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Equilibrium States of Infinite Quantum Systems

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Equilibrium States of Infinite Quantum Systems

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I. Introduction

The goal of Statistical Mechanics is to provide a theoretical explanation of the thermodynamic behavior of the macroscopic systems. By the thermodynamic behavior we mean, that the states of isolated systems, as time goes to infinity, approach the equilibrium states (consisting of one or few homogeneous phases), and that these equilibrium states may be specified by a small number of macroscopic parameters. Since the link between statistical theory and thermodynamics lies in the proper enumeration of the equilibrium states, the relevance of the classification of those, is evident.

The conventional approach has been to consider finitely extended systems calculate the equilibrium values of interesting physical quantities according to the ensemble theory, and subsequently take a thermodynamic limit, N, V, going to infinity, keeping N/V finite. V is the volume of the system, and N is the number of particles in the system. The reason for taking the thermodynamic limit is that only in that limit the intensive quantities have sharp values, with zero fluctuations around the ensemble average, and therefore we are justified in representing an actual physical system by an ensemble. Moreover, most of the phenomena which are characteristic for large systems, such as cooperative behavior for instance, can be given a presice mathematical description only in the limit of truly infinite systems.

As far as the equilibrium states are concerned we are in the following situation: in the case of finite systems the equilibrium states are given by stationary ensembles - ensemble density in the classical case and the density matrix in the quantum case.

According to the ergodic hypothesis, there is a unique extremal stationary ensemble of the isolated system, a microcanonical ensemble. If the system is not isolated, one may introduce other stationary ensembles (canonical, uniform) which however share the common feature that the density function, ρ , is a function of the energy only *), $\rho = \rho\,(H)$. In the thermodynamic limit the differences between various ensembles, which are due to the different boundary conditions, disappear.

*) We suppress here for simplicity the other constants of motion, which arise from universal conservation laws. Compare discussion of that point in (1).

Given any of the mentioned stationary ensembles, ρ, we have the means of calculating the expectation equilibrium values of the physical observables,

$$\langle A \rangle_{\varrho} = \mathrm{Tr}\,(\varrho A)$$

Taking a thermodynamic limit defines a positive, linear functional on the set of localized observables of the system (as we can expect that $\lim_{V \to \infty} \langle A \rangle_{\varrho_V}$ exists, only if A is localized in some finite region). By definition, this positive, linear functional, called a limit Gibbs' state, is an equilibrium state of an infinite system.

The necessity of taking always a thermodynamic limit makes attractive and useful the idea of dealing from the beginning with the infinitely extended systems. However, new technieques are needed for the description of a system with the infinite number of the degrees of freedom, and those are provided by the algebraic theory.

The main purpose of this investigation is to classify equilibrium states of infinite quantum systems, in the algebraic framework, without recourse to the finite systems. It is in the spirit of the algebraic approach to consider a simple model, associate an algebraic structure with it, and then to proceed to the abstraction, asking whether the found structure is the most general, or simplest possible or what additional conditions are required for that structure to be unique, or whether the model can be extracted from the general structure. We shall take this attitude here and concentrate on an example of an infinite ideal Fermi gas. Thanks to the simplicity of that example we may carry through our ideas in an exact way and we shall find the sufficient conditions for the thermodynamic equilibrium states of the system.

This condition - a certain stability property of the equilibrium state - replaces the traditional assumption of ergodicity of the system. We talk about the ergodicity of the system if there are no other <u>relevant</u> constants of motion besides energy and the other universal constants. Since a system of noninteracting particles is highly nonergodic, the example of the ideal gas illustrates particularly strongly the role of the stability condition in the determination of the equilibrium states. In the last section we shall discuss briefly the connection of our result with the Kubo-Martin-Schwinger condition for a general system.

In Section II we shall develop necessary algebraic methods, in Section III proceed to the description of an ideal Fermi gas in algebraic terms, in Section IV discuss our concept of stability, in Section V giving the classification of equilibrium states in our model.

II. Algebraic Description of Infinite Systems- Observables, States, Dynamics

1. The fundamental notions in the mathematical description of the physical systems are the observables and the states.

In the quantum mechanics of finite systems they are represented by self-adjoint operators in Hilbert space, and the states by vectors or more generally by the densitiy matrices in that Hilberg space. In the infinitely extended systems we identify the observables with the self-adjoint elements of a suitably chosen C^*-algebra, and the states with the positive linear functionals on the algebra. We would like to stress the importance and plausibility of this generalization.

Plausibility follows from the fact that in a finite system we may also consider as a relevant structure a C^*-algebra of bounded operators in a Hilbert space, $\mathcal{B}(\mathcal{H})$. We argue that most observables can be described as bounded operators (or bounded functions of unbounded operators) acting in $\mathcal{H}$, and therefore should be identified with self-adjoint elements of $\mathcal{B}(\mathcal{H})$.
As for the states, given either as vectors, $\psi \in \mathcal{H}$, or density matrices, ρ $(\rho > 0, \mathrm{Tr}\,\rho = 1)$ in , their role is to assign to every observable A of the system an expectation value, defining a positive, linear functional on $\mathcal{B}(\mathcal{H})$

$$\langle A \rangle_\psi \equiv \omega_\psi(A) = (\psi, A\psi) \tag{2.1}$$

$$\langle A \rangle_\varrho \equiv \omega_\varrho(A) = \mathrm{Tr}(\rho A) \tag{2.2}$$

The mathematical equivalence of the two descriptions follows, since given a state, ω, on a C^*-algebra, $\mathfrak{A}$, one can always construct a Hilbert space $\mathcal{H}_\omega$, and a representation of $\mathfrak{A}$ by bounded operators acting in $\mathcal{H}_\omega$,

$\pi_\omega(\mathfrak{A})$, with a cyclic vector *) $\Omega \in \mathcal{H}_\omega$, such that the expectation value of A in the state ω is given by $\omega(A) = (\Omega, \pi(A)\Omega)$.

In the finite case all physically meaningful representations (faithful) are determined by vector or density matrix states and are unitarily quasi-equivalent.

In the case of infinite systems the C^*-algebra of observables admit faithful inequivalent representations. It has been shown in particular by Takesaki (2) that the representations determined by equilibrium states at different temperatures are disjoint (inequivalent). In other words there does not exist an irreducible representation of $\mathfrak{A}$ in a Hilbert space, such that all states are either vector or density matrices in that space. Therefore in the case of infinite systems we have to use algebraic description because there is no preferred representation.

The C^*-algebras suitable for the description of physical systems have to satisfy certain general properties:
To each bounded volume $V \subset R^3$ there is associated a C^*-algebra $\mathfrak{A}(V)$, satisfying

1) causality : $V_1 \cap V_2 = \emptyset \rightarrow [\mathfrak{A}(V_1), \mathfrak{A}(V_2)] = 0$ (2.3)

2) isotony: $V_1 \subset V_2 \rightarrow \mathfrak{A}(V_1) \subset \mathfrak{A}(V_2)$ (2.4)

From 2) it follows that $\mathfrak{A}_L = \bigcup_{V \subset R^3} \mathfrak{A}(V)$ is an algebra of local observables.
The completion in norm $\mathfrak{A} = \overline{\mathfrak{A}_L}$ is then the C^*-algebra of all quasi-local observables.
For the material of this subsection we refer to (3), (4).

