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Graphical Rules for the Diagonalization of the Feynman Denominator

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Graphical rules for the diagonalization of the Feynman - denominator

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Abstract: By choosing a suitable nonsingular transformation on the external momenta of a Feynman - parametric integral simple rules can be given for the diagonalization of the Feynman - denominator V_G in terms of the new momenta. This diagonalization of V_G is the basis for a discussion of massless field theories in the framework of [3,4].
The effect of the transformation on the spin - polynomial Y_G^{ℓ} is considered too.

1. Introduction

The perturbation - theoretical treatment of quantum field theories via Feynman-parametric integrals involves - at least implicitly - a discussion of the function

$$(U_G)^{-\frac{d}{2}} \left[V_G - \sum_{\ell \in \mathcal{L}(G)} \alpha_\ell m_\ell^2 \pm i0 \right]^\lambda \qquad d, \lambda \in \mathbb{C}, \quad \sum_{\ell \in \mathcal{L}(G)} \alpha_\ell = 1, \alpha_\ell \geq 0$$

$$U_G = \sum_{T_1 \in \mathcal{T}_1} \prod_{\ell \in T_1'} \alpha_\ell \,, \qquad V_G = \sum_{\substack{\{K,K'\} \\ K \subset g}} \frac{W^{(K|K')}}{U_G} \Big(\sum_{a \in K} p_a \Big)^2 , \qquad p_a \in \mathbb{R}^4, \quad \sum_{a \in g} p_a = 0$$

In massive theories the main problem is a resolution of singularities of U_G; that means a determination of the number and order of (independent) zeros of U_G, and a representation of U_G in the form $U_G = \prod_i (t_{\sigma i})^{K_i} E(\underline{t}_\sigma)$ where $t_{\sigma i} = t_{\sigma i}(\alpha)$ are independent variables, K_i are positive integers and $E(\underline{t}_\sigma)$ does not vanish in the domain D_σ. The union of all D_σ has to cover the whole space $\sum_{\ell \in \mathcal{L}} \alpha_\ell = 1, \alpha_\ell \geq 0$. Since U_G is an analytic function of $\underline{\alpha}$ general theorems [6] assure the existence of such a resolution. Indeed it can be given explicitly in an elegant way [3] by introducing a partial order of the parameter $\underline{\alpha}$ in terms of what is called labelled singularity family in [3]. This construction moreover serves as a ground for a physically acceptable subtraction of the resulting divergences of the Feynman-parametric integrals. The function $[V_G - \sum \alpha_\ell m_\ell^2 \pm i0]^\lambda$ does not present any difficulties [3]. Due to the term $\sum_{\ell \in \mathcal{L}} \alpha_\ell m_\ell^2$, $m_\ell > 0$, $\ell \in \mathcal{L}$ it is a distribution in the external momenta $(p_a)_{a \in g}$, $\sum_{a \in g} p_a = 0$ which is an entire function of λ and smooth in $t_{\sigma i}$ in the domain D_σ.

In the massless case the situation changes considerably. Now, due to the vanishing of m_ℓ for all (or some) $\ell \in \mathcal{L}(G)$ $[V_G - \sum \alpha_\ell m_\ell^2 \pm i0]^\lambda$ produces singularities both in λ and $\underline{\alpha}$ (resp. $\underline{t}_\sigma$). This is the reason why a general discussion of massless theories in the framework of analytic and dimensional renormalization [3, e.g. 4] has not been established till now - in spite of the fact that there has been considerable interest in massless theories in recent years caused by the appearance of massless particles in field theories with certain symmetries - chiral symmetries and gauge field theories.

However it turns out that in order to control the singularities of $[V_G - \sum \alpha_\ell m_\ell^2 \pm i0]^\lambda$ in the massless case a certain diagonalization of V_G is sufficient [1] i.e. to represent V_G as a sum:

$$V_G = \sum_{i=1}^{|g|-1} \frac{V_i}{V_{i-1}} (q_i')^2 \quad , \quad V_0 = U_G$$

where $q_i' = q_i'(\underline{\alpha}, \underline{p})$ are certain linear combinations of the external momenta. The resolution of the singularities of V_i / V_{i-1} $i = 1, \dots, |g|-1$ gives a satisfactory basis for a general discussion of massless theories along the lines of [3,4] . In [1] this has been applied with great success to the determination of the scaling behavior of Feynman-amplitudes - even in Minkowski-region.

