient mathematical structure, and this would tend to handicap the development of such an integration theory.

Next, there are two methods introduced by Nelson [14]. The first amounts to interpreting the path integral in terms of Trotter's formula (valid under suitable conditions),

$$\exp\left(t(A+B)\right) = \lim_{n\to\infty} \left[\exp(tA/n)\exp(tB/n)\right]^n . \tag{10}$$

One may take $A = -iH^{(o)}$, where $H^{(o)}$ is the free Hamiltonian, and $B = -iV$. Then the successive approximations in (10) form a variant to the successive approximations which occur in Cameron's method, and the limit can be identified with the path integral.

The second method of Nelson depends on replacing m in the action A by im, so as to have a (measure-theoretic) Wiener integral. One then establishes analyticity in m, and continues from the positive-imaginary to the positive-real values.

The first of these methods has the drawback, that it does not define the integration of polynomials and of various other functionals. The second is applicable only to situations with extensive analyticity. However, both the analytically-continued form of history integrals, and the Trotter formula, have shown their usefulness in recent work on Euclidean field theory [43].

The last method which we discuss is that suggested by Itô [15]. We consider the path integral in the form (5), and we define the symbol $\mathcal{D}(\cdot)$ as a limit of measures. Explicitly: The form (5) presupposes a Hilbert space $\mathcal{H}$. Let $d\mu_{T,\alpha}$ be the Gaussian measure on $\mathcal{H}$ defined by the covariance operator T and the mean vector α. (Then T must be of trace class, symmetric, and strictly positive.) We set

$$I_{T,\alpha}(f) = \frac{1}{c_T} \int d\mu_{T,\alpha}(\xi)\, e^{\frac{1}{2}i\langle \xi, \xi\rangle} f(\xi), \qquad (11a)$$

where

$$c_T = \int d\mu_{T,0}(\xi)\, e^{\frac{1}{2}i\langle \xi, \xi\rangle}. \qquad (11b)$$

Then we define

$$\int \mathcal{D}(\xi)\, e^{\frac{1}{2}i\langle \xi, \xi\rangle} f(\xi) = \lim_{T\to\infty} I_{T,\alpha}(f), \qquad (12)$$

where the limit $T\to\infty$ must be suitably taken, and must be independent of α.

Itô established the convergence of the integrals in (12), and the validity of eq. (1), so far for a very limited class of functionals f, resp. potentials V.

This construction was adapted by the present author to the free scalar field [17]. Here the action can be put into the form

$$iA(\gamma) = \frac{1}{2}i\langle\gamma, B\gamma\rangle - \frac{1}{2}\langle\gamma, C\gamma\rangle, \tag{13}$$

where B and C are bounded symmetric operators on a suitable Hilbert space. The operator C is chosen so as to provide the increment in $m^2 \to m^2 - i\varepsilon$. Thus $C > 0$, $C \propto \varepsilon$.

It appeared in the above examples that the operators T occur in certain formulas in an awkward way, and that approximating sequences other than $\{d\mu_{T,\alpha}\}$ may be more useful. We were thus led to defining the generalized measures, quite generally, as suitable limits of measures [42].

Clearly, generalized measures can be introduced also for various measure-theoretic integrals. A measure may then factorize into a translationally-invariant generalized measure and a weight factor. There are examples where the two factors. i.e. the generalized measure and the weight, have quite different physical interpretations.

But let us return to the path integral. If a construction is like Itô's, so that one deals with the space of paths as a whole, we will say that we have the global or the Hilbert space approach. C. DeWitt also contributed to this approach in her construction of a path integral with the help of the theory of distributions [18].

For contrast, we should like to call an approach based on breaking up the time interval, the time-evolution approach. The construction of Cameron, and the one based on Trotter's formula, are of this kind. (The analytic continuation in m from the Wiener integral could be classified either way!) Other constructions based on the time-evolution approach can be found in [11 - 13, 16].

2. Some problems of integration.

We comment below on five types of problems. Only in the case of the first one are there rigorous results available for infinite-dimensional integrals.

(a) One basic problem is that of integrability. Examples of integrable functions (or functionals) can of course be found in the cited articles [10 - 18]. For the case of Itô's definition, the results are rather limited, and it appears reasonable to try to extend them by studying the finite-dimensional case. Let us consider for definiteness

$$I(f) = \frac{1}{(2\pi i)^{n/2}} \lim_{\varepsilon \downarrow 0} \int_{R^n} d^n u \, e^{\frac{1}{2}(i-\varepsilon)\langle u,u\rangle} f(u). \qquad (14)$$

C. DeWitt pointed out that the weight $e^{\frac{1}{2}i\langle u,u\rangle}$ is in L. Schwartz's spaces $\mathcal{O}_M$ and $\mathcal{O}_C'$, so that $I(f)$ is defined for $f \in \mathcal{O}_M'$ and $\in \mathcal{O}_C$ [18], the procedure with $\lim(\varepsilon \downarrow 0)$ being unnecessary here. The space $\mathcal{O}_C$ is that space of functions whose dual is $\mathcal{O}_C'$. Furthermore, the case $f \in L_1$ is trivial, and the case where f is the Fourier transform of a measure of bounded absolute variation was treated in [15]. Certain other classes of integrable functions are noted in a forthcoming paper [19].