In the following subsections we are going to describe the algebraic methods and concepts necessary for the application to Statistical Mechanics. Most of the content can be found in standard monographs on C^*-algebras (5), (6), (7), or in

*) A vector $\Omega \in \mathcal{H}_\omega$ is called cyclic, if the set $\{\pi_\omega(A)\Omega;\ A \in \mathfrak{A}\}$ is dense in $\mathcal{H}_\omega$. A representation π_ω is then called cyclic.

a treatise on algebraic theory in statistical physics (8), (9).

2. Algebras

Definition 2.1 $\mathfrak{A}$ is an algebra if it is a linear space and the product between its elements is defined, so that $A B \in \mathfrak{A}$, if $A,B \in \mathfrak{A}$. The usual laws of addition and mültiplication hold here.

Let $\mathfrak{A}$ be an algebra. An involution is a mapping $\mathfrak{A} \to \mathfrak{A}$ such that

$$(A^*)^* = A, \qquad (\lambda A)^* = \bar{\lambda} A^*, \quad (AB)^* = B^* A^* \tag{2.5}$$

A^* is called adjoint of A. If $A^* = A$, it is called self-adjoint. If $AA^* = A^*A$, A is called a normal element.

An algebra $\mathfrak{A}$ with involution is a *-algebra.

An algebra $\mathfrak{A}$ is a normed algebra if for every element $A \in \mathfrak{A}$, there is defined a norm of A, $||A||$, satisfying the usual properties of the norm. If $||A^*|| = ||A||$, it is a *-normed algebra. If a *-normed algebra is complete with respect to its norm (every Cauchy sequence A_η in $\mathfrak{A}$, has a limit A in $\mathfrak{A}$), it forms then a Banach *-algebra.

Definition 2.2 A C^*-algebra is a Banach *-algebra, for which

$$|| A^* A || = ||A||^2 \tag{2.6}$$

3. States and Representations

Definition 2.3 Any numerical functional, ω, assigning a complex number, $\omega(A)$, to the algebraic element $A \in \mathfrak{A}$, will be called a state on $\mathfrak{A}$, if it satisfies

a) linearity: $\omega(\alpha A + \beta B) = \alpha\omega(A) + \beta\omega(B)$

b) positivity: $\omega(A^*A) \geq 0$, $A \in \mathfrak{A}$

c) normalization: $||\omega|| = 1 = \omega(I)$

Definition 2.4 A representation of $\mathfrak{A}$ is a homomorphism $\mathfrak{A} \to \pi(\mathfrak{A})$ of $\mathfrak{A}$ into the bounded operators acting in a Hilbert space . A representation is faithful if the mapping is isomorphic.

Two representations are equivalent if there is a unitary mapping U, from $\mathcal{H}_1$ to $\mathcal{H}_2$:

$$U\,\pi_1\,(\mathfrak{A})\,U^{-1} = \pi_2\,(\mathfrak{A}) \tag{2.7}$$

A given representation defines many states:
every vector $\psi \in \mathcal{H}$, gives rise to a vector state

$$\omega_\psi(A) = (\psi, \pi(A)\psi), \quad A \in \mathfrak{A}$$

every density matrix ρ on $\mathcal{H}$, defines a normal state

$$\omega_\rho(A) = \mathrm{Tr}(\rho A), \quad A \in \mathfrak{A}$$

If the representation is irreducible, states given by ω_ψ are pure states; a state is called pure if it is an extremal point in the set of all states on , i.e. it is not possible to decompose ω so that

$$\omega = \lambda\omega_1 + (1-\lambda)\omega_2, \quad 0 < \lambda < 1, \quad \omega_1 \neq \omega_2$$

Theorem 2.1 To every state ω on a C^*-algebra, , there corresponds a cyclic representation $\pi_\omega(\mathfrak{A})$, unique up to unitary equivalence,

$$\omega(A) = (\Omega, \pi_\omega(A)\Omega), \quad A \in \mathfrak{A} \tag{2.8}$$

A representation is called cyclic if there is a vector $\Omega \in \mathcal{H}_\omega$ such that $\{\pi_\omega(A)\Omega;\ A \in \mathfrak{A}\}$ is dense in $\mathcal{H}_\omega$.
This representation associated with a state ω is given by the so-called GNS construction (Gelfand-Neumark-Segal).

The outline of the construction is following:

Let $\mathcal{J} = \{A \in \mathfrak{A} : \omega(A^* A) = 0\}$. $\mathcal{J}$ is a left ideal of $\mathfrak{A}$ ($\mathcal{J} \subset \mathfrak{A}$; $X \in \mathcal{J}$, $A \in \mathfrak{A} \to AX \in \mathcal{J}$).

Consider a factor space $\mathfrak{A}/\mathcal{J}$ (equivalence classes modulo $\mathcal{J}$). If $\xi_A \in \mathfrak{A}/\mathcal{J}$ denotes the equivalence class of A, then $\xi_A = \xi_B$ if $A-B \in \mathcal{J}$.
This factor space with a positive definite scalar product

$$(\xi_A, \xi_B) = \omega(A^* B)$$

is a pre-Hilbert space. Completion of $\mathfrak{A}/\mathfrak{J}$ with respect to the norm induced by that scalar product is a Hilbert space, $\mathcal{H}_\omega$. One may check that the definition of the scalar product is independent of the choice of the representants A and B in the equivalence classes. The ideal $\mathfrak{J}$ is the zero element in the factor space $\mathfrak{A}/\mathfrak{J}$. We define a representation of $\mathfrak{A}$ in $\mathcal{H}_\omega$ by

$$\pi_\omega(A)\ \xi_B = \xi_{AB}, \quad A \in \mathfrak{A}.$$

One can verify that it is really a representation, i.e.

1) $\pi_\omega(A)$ is bounded
2) π_ω preserves the algebraic structure
3) $\pi_\omega(A^*) = \pi_\omega(A)^*$
4) An equivalence class of the identify, ξ_I, is a cyclic vector in $\mathcal{H}_\omega$

$$\pi_\omega(A)\ \xi_I = \xi_A, \quad A \in \mathfrak{A}$$

$$\{\pi_\omega(A)\ \xi_I;\ A \in \mathfrak{A}\} \quad \text{is dense in } \mathcal{H}_\omega \ .$$

In the definition of a C^*-algebra we have required that it be closed in norm. This means that using the norm, we introduce a metric in $\mathfrak{A}$, defining the distance by

$$d\ (A, B) = || A - B|| \ , \qquad\qquad A\ , B \in \mathfrak{A}$$

and the notion of convergence in this metric:

$$A_n \xrightarrow{n} A \quad \text{if} \quad \lim_{n\to\infty} || A_n - A|| = 0$$

The algebra is closed in norm if every Cauchy sequence ($||A_n - A_m|| < \varepsilon_N$ for all $n,m > N$ and $\varepsilon_N \to 0$) has a limit in $\mathfrak{A}$.
In the algebra of bounded operators in a Hilbert space, $\mathcal{B}(\mathcal{H})$, we may introduce also weak topology:

$$A_n \xrightarrow{w} A \quad \text{if} \quad \lim_{n\to\infty} (\psi\ , A_n\phi\ ) = (\ \psi\ , A\phi\ ), \psi\ , \phi \in \mathcal{H}$$

Definition 2.5 A weakly-closed -algebra of bounded operators is a von Neumann algebra.