This paper describes a general procedure for the above diagonalization. A m-family $\mathfrak{h}_G$ of subsets of the set of external vertices is introduced. Linear combinations q_i' $i = 1, \dots, |g|-1$ of the external momenta corresponding to the elements of $\mathfrak{h}_G$ are chosen. V_G is diagonal in the new momenta q_i'. The coefficients of $(q_i')^2$ have a

simple connection to certain graphs G_i obtained from the original graph G. The effect of the transformation $(p_a)_{a\in g_G} \to (q_i^l)_{i=1,\dots,|g|-1}$ on the spin - polynomial Y_G^l for line l (of degree 1) is considered too. The graphs G_i again show up in a natural way.
In a subsequent paper [7] the result will be applied to the analytic and dimensional renormalization of massless theories.

The author is greatly indebted to Prof. K. Pohlmeyer whose idea of a diagonalization of V_G in massless field theories started the investigation. The author wishes to thank him for many discussions on the subject.

2. Notation

For convenience of the reader the basic definitions of the theory of (Feynman-) graphs will be given. The notation heavily relies on [2] and proofs may be found there.

A graph $G = (\mathcal{L}(G), v(G), \varphi_G)$ is a triplet consisting of a (finite) set of lines $\mathcal{L}(G)$, a (finite) set of vertices v(G) and a mapping φ_G

$$\varphi_G : \begin{cases} \mathcal{L}(G) \longrightarrow v(G) \times v(G) \\ l \longrightarrow (\varphi_i(l), \varphi_f(l)) \end{cases}$$

$\varphi_i(l)$, $\varphi_f(l)$ are the endpoints of the line $l \in \mathcal{L}(G)$

The graph will be assumed to be oriented ;that is $(a_1, a_2) \neq (a_2, a_1)$ if $a_1 \neq a_2 \in v(G)$.

A subgraph $H \subset G$ is a graph $H = (\mathcal{L}(H), v(H), \varphi_H)$ satisfying $\mathcal{L}(H) \subset \mathcal{L}(G)$, $v(H) \subset v(G)$ and $\varphi_H = \varphi_G|_{\mathcal{L}(H)}$.

Let a be an element of v(G). The sets S(a) and L(a) are introduced according to:

$$S(a) = \{ \ell \in \mathcal{L}(G) \mid \varphi_i(\ell) = a \text{ or } \varphi_f(\ell) = a \}$$
$$L(a) = \{ \ell \in \mathcal{L}(G) \mid \varphi_i(\ell) = \varphi_f(\ell) = a \}$$

Two distinct vertices $a \neq b$ are said to be adjacent if $S(a) \cap S(b) \neq \emptyset$

The graph is connected iff for each pair $a \neq b$ of vertices there is a sequence of vertices $a = a_0, a_1, \ldots, a_k = b$ such that a_j and a_{j+1} $j=0,..,k-1$ are adjacent.

Let k be a subset of v(G) containing more than one element; $|k| \geq 2$. G(k) is the graph $(\mathcal{L}(G), (v(G) \setminus k) \cup \{k_0\}, \varphi_{G(K)})$ obtained by identifying the vertices in k ,i.e.

$$\varphi_{G(K)}(\ell) = \begin{cases} (\varphi_i(\ell), \varphi_f(\ell)) & \varphi_i(\ell), \varphi_f(\ell) \in v(G) \setminus K \\ (\varphi_i(\ell), K_0) & \varphi_i(\ell) \in v(G) \setminus K, \ \varphi_f(\ell) \in K \\ (K_0, \varphi_f(\ell)) & \varphi_i(\ell) \in K, \ \varphi_f(\ell) \in v(G) \setminus K \\ (K_0, K_0) & \varphi_i(\ell), \varphi_f(\ell) \in K \end{cases} \qquad \begin{array}{l} \ell \in \mathcal{L}(G), \\ K_0 \in k \end{array}$$

If $a \in v(G)$, then a^k will denote the corresponding vertex in G(k). One has $k_0 = a^K$ for all $a \in k$.

Let $k' = \{a_1, \ldots, a_{k'}\}$ be a subset of v(G) satisfying $|\{a_1^K, .., a_{k'}^K\}| \geq 2$. Then G(k|k') stands for the graph $G(k)(\{a_1^K, .., a_{k'}^K\})$ obtained from the graph G(k) by identifying the vertices $a_1^K, .., a_{K'}^K$.