It would also appear desirable to have some more general criteria for integrability, e.g. an adaptation of the bounded convergence theorem. We observe that this theorem cannot hold in the usual form, and it is instructive to look at two counterexamples.

For the first, we have the integrable function $f_o(u) = 2$, and the nonintegrable one $e^{-\frac{1}{2}i\langle u,u\rangle}$. The second example is that of a real, non-negative, non-decreasing function f on R^1, which is bounded by the integrable $f_o(u) = u^2$, and which we will define presently.

Consider the real part of $I(f)$, and the zeros of the weight $\cos(\frac{1}{2}u^2)$. They are at $|u|$ equal to

$$a_n = (3\pi + 4n\pi)^{\frac{1}{2}} \quad \text{and} \quad b_n = (\pi + 4n\pi)^{\frac{1}{2}}, \; n = 0, 1, 2, \ldots \tag{15}$$

At a_n the cosine becomes positive as $|u|$ increases, and at b_n, negative. Let

$$f(u) = 0 \quad \text{for} \quad u < a_o, \tag{16a}$$

$$= a_n^2 \quad \text{for} \quad a_n \leq u < a_{n+1}. \tag{16b}$$

The following estimate is adequate for us,

$$c_n \underset{\text{df}}{=} \int_{a_n}^{a_{n+1}} du \cos\left(\tfrac{1}{2}u^2\right) = a_n \left(\int_0^{\frac{b_{n+1}}{a_n}-1} dv + \int_{\frac{b_{n+1}}{a_n}-1}^{\frac{a_{n+1}}{a_n}-1} dv \right) \times \sin\left[\tfrac{1}{2}a_n^2(2v + v^2)\right] \geq (\text{const.})\, n^{-3/2} > 0, \tag{17}$$

for sufficiently large n. (In the first integral with respect to v the sine is positive, and in the second negative. If n is large then v remains small, and an expansion is effective.) Hence $\sum a_n^2 c_n$ diverges. The divergence as $\varepsilon \downarrow 0$ in (14) can now be easily established, and also follows from the results of [19].

It is clear that the function f can be modified to a C^∞, strictly increasing function, without changing the conclusion.

(b) In the case of infinite-dimensional integrals, a basic problem is that of the space of integration: On what space must f be defined, for the integral $I(f)$ to be meaningful? Alternatively, one may ask: What topology is to be imposed on the functions, when they are defined on a Hilbert space?

Of course, only experience can teach us what choices are convenient here. We should therefore like to present two illuminating examples. These examples in fact emphasize the close correspondence between the positive-definite Gaussian integrals and the Feynman-type Gaussian integrals, even though there is no countably-additive measure associated with the latter.

The first example is the functional form of the formula $[p,q] = i^{-1}$,

$$\lim_{\varepsilon \downarrow 0} \int \mathcal{D}(\gamma)\, e^{\frac{1}{2} i \langle \dot{\gamma}, \dot{\gamma} \rangle} e^{-i\int d\tau' V(\gamma(\tau'))} \times \left[\dot{\gamma}(\tau+\varepsilon)\gamma(\tau) - \gamma(\tau)\dot{\gamma}(\tau-\varepsilon) - i^{-1} \right] = 0. \quad (18)$$

See [27] for a heuristic derivation. (We assumed here a one-dimensional system with a continuous potential V and mass unity.) This equation harmonizes with the choice of the Hilbert space $\mathcal{H}$ determined by the norm $\langle \dot{\gamma}, \dot{\gamma} \rangle$ as the space of integration. Indeed, the elements of $\mathcal{H}$ have derivatives which are measurable but in general not continuous, while eq. (18) expresses a mean value (in some sense) of the discontinuity of the derivative in the sample paths.