We shall consider only algebras with the identity, I. Then:

The commutant of $\mathfrak{A}$ is a set $\mathfrak{A}'$

The bicommutant of $\mathfrak{A}$, $\mathfrak{A}''$, is defined as a commutant of a commutant. Von Neumann algebras satisfy $\mathfrak{A} = \mathfrak{A}''$.

The center of the algebra is a set of elements in the algebra commuting with every $A \in \mathfrak{A}$, i.e.

$$\mathfrak{z} = \mathfrak{A}'' \cap \mathfrak{A}' = \{ A \in \mathfrak{A} : CA = AC , \ C \in \mathfrak{A} \}$$

If a center of a von Neumann algebra is trivial, i.e. $\mathfrak{z} = \{ \lambda I \}$ we call that algebra a factor.

Let $\pi(\mathfrak{A})$ be a representation of $\mathfrak{A}$ in $\mathfrak{B}(\mathcal{H})$. $\pi(\mathfrak{A})$ is a normclosed subalgebra of $\mathfrak{B}(\mathcal{H})$. We can enlarge $\pi(\mathfrak{A})$ by adding all weak limit points of $\pi(\mathfrak{A})$:

$$\pi(\mathfrak{A}) \to \pi(\mathfrak{A})^{-} = \pi(\mathfrak{A})''$$

Definition 2.6 The representation $\pi(\ )$ is called irreducible if

$$\pi(\mathfrak{A})'' = \mathfrak{B}(\mathcal{H})$$

Theorem 2.2 The cyclic representation $\pi(\mathfrak{A})$ determined by a state ω is irreducible if and only if ω is pure.

Definition 2.7 $\pi(\mathfrak{A})$ is called a primary representation if $\pi(\mathfrak{A})''$ is a factor. A state ω which by the GNS construction gives rise to a factor representation is called primary.

The last general concept we want to describe is the notion of states normal with respect to each other (belonging to the same family). From every state, ω, we can construct a class of normal states of $\pi_\omega(\mathfrak{A})''$ i.e. states represented by the density matrices. They are the normclosure of the set of states $\{\omega_A , A \in \mathfrak{A}\}$, where

$$\omega_A(B) = \frac{\omega(ABA^*)}{\omega(AA^*)} , \qquad \omega(AA^*) > 0$$

4. Symmetry transformations

The symmetry transformations of space-time are represented in the algebraic frame as automorphism groups of the algebra. The most interesting for us are

Space translations:

$$R^3 \ni x \to \alpha_x : \quad A \to \alpha_x(A), \quad \alpha_x(\mathfrak{A}) = \mathfrak{A}$$

Gauge transformations:

$$\langle 0, 2\pi \rangle \ni \Theta \to \alpha_\Theta : \quad A \to \alpha_\Theta(A), \quad \alpha_\Theta(\mathfrak{A}) = \mathfrak{A}$$

Time evolution:

$$R^1 \ni t \to \alpha_t : \quad A \to \alpha_t(A), \quad \alpha_t(\mathfrak{A}) = \mathfrak{A}$$

which are strongly continuous, i.e. for instance for time evolution

$$\lim_{t \to 0} \|\alpha_t(A) - A\| = 0 \quad , \quad A \in \mathfrak{A} .$$

The algebra, $\mathfrak{A}$, of quasi-local observables will be in general noncommutative. However, on physical grounds we assume that the system $(\mathfrak{A}, \alpha_t)$ has an assymptotically abelian structure. This means that for any A, B $\in \mathfrak{A}$,

$$\lim_{t \to \infty} \| [\alpha_t(A), B] \| = 0 \tag{2.9}$$

The motivation behind this assumption is that the measurements of two non commuting local observables become more and more compatible, if they are performed at different times and the time difference tends to infinity.

The physically most interesting states are the states invariant under the transposed action of the automorphism groups. Without specifying the details, let α_g be an automorphism group of $\mathfrak{A}$, representing a symmetry group G of a physical system:

We say that a state, ω, is invariant under α_g if

$$\omega(\alpha_g(A)) = \omega(A) \quad , \quad A \in \mathfrak{A} \tag{2.10}$$

We say that a state, ω, is extremal invariant under α_g if it is an extremal point in the set of all α_g-invariant states.

The interesting characterization of the states is by their cluster property with respect to the group of automorphisms which act in an asymptotically abelian way:

1. Strong clustering

$$\lim_{|g|\to\infty} \left| \omega(\alpha_g(A)\, B - \omega(\alpha_g(A))\, \omega(B) \right| = 0 \tag{2.11}$$

2) Weak clustering

$$\lim_{n\to\infty} \frac{1}{G_n} \int d\, g\, (\quad \omega(\alpha_g(A)B) \quad - \quad \omega(\alpha_g(A)) \quad \omega(B) = 0 \tag{2.12}$$

The notion of weak clustering is defined only for amenable groups i.e. groups possesing invariant means. The previously mentioned primary states are strongly clustering with respect to any asymptotically abelian group, whereas the extremal invariance is equivalent to the weak clustering of the state. Compare (10).

III. The Infinite Free Fermi System

1. Algebra of Observables and Dynamics

An infinite system of non-interacting fermions can be idealized as a CAR - Canonical Anticommutation Relations - algebra, $\mathfrak{A}$. This algebra is defined as the uniform closure of the algebra generated by $a(f), a(f)^*$, where $a(f)$, $a(f)^*$, are the annihilators and creators for one particle with the wave function f. They are defined as mappings from the test functions space (space of all square integrable functions, $\mathcal{L}^2(R^3)$) to the operators $a(f)$,

$$\mathcal{L}^2(R^3) \ni f \rightarrow \qquad a(f) \quad ,$$

linear in f, and satisfying CAR

$$\begin{gathered} \{ a(f), a(g)\} = 0 = \{a(f)^*, a(g)^*) \\ \{ a(f)^*, a(g)\} = (f, g) \quad , \quad f, g \in \mathcal{L}^2(R^3) \end{gathered} \tag{3.1}$$

where $\{\ ,\ \}$ denotes the anticommutator, $(\ ,\ )$ is the scalar product in the test function space $\mathcal{L}^2(R^3)$,

$$(f(\underline{x}),\ g(x)) = \int f(\underline{x})^* g(x)\, d^3x$$

and $a(f) = \int f(\underline{x})\, a(\underline{x})\, d^3x$, $a(f)^* = \int f(\underline{x})^*\, a(x)^*\, d^3x$

are the smeared values of the local fields

$$\{a(\underline{x})^*,\ a(\underline{y})\} = \delta(\underline{x}-\underline{y}) \tag{3.2}$$

From the CAR it follows that $a(f)$ and $a(f)^*$ are bounded, in fact

$$\|a(f)\| = \|a(f)^*\| = \|f\| \tag{3.3}$$

where $\|f\|$ is the $\mathcal{L}^2$-norm of the test function f . The local algebras $\mathfrak{A}(V)$ are defined as the norm closures of the *-algebras generated by all $a(f)$, $a(f)^*$, with support (f) in V.