Let l_0 be a line $l_0 \in \mathcal{L}(G)$. The reduced graph G/l_0 is the graph with line l_0 being contracted,i.e. $G/l_0 = (\mathcal{L}(G/l_0), v(G/l_0), \varphi_{G/l_0})$ with

$$\mathcal{L}(G/l_0) = \mathcal{L}(G) \setminus \{l_0\},$$
$$v(G/l_0) = (v(G) \setminus \{\varphi_i(\ell_0), \varphi_f(\ell_0)\}) \cup \{a_0\} \quad , a_0 \in \{\varphi_i(\ell_0), \varphi_f(\ell_0)\}$$

and

$$\varphi_{G/l_0}(l) = \begin{cases} (\varphi_i(l), \varphi_f(l)) & \varphi_i(l), \varphi_f(l) \in v(G) \setminus \{\varphi_i(l_0), \varphi_f(l_0)\} \\ (a_0, \varphi_f(l)) & \varphi_i(l_0) = \varphi_i(l_0) \\ (\varphi_i(l), a_0) & \varphi_f(l) = \varphi_f(l_0) \\ (a_0, a_0) & \varphi_i(l) = \varphi_i(l_0),\ \varphi_f(l) = \varphi_f(l_0) \end{cases} \qquad l \in \mathcal{L}(G) \setminus \{l_0\}$$

If $L = \{l_1, \dots, l_n\}$ is a subset of $\mathcal{L}(G)$, then G/L is defined to be $(-((G/l_1)/l_2)/\dots/l_n$.

Let $H \subset G$ be a subgraph and $a \in v(G)$ a vertex. Define a number

$$\underline{D(a,H)} = |S(a) \cap \mathcal{L}(H)| + |L(a) \cap \mathcal{L}(H)|$$

A <u>path</u> P joining two distinct vertices a,b is a minimal connected subgraph $P \subset G$ satisfying

$$c \in v(G) \Rightarrow D(c,P) = \begin{cases} 0 \text{ or } 2 & c \neq a,b \\ 1 & \text{otherwise} \end{cases}$$

$\mathbf{P}_G(ab)$ is the set of all paths joining a and b .

A <u>loop</u> C is a minimal nonempty connected subgraph $C \subset G$ with $a \in v(G) \Rightarrow D(a,c) = 0$ or 2

<u>Remark:</u> From now on the graph G is always understood to be a connected graph. The subgraphs may be disconnected of course. Furthermore sometimes a graph H will be identified with its set of lines $\mathcal{L}(H)$ and vice versa a set of lines $L \subset \mathcal{L}(G)$ will denote the graph $L = (L, v(L), \varphi_L)$ with

$$v(L) = \{\varphi_i(l) \mid l \in L\} \cup \{\varphi_f(l) \mid l \in L\}; \quad \varphi_L(l) = \varphi_G(l),\ l \in L$$

A (1-) <u>tree</u> T_1 is a connected subgraph $T_1 \subset G$ satisfying $v(T_1) = v(G)$ and including no loop. It possesses the properties

<u>2.1.1.:</u> For any pair $a \neq b \in v(G)$ there is a unique path $P \subset T_1$, $P \in \mathbb{P}_G(ab)$

<u>2.1.2.:</u> $I \subset T_1$ implies $T_1 \setminus I$ is a tree in G/I.

$\mathbf{T}_G$ is the set of all trees in G.

A <u>2 - tree</u> T_2 is a tree T_1, one line being omitted. If h_1, h_2 are disjoint subsets of v (G) a 2-tree T_2 is said to <u>separate h_1 and h_2</u> if it connects all the vertices in each set h_1 and h_2 resp., without connecting h_1 and h_2.

A <u>co - r-tree</u> T_r' (r = 1,2) related to the r - tree T_r is the subgraph obtained from the set $\mathcal{L}(G)\setminus\mathcal{L}(T_r)$.

Numbers $\alpha_\ell \geq 0$ and $\beta_\ell (= \alpha_\ell^{-1})$ called Feynman-parameter are assigned to each line $l \in \mathcal{L}(G)$.