On the other hand, Gaussian integrals lead to familiar integrability conditions, which require that various functionals be determined by Hilbert-Schmidt or by trace-class operators. We expect such conditions to arise also in the case of the Feynman-type integrals. We find, explicitly, for a symmetric operator B with discrete spectrum and eigenvalues $\lambda_j \geq 0$,

$$\int \mathcal{D}(\gamma)\, e^{\frac{1}{2}i\langle\dot\gamma,\dot\gamma\rangle}\, e^{\frac{1}{2}i\langle B\dot\gamma, B\dot\gamma\rangle} = \left[\prod_j \left(1+\lambda_j^{\,2}\right)\right]^{-\frac{1}{2}} . \tag{19}$$

The product converges if and only if B is Hilbert-Schmidt. This amounts to saying that the functional $e^{\frac{1}{2}i\langle B\,.\,.\rangle}$ must be extendible to a larger space, of the form $C^{-1}\mathcal{H}$ (where C is nonsingular and Hilbert-Schmidt), in order to be integrable. This is also equivalent to the condition that the functional $e^{\frac{1}{2}i\langle B\,.\,.\rangle}$ be continuous in the $\mathcal{J}$-topology of L. Gross [39].

(c) The problem of the space of integration appears in a different light when degenerate Lagrangians are considered. To illustrate, let us use the form (6) for the path integral, and let us choose the canonical variables P, Q in such a way that the Hamiltonian vanishes. Then

$$G\left(t_1, Q_1; t_0, Q_0\right) = N \int_{Q(t_j)=Q_j} \mathcal{D}(Q)\,\mathcal{D}(P)\, e^{i\langle P, \dot Q\rangle} . \tag{20}$$

Heuristically the P-integration yields $\delta(\dot Q)$, so that, with an appropriate normalization, one has

$$G\left(t_1, Q_1; t_0, Q_0\right) = \delta\left(Q_1 - Q_0\right) . \tag{21}$$

This result [24], which is expected for a theory where $H = 0$, shows an effective reduction of the space of integration.

Degenerate Lagrangians also occur in static models (where $m \to \infty$ and $H^{(c)} = 0$), in some formulations of spin [35], and in gauge theories. Further discussion can be found in [26].

(d) There has been some interest recently in the special problems which are brought about by curvilinear coordinates in Feynman-type integrals (e.g. [20, 25, 32]). We confine ourselves to recalling the close correspondence between the path and the Wiener integral, and the available extension of the latter to Riemannian manifolds [36].

(e) The asymptotic behavior of Feynman-type integrals was investigated by the saddle-point method, in several recent papers. However, with regard to rigorous results, it appears that only the case of one-dimensional integrals has been fully treated [41]. We confine ourselves to stating Kelvin's formula,

$$\int_a^b ds\, e^{iuf(s)} g(s) \approx e^{iuf(\bar{s})} g(\bar{s}) \left(\frac{2\pi}{|uf''(\bar{s})|}\right)^{\frac{1}{2}} e^{\pm i\pi/4}. \qquad (22)$$

Here one assumes that f has exactly one extremum on (a,b), namely at $\bar{s}$, that $f''(\bar{s})$ is defined and $\neq 0$, and that g is continuous at $\bar{s}$. The large parameter u is assumed real. The sign in $\pm i\pi/4$ agrees with $\pm uf''(\bar{s}) > 0$.

3. Applications.

Historically the most significant application of Feynman-type integrals has been to quantum electrodynamics [3]. There the integrals served to streamline the manipulations and therefore helped to clarify the nature of various difficulties like the divergences. These integrals continue to be exploited in quantum field theory, where they may facilitate in particular the isolation of dynamical components of gauge fields [27], and the investigation of partial sums of the perturbation series (e.g. [21]).

An (apparently) minor application of a foundational nature has been to nonrelativistic problems involving "bad" potentials, i.e. those where the sum $H^{(o)} + V$ is not densely defined, so that the Schrödinger equation does not provide a sufficient description

[13, 14, 16]. An alternative approach to such potentials is by way of quadratic forms [40].

We will now describe three other types of applications in a little more detail. (But we make no attempt to provide a complete survey of applications.)

(a) The asymptotic evaluations depend on adapting the formula (22) to function space integrals. The procedure is typically as follows. In the integrand one separates the factor $e^{iB(\eta)}$, where B depends linearly on the large parameter under consideration. Then the path $\bar{\eta}$ defined by

$$[\delta B/\delta\eta(\tau)]_{\eta=\bar{\eta}} = 0 \tag{23}$$

is expected to provide the dominant effect. One thus supposes that

$$\int \mathcal{D}(\eta)\, e^{iB(\eta)} C(\eta) \approx e^{iB(\bar{\eta})} C(\bar{\eta}) \int \mathcal{D}(\chi)\, e^{i\langle(\delta^2 B)_{\bar{\eta}}\chi,\chi\rangle}, \tag{24}$$

where the second functional derivative $(\delta^2 B)_{\bar{\eta}}$ at $\bar{\eta}$ is a kernel which defines an operator, and we denote the latter by the same symbol.

The last integral is Gaussian, and so can be done in closed form. Its value can be interpreted as the Jacobian resulting from a change of variable. Cf. $|uf''(\bar{s})|^{-\frac{1}{2}} e^{\pm i\pi/4}$ in (22).