Because of the anticommutation relations the local algebras, $\mathfrak{A}(V)$, do not satisfy causality condition. Therefore $\mathfrak{A}$ is not an algebra of quasi-local observables. One can, however, define even and gauge-invariant subalgebras of $\mathfrak{A}$:

$\mathfrak{A}_e$ - subalgebra generated by even polynomials in a and a^*

$\mathfrak{A}_g$ - subalgebra generated by polynomials with equal number of a and a^* in each term.

We have the following relations among them

$$\mathfrak{A}_g \subset \mathfrak{A}_e \subset \mathfrak{A}$$

We can choose either $\mathfrak{A}_e$ or $\mathfrak{A}_g$ as the relevant algebra of observables. The time evolution is described by the action of α_t-automorphism on the generators $a(f)$:

$$\alpha_t\,(a(f)) = a(f_t) \tag{3.4}$$

where f_t is the solution of the Schrödinger equation for a free single particle, with the initial condition $f_0 = f$. In the momentum space we may write symbolically

$$\alpha_t\left(a^*(p)\right) = e^{i\varepsilon(p)t}\, a(p)^* \tag{3.5}$$

where $\varepsilon(p)$ is the energy of a single particle and $a(p)^*$ is the creator of a particle with momentum $\underline{p}$.

2. States

Since every element in $\mathcal{O}\!\mathcal{l}$ can be uniformely approximated by the algebraic expressions in terms of $a(f)$, $a(f)^*$, with $f \in \mathcal{L}^2(R^3)$ with compact support, a state, ω, on $\mathcal{O}\!\mathcal{l}$ due to linearity and continuity is uniquely determined by the expectation values of any finite products of a's and a^*'s. These are so-called n-point functions

$$\omega\left(a(f_n)^* \cdots a(g_m)\right) = W_{nm}(f_1 \ldots g_m) \tag{3.6}$$

or in terms of the ditributions

$$\omega\left(a^*(p_n) \ldots a(q_m)\right) = W_{nm}(p_1, \ldots q_m) \tag{3.7}$$

There is a special class of states deserving our attention - the quasifree states - which are defined as even states (vanishing on any odd element in $\mathcal{O}\!\mathcal{l}$), with vanishing truncated functions, W^T_{nm}, for all n, m, with $n + m > 2$. The truncated n-point functions (correlation functions) are defined by

$$W_{nm}(f_1 \ldots g_m) = \sum_{\pi} \sigma(\pi)\, W^T_{r_1 s_1}(f_{i_a} \cdots g_{j_d}) \ldots W^T_{r_k s_k}(f_{i_\cdot} \cdots g_{j_q}) \tag{3.8}$$

where the sum is taken over all partitions of $(f_1, \ldots, f_n, g_1, \ldots, g_m)$ into disjoint subsets, including the identity partition. $\sigma(\pi) = (-1)^k$, where k is the number of permutations needed to rearrange the elements $(f_1, \ldots, g_m)$ in the order in which they appear in the partitions.

W_{nm}- functions of a quasi-free state cann all be expressed in terms of 2-point functions and if in addition the state is gauge invariant *) then $W_{20} = W_{02} = 0$, and

---

*) A gauge invariant state is a state invariant under the action

of the group α_θ, $0 \leq \theta \leq 2\pi$, $\alpha_\theta\,(a(f)) = \exp(i\theta)\, a(f)$, $f \in \mathcal{L}^2(R^3)$,

$$\omega\,(\alpha_\theta\,(A)) = \omega\,(A)\ ,$$

It follows that the expectation values of any monomial $a_1^* \cdots a_n^* \cdots a_m$ for $n \neq m$, vanish.

$$W_{nm} = \det \left\{ W_{11} \ (f_i , g_j) \right\} \tag{3.9}$$

For a thorough discussion of a CAR-algebra and its states we refer to the thesis of R.T. Powers (11).

A homogeneous (translationally invariant) gauge invariant quasi-free state is given uniquely by its 2-point functions of the form

$$\omega \left(a(p)^* a(q) \right) = W(p, q) = \delta(p-q)\rho(p) \tag{3.10}$$

where ρ is any function satisfying $0 \leq \rho(p) \leq 1$.

Any such state is automatically stationary

$$i \frac{\partial}{\partial t} W(p, q, t) = \frac{p^2 - q^2}{2m} W(p, q, t) = 0 , \tag{3.10a}$$

and furthermore as shown by Dell'Antonio (12) it is primary.

If $\rho(\underline{p})$ is a Fermi distribution, this quasi-free state describes statistical equilibrium state of an infinite ideal Fermi gas. This poses the question why the other stationary, primary, quasi-free states, with different one-particle momentum distribution $\rho(p)$ are not equilibrium states.
It is well recognized fact in standard statistical mechanics and it is always pointed out, in somewhat vague words, that one should imagine adding of some impurities to the noninteracting system, to provide the mechanism for the adjustment of the moments of the particles to the equilibrium distribution. How this mechanism could be implemented in a precise way in the infinite systems was however not known.
We have found that Fermi distributions are singled out among all primary, stationary states by the requirement of stability under local perturbations of the dynamics (which we shall describe in the next section).
We propose thus the following definition

A state, ω, on a C^* - algebra, $\mathfrak{A}$, representing a physical system, is a statistical equilibrium state, if

1) ω is stationay, i.e. invariant under time evolution

$$\omega\ (\alpha_t(A)) = \omega\ (A), \qquad A \in \mathfrak{A}$$

2) ω is primary
3) ω is stable under local perturbations of the dynamics

We shall analyse the consequences of these requirements in section V. The next section will be devoted to the precise definition of local perturbations of the dynamics and the notion of stability.

IV. Stability

1. Perturbations of the dynamics

Consider a C^* -algebra, $\mathfrak{A}$, of bounded operators. Let α_t be a strongly continuous, one-parameter group of automorphisms of $\mathfrak{A}$. And let ω be an α_t -invariant state on $\mathfrak{A}$.
By the GNS prescription we construct a cyclic representation of $\mathfrak{A}$, $\pi\ (\mathfrak{A})$, with a cyclic vector Ω, such that

$$\omega\ (A) = (\ \Omega,\ \pi(A)\Omega\ )\ ,$$

Since ω is α_t -invariant, it follows that α_t can be unitarily implemented in the representation π :

$$\pi\ (\alpha_t(A)) = U_t\ \pi\ (A)\ U_t^* \tag{4.1}$$

with $U_t\Omega = \Omega$,
by defining the action of U_t on a dense set of vectors by

$$U_t\ \pi(A)\ \Omega = \pi\ (\alpha_t\ (A)\ )\ \Omega \tag{4.2}$$

Let us denote the infinitezimal generator of U_t by H :

$$H = (\ -i\ \frac{d}{dt}\ U_t\ )_{t=0}\ ; \qquad U_t = e^{iHt} \tag{4.3}$$

H is of course a global operator, not in general affiliated with $\mathfrak{A}$.
It is also not the physical energy operator, since the latter defined as a

space integral of the energy density

$$E = \int d^3x \varepsilon (\underline{x})$$

has infinite fluctuations which are subtracted out in the construction of H ($H = \int d^3x \, (\pi(\varepsilon(\underline{x}) - \pi(\varepsilon(\underline{x})'))$, see (13).

From now on to simplify the notation , we shall omit the representation symbol, π , writing simply A instead π (A) etc.