Certain functions can be defined:

$$\tilde{\mathcal{U}}_G = \sum_{T_1 \in \mathcal{T}_G} \prod_{\ell \in T_1} \beta_\ell \quad , \quad \mathcal{U}_G = \sum_{T_1 \in \mathcal{T}_G} \prod_{\ell \in T_1'} \alpha_\ell$$

called resp. <u>tree-product sum</u> and <u>co-tree-product sum</u> of the graph G, and similarly for $h_1, h_2 \subset$ v (G)

$$\tilde{W}_G^{(h_1|h_2)} = \sum_{T_2 \in \mathcal{T}_2(h_1|h_2)} \prod_{\ell \in T_2} \beta_\ell \quad , \quad W_G^{(h_1|h_2)} = \sum_{T_2 \in \mathcal{T}_2(h_1|h_2)} \prod_{\ell \in T_2'} \alpha_\ell$$

resp. <u>2 - tree - product sum</u> and <u>co- 2-tree-product sum</u> separating h_1 and h_2.

Here the summation goes over all 2-trees $T_2 \in T_2 \ (h_1|h_2)$ separating h_1 and h_2.

<u>Remark:</u> The sum equals zero if $h_1 \cap h_2 \neq \emptyset$. In general the resp. α- and β - dependences will be suppressed. Furthermore vertices a,b,c... and the set $\{a,b,c\ ..\}$ will be identified.

Some properties of the W - functions are of use.

<u>Remark 2.2.1.:</u> $a \in$ v (G), $h_1, h_2 \subset$ v (G)

$$\tilde{W}_G^{(h_1|h_2)} = \tilde{W}_G^{(h_1 a|h_2)} + \tilde{W}_G^{(h_1|h_2 a)}$$

<u>2.2.2.:</u> $h_1, h_2, g \subset$ v (G)

$$\tilde{W}_G^{(h_1|h_2)} = \sum_{\substack{K \subset g \\ h_1 \subset K \\ h_2 \subset g\setminus K}} \tilde{W}^{(K|g\setminus K)}$$

Similar formulas hold for the functions with tilde.
The <u>incidence matrix</u> $\mathfrak{A}_G = \left([a:l]\right)_{\substack{l\in\mathcal{L}(G)\\ a\in v(G)}}$ is defined by

$$[a:l] = \begin{cases} +1 & a = \varphi_f(l) \\ -1 & a = \varphi_i(l) \\ 0 & \text{otherwise} \end{cases} \qquad l \in S(a)\setminus L(a)$$

Let the matrix $\mathfrak{M}_G$ be given by

$$\mathfrak{M}_G = \left(\sum_{l\in\mathcal{L}(G)} [a_1:l]\,\delta_l\,[a_2:l]\right)_{a_1,a_2\in v(G)}$$

If a is an arbitrary vertex, $\mathfrak{M}_G^a$ is the matrix constructed from $\mathfrak{M}_G$ by deleting the row and the column belonging to a. $[\mathfrak{M}_G^a]$ will denote the determinant of the matrix $\mathfrak{M}_G^a$. $[\mathfrak{M}_G^a]^{(b|c)}$ $a \neq b,c$ is the <u>co - factor</u> corresponding the <u>minor</u> $\underline{[\mathfrak{M}_G^a]_0^{(b|c)}}$ – the determinant of the matrix obtained by deleting the rows a,b and columns a,c in $\mathfrak{M}_G$. $\underline{[\mathfrak{M}_G^a]_0^{(b_1 b_2|c_1 c_2)}}$ will denote the $(a,b_1,b_2 \mid a,c_1,c_2)$ <u>minor</u> of $\mathfrak{M}_G$ ($a \neq b_1, b_2, c_1, c_2$; $b_1 \neq b_2$; $c_1 \neq c_2$).

A number $\underline{\tau(a,b)}$ is assigned to each pair (a,b) of vertices:

$$\tau(a,b) = \begin{cases} 0 & a = b \\ +1 & \text{otherwise.} \end{cases}$$

The following formulas are contained in [2].

<u>Remark: 2.3</u>. Let a be a vertex of G. Then

$$[\mathfrak{M}_G^a] = \tilde{\mathfrak{U}}_G \qquad a \in v(G)$$

Let $a \neq b$ be vertices of G.