A detailed presentation of asymptotic estimates can be found in [31]. A short but more mathematical discussion is in [30]. Here we confine ourselves to a few simple examples.

(i) Take for e^{iB} the contribution of the kinetic energy, and for C, that of the potential energy. One recovers the result of ordinary perturbation theory.

(ii) In case of a potential gV, with g being the large parameter, one cannot simply interchange the roles of e^{iB} and C in (i). However, a clever change of variables leads to meaningful results [31].

(iii) For the semiclassical approximation we take $B = A$ (the action) and $C = 1$. This approximation was first investigated in [4], and was applied in recent years by Pechukas [33] and others to problems in atomic collisions.

(iv) These methods have been applied to the classical problem of scattering of waves by a random medium [23]. One starts here by imbedding the problem of the reduced wave equation into the initial-value problem defined by a Schrödinger-like equation.

(b) Sometimes the usefulness of a perturbation expansion can be enhanced by rearranging the terms. A closed expression for the quantity of interest (e.g. for a Green's function) might thus serve as a useful guide for first making and then investigating the rearrangements. We give two examples from many-body theory.

(i) The perturbation expansion of a path integral depends on expanding $\exp(-i\int d\tau V)$, evaluating the successive integrals, or moments, which we denote by M_j, and summing. One may rearrange this sum by introducing the series of cumulants K_j, where

$$1+\sum_{j=1}^{\infty} M_j = \exp\left(\sum_{j=1}^{\infty} K_j\right), \tag{24a}$$

so that

$$K_1 = M_1 , \quad K_2 = M_2 - M_1^2, \text{ etc.} \tag{24b}$$

The article [28] points out some possible advantages of the cumulants over the moments. E.g., for certain problems the path integral yields the following estimate for an asymptotic region,

$$K_2 \approx K_3 \approx \ldots \approx 0. \qquad (25)$$

Moreover, the cumulants appear to give better numerical results (than the moments) for the tail of the density of states in disordered systems.

(ii) In another approximation scheme [22] one starts with the usual expression for Green's function in quantum field theory [eq. (7)],

$$G(x-y) = \int \mathcal{D}(\bar{\eta})\,\mathcal{D}(\eta)\, e^{iA}\, \eta(x)\, \bar{\eta}(y). \qquad (26)$$

The action A contains a quadratic part $A^{(o)}$ with a kernel function and an interaction A_I. The scheme depends on expanding e^{iA_I} and adjusting the kernel in $A^{(o)}$ so that G be determined by $A^{(o)}$ alone.

The resulting expansion has led to some useful results concerning the behavior of systems near their critical points.

(c) The last application is of an abstract character.

The problem of possible statistics of given particles has been treated by functional methods in two different ways (which we summarize below). In both of these one integrates over a multiply-connected space of functions (cf. also [34]). The basic premises in the two cases are rather different, but in each case one concludes that only bose and fermi statistics are possible.

(i) One way to proceed is to take as the configuration space for n identical particles the space $R^{3n} = (\underline{x}_1, \ldots, \underline{x}_n)$, with the elimination of the points where two of the vectors $\underline{x}_i$ coincide, i.e. where $\underline{x}_j = \underline{x}_k$ for some $j \neq k$. Then we form the space of equivalence classes, where

$$(\underline{x}_1, \ldots, \underline{x}_n) \equiv (\underline{x}_{\pi(1)}, \ldots, \underline{x}_{\pi(n)}) \text{ for some } \pi \in S_n. \qquad (27)$$

The resulting space $\bar{R}$ has S_n as its covering group.

The path-integral expression for the propagator G now includes contributions G_α from different homotopy classes. One may express G as follows,

$$G\left(t_1, \bar{X}_1; t_0, \bar{X}_0\right) = e^{i\delta} \sum_{\pi \in S_n} \chi(\pi)\, G_\pi\left(t_1, \ldots\right), \qquad (28)$$

where $X_k \in \bar{R}$, $e^{i\delta}$ is an irrelevant phase factor, and the $\chi(\pi)$ form a one-dimensional unitary representation of S_n. There are only two such representations, giving bosons and fermions.

(ii) Another way depends on assuming, for particles with spin, a history integral over a suitable space of functions with spin [35]. In the case of integral spin, and for a fixed time, such a formulation yields directly commutativity, hence symmetry under exchange. On the other hand, for half-odd-integer spin one is led to integrating over spaces of pairs $\{\eta, -\eta\}$ of functions, which transform as spinors. Such spaces are not simply connected. Then if one formulates the exchange of two particles in a natural way, an easy calculation (or a topological argument [37]) shows the following: The functional $F(\cdot)$ which corresponds to the two particles in question undergoes a change of sign.

Thus, in both (i) and (ii) parastatistics are ruled out.

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