Let $h = h^* \in \mathfrak{A}$. For any such h, we define a perturbed automorphism of $\mathfrak{A}$, in the following way (14) :

$$\alpha_t^{(h)} (A) = \quad X_t^{(h)} \alpha_t (A) \quad X_t^{(h)*}, \tag{4.4}$$

where $X_t^{(h)}$ satisfies the following

1) $X_t^{(h)} \in \mathfrak{A}$

2) $$i \frac{d X_t^{(h)}}{d t} = X_t^{(h)} \alpha_t (h) \quad , \quad X_0^{(h)} = I \tag{4.5}$$

3) $$X_t^{(h)} = \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_n \cdots \int_0^{t_2} dt_1 \, \alpha_{t_1}(h) \cdots \alpha_{t_n}(h)$$

is a unique solution of 2) and satisfies the so-called cocycle property

4) $$X_{t+s}^{(h)} = X_t^{(h)} \alpha_t \left(X_s^{(h)} \right) \tag{4.6}$$

5) $$X_t^{(h)*} = \left(X_t^{(h)} \right)^{-1} \tag{4.7}$$

The infinitezimal generator H defined by

$$H' = -i \frac{d}{dt} \left(X_t^{(h)} e^{iHt} \right)_{t=0} = - X_t^{(h)} \alpha_t(h) e^{iHt} + X_t^{(h)} H e^{iHt}$$

is

$$H' = H - h \tag{4.8}$$

2. Perturbation of the state and the stability condition

We discuss now the effect of the above described local perturbations on the state of the system. We assume that the state ω , is not only invariant under α_t , but also primary. It implies in particular the vanishing of the temporal correlation functions for time going to infinity

$$| \omega(B \; \alpha_t(A)) \; - \; \omega(A) \; \omega(B) | \underset{t\to\infty}{\to} 0 \tag{4.9}$$

and, a forteriori

$$\lim_{T\to\infty} \frac{1}{2T} \int \{ \omega(B \, \alpha_t(A) - \omega(A) \; \omega(B) \} dt = 0 \tag{4.10}$$

This latter property, equivalent to the state being extremal invariant in time must be demanded on physical grounds, because it means that time averages of the localized observables

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \alpha_t(A) \; dt$$

have sharp values in the state ω .

From (4.9) follows the important property that any weak limit of $\alpha_t(A)$ for all $A \in \mathfrak{A}$, is a multiple of the identity, $\omega(A)I$.

This can be seen as follows

For ψ , ψ , any vectors in $\mathcal{H}$, A, B, C in $\mathfrak{A}$,

$$\lim_{|t|\to\infty} (\psi, \alpha_t(A)\, \psi) = \lim_{|t|\to\infty} (\Omega B, \alpha_t(A) C (\Omega) =$$

$$\lim_{|t|\to\infty} \omega(B \, \alpha_t(A) \, C) = \lim_{|t|\to\infty} \omega(\alpha_t(A) \, BC) = \omega(A) \, \omega(BC)$$

$$\text{i.e. } w \lim_{|t|\to\infty} (\alpha_t(A)) = \omega(A) \, I \tag{4.11}$$

We are prepared now to discuss the central idea in our approach, namely the stability of the state under local perturbations of the dynamics.

Let ω be an α_t - invariant state, $\omega(\alpha_t(A)) = \omega(A)$, and let $\alpha_t^{(\lambda h)}$ be an automorphism of $\mathfrak{A}$ obtained from α_t in the manner described in subsection 1. by the local perturbation $\lambda h (h \in \mathfrak{A}$, λ a real number, which we introduce in order to be able to control the strength of ther perturbation).

We say that ω is <u>stable</u> if for any $h = h^+$ in $\mathfrak{A}$, there exist an $\alpha_t^{(\lambda h)}$ - invariant state, $\omega^{(\lambda h)}$, such that $\omega^{(\lambda h)}$ is weakly differentiable with respect to λ, at the origin, with the derivative

$$\omega_1^{(h)} = \left. \frac{d \, \omega^{(\lambda h)}}{d \lambda} \right|_{\lambda = 0} \tag{4.12}$$

a normal form in the representation.

Let us compute $\omega_1^{(h)}$. By the invariance of the state $\omega^{(\lambda h)}$, we have

$$\frac{d}{dt}\,\omega^{(\lambda h)}\left(\alpha_t^{(\lambda h)}(A)\right) = \omega^{(\lambda h)}\left(\frac{d}{dt}\alpha_t^{(\lambda h)}(A)\right) = 0 \tag{4.13}$$

From the definition of $\alpha_t^{(\lambda h)}$ we find

$$i\frac{d}{dt}\,\alpha_t^{(\lambda h)}(A) = i\frac{d}{dt}\left(X_t^{(\lambda h)}\alpha_t(A)\,X_t^{(\lambda h)*}\right) =$$

$$= X_t^{(\lambda h)}\alpha_t(\lambda h A)X_t^{(\lambda h)*} + X_t^{(\lambda h)}\,i\,\frac{d\alpha_t(A)}{dt}\,X_t^{(\lambda h)*} - X_t^{(\lambda h)}\alpha_t(\lambda h A)\,X_t^{(\lambda h)*}$$

$$i\frac{d}{dt}\,\alpha_t^{(\lambda h)}(A)\Big|_{t=0} = \lambda\,[h, A] - [H, A] \tag{4.14}$$

Substituting into (4.13),

$$\omega^{(\lambda h)}\left(\lambda\,[h, A]\right) + \omega^{(\lambda h)}\left([H, A]\right) = 0$$

The first order perturbation in λ, gives then

$$\omega\left([h, A]\right) = \omega_1^{(h)}\left([H, A]\right) \tag{4.15}$$

We put now $A \to \alpha_t(A)$ and find

$$\omega_1^{(h)}\left([H, \alpha_t(A)]\right) = \omega\left([h, \alpha_t(A)]\right)$$

$$\omega_1^{(h)}\left(\frac{d}{dt}\alpha_t(A)\right) = i\,\omega\left([h, \alpha_t(A)]\right) \tag{4.16}$$

Integrating both sides of (4.16),

$$\int_T^0 \omega_1^{(h)}\left(d\,\alpha_t(A)\right) = i\int_T^0 \omega\left([h, \alpha_t(A)]\right)dt$$

and taking the limit $T \to \pm\infty$, leads to

$$\omega_1^{(h)}(A) - \lim_{T\to\pm\infty}\omega_1^{(h)}\left(\alpha_T(A)\right) = i\int_{\pm\infty}\omega\left([h, \alpha_t(A)]\right)dt$$

Due to the vanishing of the temporal correlation functions , we have found, that,(4.11) ,

$$\underset{|t|\to\infty}{\text{w-lim}} \; (\alpha_t(A)) = \omega(A)I.$$

On the other hand from the normalization of the state we find

$$\omega^{(\lambda h)}(I) = 1 = \omega(I) + \omega_1^{(h)}(I) \rightarrow \qquad \omega_1^{(h)}(I) = 0 \quad (4.17)$$

Thus ,

$$\lim_{T\to\pm\infty} \omega_1^{(h)}(\alpha_T(A)) = \omega_1^{(h)}(I)\omega(A) = 0 \qquad (4.18)$$

The first order correction to the state is then given , for any $A \in \mathfrak{A}$, by the formula

$$\omega_1^{(h)}(A) = \int_{\pm\infty}^{0} \omega([h, \alpha_t(A)])\,dt \qquad (4.19).$$

The two alternative expressions for $\omega_1^{(h)}$ give us the stability condition for the unperturbed state ω :

$$\int_{-\infty}^{+\infty} \omega([h, \alpha_t(A)])\,dt = 0 \qquad (4.20).$$

V. Classification of Equilibrium States for Free Fermi System

In this section we discuss the implications of the stability condition (4.20) for the infinite system of non-interacting Fermi particles.