<u>2.4</u>. $\mathfrak{A}_{G(ab)} = \left([c^{ab}:l]\right)_{c^{ab}\in v(G(a,b)),\ l\in\mathcal{L}(G)}$ satisfies

$$[c^{ab}:l] = \begin{cases} [c:l] & c^{ab} \in v(G)\setminus\{a,b\} \\ [a:l]+[b:l] & c^{ab} = a^{ab} \equiv b^{ab} \end{cases} \qquad l \in \mathcal{L}(G)$$

<u>2.5</u>.

$$\tilde{W}_G^{(a|b)} = [\mathfrak{M}_G^a]^{(b|b)} = \tilde{\mathfrak{U}}_{G(ab)}$$

$$= [\mathfrak{M}_{G(ab)}^{c^{ab}}] \qquad c^{ab} \in v(G(ab))$$

Let a,b,c,d,e be arbitrary vertices of G.

2.6. A number $\sigma(b,c) = \sigma(c,b) = \pm 1$ exists (depending on a) such that

$$\tau(a,b)\tau(a,c)\,[\mathcal{M}_G^a]^{(b|c)} = \tau(a,b)\tau(a,c)\,\sigma(b,c)\,[\mathcal{M}_G^a]_0^{(b|c)}$$

and

$$\sigma(b,c)\cdot\sigma(c,d) = \sigma(b,d) \qquad b,c,d \in v(G)\setminus\{a\}$$

2.7. The following identity holds

$$\tau(a,b)\,\tau(a,c)\,[\mathcal{M}_G^a]^{(b|c)} = \widetilde{W}_G^{(a|bc)} = \widetilde{W}_G^{(ad|bc)} + \widetilde{W}_G^{(a|bcd)}$$
$$= \widetilde{W}_G^{(ade|bc)} + \widetilde{W}_G^{(ad|bce)} + \widetilde{W}_G^{(a|bcd)}$$

One distinguishes a certain subset g_G of $v(G)$ -called the set of external vertices. If k_G is a subset of g_G, k' will denote the set $g_G \setminus k_G$. A vector-momentum - $p_a \in R^4$ is assigned to each $a \in g_G$ such that $\sum_{a\in g_G} p_a = 0$. Thus $(p_a)_{a\in g_G}$ is an element of $R^{4|g_G|-4}$. Moreover a scalar product $(p_a \cdot p_b)$ will be assumed to exist on R^4. (The subscript G may be omitted sometimes).

The function V_G is defined to be

$$V_G = \frac{1}{U_G} \sum_{\substack{\{K,K'\}\\ K\subset g}} W_G^{(K|K')} \Big(\sum_{a\in K} p_a\Big)^2$$

Choose arbitrarily $a \neq b \in g_G$, $l \in \mathcal{L}(G)$ and $P \in \mathbf{P}_G(a\,b)$. The path P will be taken to be oriented from a to b. Then a number $[P:l]$ can be introduced by

$$[P:l] = \begin{cases} 0 & l \notin P \\ +1 & l \in P \text{ and the orientation of } l \text{ and } P \text{ coincide} \\ -1 & \text{otherwise} \end{cases}$$

The spin-polynomial Y_G^l (of degree 1) for line l is a certain linear combination of external momenta

$$Y_G^l = \frac{1}{U_G} \sum_{b\in g\setminus\{a\}} p_b \cdot \sum_{P\in \mathbf{P}_G(ab)} [P:l]\cdot U_{G/P}$$

Because of the conservation of momenta $\sum_{a\in g} p_a = 0$ Y_G^l does not depend on the peculiar choice $a \in g_G$.

3. Two identities

In this section two identities will be proved. They are crucial for the general discussion in section 6. .

Proposition 3.1. Let $a_1 \neq a_2$, $b_1 \neq b_2$, $c_1 \neq c_2$ be some vertices of the graph G. Then

$$W_G^{(a_1|a_2)}\left[W_G^{(b_1c_1|b_2c_2)} - W_G^{(b_1c_2|b_2c_1)}\right] -$$

$$- \left[W_G^{(a_1b_1|a_2b_2)} - W_G^{(a_1b_2|a_2b_1)}\right]\left[W_G^{(a_1c_1|a_2c_2)} - W_G^{(a_1c_2|a_2c_1)}\right]$$

$$= \mathcal{U}_G \left[W_{G(a_1a_2)}^{(b_1^{a_1a_2}\, c_1^{a_1a_2}|b_2^{a_1a_2}\, c_2^{a_1a_2})} - W_{G(a_1a_2)}^{(b_1^{a_1a_2}\, c_2^{a_1a_2}|b_2^{a_1a_2}\, c_1^{a_1a_2})}\right]$$