We put

$$h = a^*(g)a(g'), \quad A = a^*(f_1)\cdots a^*(f_n)a(f'_m)\cdots a(f'_1), \quad n+m \text{ even} \qquad (5.1)$$

We shall use the momentum representation , so that

$$a^*(g) = \int a^*(q)\,g(q)^*\,d^3q \qquad \text{etc.}$$

To simplify the calculation, we shift in (4.20) the action of α_t from A to h , what is allowed, because ω is α_t-invariant.

The time translate of h is given by

$$\alpha_t(h) = a^*(g_t)\,a(g'_t) .$$

Working out the commutator $[\alpha_t(h), A]$ gives

$$[a^*(g_t)a(g_t'),\ a^*(f_1)\cdots a^*(f_n);a(f_m')\cdots a(f_1')] =$$

$$\sum_{k=1}^{n}(-1)^{k-1}(f_k, g_t')\prod_{\substack{i,j\\ i\neq k}} a^*(g_t)a^*(f_i)a(f_j') +$$

$$\sum_{l=1}^{m}(-1)^{l}(g_t, f_l')\prod_{\substack{i,j\\ j\neq l}} a^*(f_i)a(f_j')a(g_t') \qquad (5.2)$$

The stability condition then takes the form :

$$\int_{-\infty}^{\infty} dt \int d^3q \int d^3q'\, e^{i(\varepsilon(q)-\varepsilon(q'))t} \times$$

$$\times\left\{\sum_{k=1}^{n}(-1)^{k-1} f_k^*(q')\,g'(q')\,g^*(q)\prod_{\substack{i,j\\ i\neq k}}\omega\left(a^*(q)a^*(f_i)a(f_j')\right)+\right.$$

$$\left.\sum_{l=1}^{m}(-1)^{l} g^*(q)\,f_l'(q)\,g'(q')\prod_{\substack{i,j\\ j\neq l}}\omega\left(a^*(f_i)a(f_j')a(q')\right)\right\} = 0 \qquad (5.3)$$

The time integration may be performed, yielding $\delta(\varepsilon(q)-\varepsilon(q'))$.
Let us choose

$$\mathrm{Supp}\,(g') \wedge \left(\bigcup_{k=2}^{n} \mathrm{Supp}\,(f_k)\right) = \emptyset \qquad (5.4)$$

$$\mathrm{Supp}\,(g) \wedge \left(\bigcup_{k=1}^{m} \mathrm{Supp}\,(f_k')\right) = \emptyset \qquad (5.5),$$

Supp(g) means: support of the function g ; and introduce the abbreviation

$$F(q) = g(q)\int g'^*(q')f_1(q')\delta(\varepsilon(q')-\varepsilon(q))d^3q' \qquad (5.6)$$

Then the relation (5.3) becomes

$$\omega\left(a^*(F)a^*(f_2)\ldots a(f_1')\right) = 0 \qquad (5.7)$$

which holds under the conditions (5.4) and (5.5).

However we see that the conditions (5.4 - 5.6) impose no other restrictions on the functions F, $f_2, \ldots, f_n, f_1', \ldots, f_m'$, besides the following

$$\mathrm{Supp}\,(F) \cap \{0\} = \emptyset$$
$$\mathrm{Supp}\,(F) \cap \left(\bigcup_{k=1}^{m} \mathrm{Supp}\,(f_k') \right) = \emptyset$$

and
$$\left(\bigcup_{k=2}^{n} \mathrm{Supp}\,(f_k) \right)^c \cap S_R \neq \emptyset \tag{5.8}$$

where S_R denotes the surface of a sphere around the origin with radius R, $\min_{q \in Supp(F)} |q| \leq R \leq \max_{q \in Supp(F)} |q|$. For, under these circumstances,

$$\varphi(q) = \int \delta(\varepsilon(q) - \varepsilon(q'))\, g'^{*}(q') f_1(q')\, d^3q' \neq 0 \text{ in } \mathrm{Supp}\,(F)$$

for some allowed choice of g', f_1, and therefore $g = \varphi^{-1} F$, exists and satisfies (5.5). The last condition (5.8) is not really restrictive because we can always choose the supports of the functions f_i small enough.
This means that

$$W_{nm}(f_1, \ldots, f_n; f_1', \ldots, f_m') = W_{nm}(F, f_2, \ldots, f_n; f_1', \ldots, f_m') = 0$$

whenever

$$\mathrm{Supp}\,(f_1) \cap \left(\bigcup_{k=1}^{m} \mathrm{Supp}\,(f_k') \right) = \emptyset \;, \quad \mathrm{Supp}\,(f_1) \cap \{0\} = \emptyset \text{ and}$$

supports of the functions f_k, $k=2,3,\ldots,n$, are small enough.
In other words, the distributions

$$W_{nm}(\underline{k}_1, \ldots, \underline{k}_n; \underline{k}_1', \ldots, \underline{k}_m') = \omega\,(a^*(\underline{k}_1) \ldots a(k_1')) \text{ must have}$$

point supports in at least one of the momentum variables $\underline{k}_1$ or $(\underline{k}_1 - \underline{k}_{p_i}')$, where $\underline{k}_{p_i}$ is any of $\underline{k}_1, \ldots, \underline{k}_n$.
Since the index 1 was chosen arbitrary, we may repeat the same argument for all indices $2,3,\ldots,n$, concluding that W_{nm} must be of the form

$$W_{nm}(\underline{k}_1, \ldots, \underline{k}_n; \underline{k}_1', \ldots, \underline{k}_m') = \prod_{i,j} (-1)^P \delta\,(\underline{k}_i - \underline{k}_{p_i}')\, \delta(\underline{k}_j) F(\underline{k}_3, \ldots, \underline{k}_n).$$

However we will show that actually ω has to be gauge-invariant, i.e. $W_{nm} = 0$ for $n \neq m$.
Consider for instance $W_{20}(f_1, f_2) = \omega\,(a^*(f_1) a^*(f_2))$.
Since ω is bounded we must have

$$|\,\omega\,(a^*(f_1) a^*(f_2))\,| \leq ||f_1|| \cdot ||f_2|| \text{ for any } f_1, f_2 \in \mathcal{L}^2 .$$

Let us compute $\omega\,(a^*(f_1) a^*(f_2))$:

$$\omega\,(a^*(f_1) a^*(f_2)) = \int d^3k_1 \int d^3k_2\, f_1^*(\underline{k}_1) f_2^*(\underline{k}_2) W_{20}(\underline{k}_1, \underline{k}_2)$$

but if $W_{20}(\underline{k}_1, \underline{k}_2) = \delta(\underline{k}_1)\,\delta(\underline{k}_2)F(\underline{k}_1,\underline{k}_2)$, then

$$\omega\ (a^*(f_1)\,a^*(f_2)) = f_1^*(0)\, f_2^*(0)F(0,0),$$

which clearly is not bounded for all f_1, $f_2 \in \mathcal{L}^2(R^3)$. Therefore

$$W_{20}(f_1, f_2) = 0 \tag{5.9}$$

(same argument may be applied to any non-gauge invariant expectation value).
Let us summarize: an α_t-invariant state ω, stable under local perturbations must be

1) gauge invariant
2) $$W_{2n}(\underline{k}_1,\ldots,\underline{k}_n;\underline{k}'_1,\ldots,\underline{k}'_n) = \sum_P (-1)^P \delta(\underline{k}_1-\underline{k}'_{j_1})\ldots\delta(\underline{k}_n-\underline{k}'_{j_n})F(\underline{k}_1,\ldots,\underline{k}_n) \tag{5.10}$$

The same formula holds for the truncated functions. Thus for $n > 1$, W^T_{2n} must have point support in more than one momentum variable $(\underline{k}_i-\underline{k}'_{j_i})$. However in that case for some pair of indices (i,j) W^T_{2n} (time dependent) will not vanish as $(t_i - t_j) \to \infty$, contradicting the primariness of the state.
To see this, consider for instance the 4-point function.

In $$W^T_4(\underline{k}_1,\underline{k}_2;\underline{k}'_2,\underline{k}'_1) = \{\delta(\underline{k}_1-\underline{k}'_1)\delta(\underline{k}_2-\underline{k}'_2) - \delta(\underline{k}_1-\underline{k}'_2)\delta(\underline{k}_2-\underline{k}'_1)\}\times$$

$$\times\{F(\underline{k}_1,\underline{k}_2) - F(\underline{k}_1)F(\underline{k}_2)\} = \omega^T(a^*(\underline{k}_1)\,a^*(\underline{k}_2)\,a(\underline{k}'_2)\,a(\underline{k}'_1))$$

let us take the time translates $e^{i\varepsilon(k_1)t_1}a^*(\underline{k}_1)$, $e^{-i\varepsilon(k'_1)t'_1}a(\underline{k}'_1)$, with $(t_1 - t'_1)$ going to infinity and the other factors kept constant. Due to the presence of $\delta(\underline{k}_1-\underline{k}'_1)\,\delta(\underline{k}_2-\underline{k}'_2)$, the first term is time translationally invariant and the second will vanish, when smeared out with the smooth test functions, so the whole expression will vanish.
Therefore W^T_{2n} must be identically zero for $n > 1$ i.e.

3) $F(\underline{k}_1,\ldots,\underline{k}_n) = \prod_{i=1}^{n} F(\underline{k}_i)$, that is ω must be a gauge invariant, quasi-free state. It follows that ω is uniquely determined by its 2-point function

$$W(\underline{p},\underline{q}) = \omega\ (a^*(\underline{p})\,a(\underline{q})) = \delta(\underline{p}-\underline{q})\rho(\underline{p}) \tag{5.11}$$

where $\rho(\underline{p})$ is a positive function(because the state has to satisfy positivity condition).
$\rho(\underline{p})$ is the one-particle momentum distribution. In order to investigate the form of $\rho(\underline{p})$ we put in (4.20)

$$A = \int A(\underline{k}_1, \underline{k}_2;\underline{k}'_2, \underline{k}'_1)\,a^*(\underline{k}_1)\,a^*(\underline{k}_2)\,a(\underline{k}'_2)\,a(\underline{k}'_1)\,d^3k_1 d^3k_2 d^3k'_2 d^3k'_1$$

$$h = \int h\,(\underline{q}_1,\underline{q}_2,\underline{p}_2,\underline{p}_1)\,a^*(\underline{q}_1)\,a^*(\underline{q}_2)\,a\,(\underline{p}_2)\,a\,(\underline{p}_1)\ d^3q_1 d^3q_2 d^3p_1 d^3p_2$$

and obtain

$$\int d^3k_1..d^3p_2 A(\underline{k}_1,\underline{k}_2;\underline{k}_2',\underline{k}_1')h(\underline{q}_1,\underline{q}_2;\underline{p}_2,\underline{p}_1)\ \delta\ (\varepsilon\ (\underline{k}_1)+\varepsilon\ (\underline{k}_2)-\varepsilon\ (\underline{k}_2)-$$
$$\varepsilon\ (\underline{k}_1)\ )\ \omega\ ([a^*(\underline{k}_1)\,a^*(\underline{k}_2)\,a\,(\underline{k}_2')\,a\,(\underline{k}_1')\ ,\ a^*(\underline{q}_1)\,a^*(\underline{q}_2)\,a\,(\underline{p}_2)\,a\,(\underline{p}_1)])=0 \tag{5.12}$$

Using the quasi-free property of ω ,

$$\omega(\,a^*(\underline{k}_1)\,a^*(\underline{k}_2)a(\underline{k}_2')\,a\,(\underline{k}_1')\,a^*(\underline{q}_1)\,a^*(\underline{q}_2)\,a\,(\underline{p}_2)\,a\,(\underline{p}_1)) =$$

$$\sum_{\text{pairings}}(-1)^P\ \prod_{i,j}\omega(\,a^*(\underline{k}_i)\,a\,(\underline{p}_j))\ \omega\,(\,a\ (\underline{k}_i')\,a^*(\underline{q}_j)) \tag{5.13}$$

where

$$\omega\ (\,a^*\ (\underline{k}_i)\ a\ (\underline{p}_j))=\ \delta\ (\underline{k}_i-\underline{p}_j)\rho\ \ (\underline{p})$$
$$\omega\ (\,a\ \ (\underline{k}_i')\,a^*\ (\underline{q}_j))=\ \delta\ (\ \underline{k}_i'-\ \underline{q}_j)(1-\ \rho\ (\underline{q}_j)) \tag{5.14}$$

The second term in the commutator we obtain interchanging $\underline{q}_i \leftrightarrow \underline{k}_i$, $\underline{p}_i \leftrightarrow \underline{k}_i'$

This inserted to (5.12), choosing the support of the function h in the variable $\underline{p}_2$ disjoint from that of A in the variable $\underline{k}_1$, gives

$$\int d^3k_1\ d^3k_2\ d^3k_1'\ d^3k_2'\ \ A\ (\underline{k}_1,\ \underline{k}_2;\ \underline{k}_2',\ \underline{k}_1')\ h\ (\underline{k}_1,\ \underline{k}_2;\ \underline{k}_2',\ \underline{k}_1')\ \mathrm{x}$$
$$\delta\ (\ \varepsilon\ (\underline{k}_1)\ +\ \varepsilon\ (\underline{k}_2)\ -\ \varepsilon\ (\underline{k}_2')\ -\ \varepsilon\ (\underline{k}_1')\ )\ \{\ \rho\ (\underline{k}_1')\ \rho\ (\underline{k}_2')\ \ (1-\rho\,(\underline{k}_1))(1-\rho\,(\underline{k}_2))$$
$$-\ \rho\ (\underline{k}_1)\ \rho\ (\underline{k}_2)\ \ (\ 1\ -\ \rho\ (\underline{k}_1')\ )\ (\ 1\ -\ \rho\ (\underline{k}_2')\ )\}\ \ =\ 0 \tag{5.15}$$

To satisfy this condition for essentially arbitrary functions $A(\underline{k}_1\underline{k}_2,\ \underline{k}_2'\underline{k}_1')$ and h $(\underline{k}_1\underline{k}_2,\ \underline{k}_2'\underline{k}_1')$, the expression in the curly bracket must vanish on the energy shell.

We have then the relation

$$\frac{1\ -\ \rho\ (\underline{k}_1)}{\rho\ (\underline{k}_1)}\ \cdot\ \frac{1\ -\ \rho\ (\underline{k}_2)}{\rho\ (\underline{k}_2)}\ =\ \frac{1\ -\ \rho\ (\underline{k}_1')}{\rho\ (\underline{k}_1')}\ \cdot\ \frac{1\ -\ \rho\ (\underline{k}_2')}{\rho\ (\underline{k}_2')} \tag{5.16}$$

or, with

$$R(\underline{k})\ =\ln\frac{1\ -\ \rho\ (\underline{k})}{\rho\ (\underline{k})}\ ,\ R\ (\underline{k}_1)\ +\ R\ (\underline{k}_2)\ =\ R\ (\underline{k}_1')\ +\ R\ (\underline{k}_2')$$

whenever $\varepsilon\ (\underline{k}_1)\ +\ \varepsilon\ (\underline{k}_2)\ =\ \varepsilon\ (\underline{k}_1')\ +\ \varepsilon\ (\underline{k}_2')$ (5.17)

Thus

$$R\ (\underline{k}) = \alpha + \beta\ \varepsilon\ (\underline{k})\ ,$$

α , β - constants.

For the one-particle momentum distribution $\rho(k)$, we find then

$$\rho\ (\underline{k}) = \frac{1}{1 + \exp\ (\alpha+\beta\varepsilon(\underline{k})\ )} \qquad (5.18)$$

The last problem we want to investigate in this section is the behavior of an equilibrium state under more general kind of perturbations. Since we have chosen the even part of the CAR algebra as an algebra of observables, we may consider a system in which the particle number is not conserved, allowing, for instance, creation of pairs of particles. We could think for example about a neutrino-antineutrino system, in which the difference $N_\nu - N_{\bar\nu}$ is conserved, but pairs (ν , $\bar\nu$) can be created or destroyed.
In our idealized example we consider the particle identical with its antiparticle. As we have already found, an equilibrium state will be gauge - invariant, quasi-free, with one-particle momentum distribution of Fermi type.
We expect that the stability under non-gauge invariant perturbations will restrict the state furthermore or that there does not exist any primary state on $\mathfrak{A}_e$, stable under such general perturbations. With the same technique as above, putting in (4.20)

$$A = a^*(f_1)\,a^*(f_2)\ , \qquad h = a(g_1)\,a(g_2)$$

and using the properties of ω : (5.11), (5.12), (5.14), (5.15), we find the following :

$$\int_{-\infty}^{\infty} \omega\ (\,[A,\ \alpha_t\,(h)]\,)\ dt =$$

$$\int_{-\infty}^{\infty} e^{\ i(\varepsilon(\underline{q}_1) + \varepsilon(\underline{q}_2))\ t}\ dt \int d^3q_1\ d^3q_2\ \ (1 - \rho(\underline{q}_1) - \rho\ (\underline{q}_2)) \times$$

$$g_1(\underline{q}_1)\ g_2(\underline{q}_2)\ \{\ \ f_1^*\,(\underline{q}_1)\ \ f_2^*\,(\underline{q}_2) -\ \ f_1^*(\underline{q}_2)\ \ f_2^*\ (\underline{q}_1)\}\ \ = 0$$

which we rewrite in the form :

$$\lim_{T\to\infty}\int_{-T}^{T} e^{i\ (\varepsilon(\underline{q}_1) + \varepsilon(\underline{q}_2))\ t}\ dt \int d^3q_1 \int d^3q_2\ F\ (\underline{q}_1,\underline{q}_2)(1-\rho(\underline{q}_1)-\rho(\underline{q}_2)) = 0 \qquad (5.19)$$

where

$$F\ (\underline{q}_1,\ \underline{q}_2) = g_1(\underline{q}_1)\,g_2(\underline{q}_2)\ \{\ f_1^*(\underline{q}_1)\ f_2^*(\underline{q}_2) - f_1^*(\underline{q}_2)\ f_2^*\,(\underline{q}_1)\ \}$$

is an integrable function.

It may be seen that this is equivalent to the bracket $(1-\rho\ (\underline{q}_1) - \rho(\underline{q}_2)) = 0$ at $\underline{q}_1 = \underline{q}_2 = 0$.

Thus the stability condition is satisfied if and only if

$$1 - 2\rho\ (0) = 0 \tag{5.20}$$

Comparing with (5.18) we see that this means that $\alpha = 0$.

The conclusion is therefore that an equilibrium state ω, of the infinite free Fermi system, if it is supposed to be stable also under non-gauge invariant perturbations, is given uniquely by one-particle momentum distribution

$$\rho\ (\underline{p}) = \frac{1}{1 + \exp\ (\beta\varepsilon(\underline{p}))} \tag{5.21}$$

i.e. the Fermi distribution with vanishing chemical potential. We are then in a situation similar to the black body radiation, in the sense, that an equilibrium state is characterized by only one thermodynamic parameter, β.

VI. General Consequences of the Stability Condition

The stability condition (4.20), is not limited to free systems but can be applied without any change to general infinite quantum systems.
In recent years it has become customary to characterize equilibrium states of infinite systems by the so-called Kubo-Martin-Schwinger (KMS) condition. The origin of this is the observation that Gibbs' states of finitely extended systems enjoy certain analyticity properties (15). In fact, one has as a consequence of the invariance of the trace under cyclic permutations of its arguments, the following theorem:

> Let $F(t) = \omega\ (B\ \alpha_t(A))$, $G(t) = \omega\ (\alpha_t(A)B)$ where A, B are bounded operators acting on a Hilbert space, $\omega(A) = Tr(\rho A)$ (6.1)
>
> $$\rho = \text{const} \exp\ (-\beta\ (H - \mu N)).$$
>
> There exists an analytic function G(z), uniformely bounded in the strip $0 < \text{Im } z < \beta$, such that F(t) is the boundary value of F(z) on the real axis (similar statements hold for G(z) in the strip $-\beta < \text{Im } z < 0$), and
>
> $$F\ (t + i\beta) = G(t) \tag{6.2}$$

In terms of Fourier transforms, this relation between the functions F and G, becomes

$$\hat{F}\ (E) = \exp\ (\beta\ E)\ \hat{G}\ (E) \tag{6.3}$$

It happens that this property is preserved in the thermodynamic limit and therefore can be taken as an independent definition of the equilibrium states of infinite systems (16). For the discussion of the condition (6.3) in the thermodynamic limit we refer to (13).

Seeking for a description of the equilibrium states of infinite systems in simple physical terms, we have applied in reference (17) the stability requirement to states of general infinite quantum systems. It has been found there that for a system ($\mathfrak{A}$, α_t) with an asymptotically abelian structure, any primary, stationary state, with absolutely integrable temporal correlation functions, which satisfies (4.20), is either a ground state (temperature zero) or it is a KMS state for some value of β.

The gratifying result was, that the whole information contained in (6.3), followed already from the condition (4.20), if one assumed a sufficiently fast rate of decrease of the temporal correlation functions.

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