Proof: It is sufficient to prove the identity for the functions with tilde. By applying 2.7. (twice) and 2.2.1. the following formula can easily be derived.

$$\widetilde{W}_G^{(a_1c_2b_2|c_1b_1)} - \widetilde{W}_G^{(a_1c_1b_2|c_2b_1)} = \tau(a_1,b_1)\left[\tau(a_1,c_1)\left[\mathfrak{M}_G^{a_1}\right]^{(b_1|c_1)} - \tau(a_1,c_2)\left[\mathfrak{M}_G^{a_1}\right]^{(b_1|c_2)}\right]$$

$$\widetilde{W}_G^{(c_2b_2|a_1c_1b_1)} - \widetilde{W}_G^{(c_1b_2|a_1c_2b_1)} = \tau(a_1,b_2)\left[\tau(a_1,c_2)\left[\mathfrak{M}_G^{a_1}\right]^{(b_2|c_2)} - \tau(a_1,c_1)\left[\mathfrak{M}_G^{a_1}\right]^{(b_2|c_1)}\right]$$

Application of 2.2.1. to the sum gives (3.1.1.):

$$\widetilde{W}_G^{(b_1c_1|b_2c_2)} - \widetilde{W}_G^{(b_1c_2|b_2c_1)} = \sum_{(\alpha,\beta)} \tau(a_1,\alpha)\tau(a_1,\beta)\,\varepsilon_{\alpha\beta}\left[\mathfrak{M}_G^{a_1}\right]^{(\alpha|\beta)}$$

Here the summation goes over the pairs

$$(\alpha,\beta) = \begin{cases} (b_1,c_1)\,,\ (b_2|c_2) & \text{with } \varepsilon_{\alpha\beta} = +1 \\ (b_1,c_2)\,,\ (b_1,c_1) & \text{"} \quad \varepsilon_{\alpha\beta} = -1 \end{cases}$$

By choosing $b_1 = a_1$, $b_2 = a_2$ 3.1.1. shows (3.1.2.):

$$\widetilde{W}_G^{(a_1c_1|a_2c_2)} - \widetilde{W}_G^{(a_1c_2|a_2c_1)} = \tau(a_1,c_2)\left[\mathfrak{M}_G^{a_1}\right]^{(a_2|c_2)} - \tau(a_1,c_1)\left[\mathfrak{M}_G^{a_1}\right]^{(a_2|c_1)}$$

A similar formula holds with $c_1 = b_1$, $c_2 = b_2$.
Inserting these identities and using 2.6. the l.h. side of the identity can be written as

$$l.h.s. = \sum_{(\alpha,\beta)} \varepsilon_{\alpha\beta}\,\tau(a_1,\alpha)\tau(a_1,\beta)\,\sigma(\alpha,\beta)\left\{ \left[\mathfrak{M}_G^{a_1}\right]_0^{(a_2|a_2)}\left[\mathfrak{M}_G^{a_1}\right]_0^{(\alpha|\beta)} - \left[\mathfrak{M}_G^{a_1}\right]_0^{(a_2|\alpha)}\left[\mathfrak{M}_G^{a_1}\right]_0^{(a_2|\beta)}\right\}$$

Now, Jacobi's Theorem [5] states: "Let $A = (a_{iK})$ $i,K = 1,\dots,n$ be a $n \times n$ matrix and $i_1 < i_2$, $K_1 < K_2$. Then the following formula is valid:

$$[A]_0^{(i_1|K_1)}[A]_0^{(i_2|K_2)} - [A]_0^{(i_2|K_1)}[A]_0^{(i_1|K_2)} = [A]\,[A]_0^{(i_1 i_2|K_1 K_2)}\text{"}$$

Without any loss of generality it may be assumed that the rows and columns of $\mathcal{W}_G^{a_1}$ are ordered such that those corresponding to a_1, a_2 are placed in front of those related to α, β. It should be noted that in l.h.s. only those terms contribute with $\alpha, \beta \neq a_1, a_2$. Thus Jacobi's Theorem proves

$$l.h.s. = [\mathcal{W}_G^{a_1}] \sum_{(\alpha,\beta)} \varepsilon_{\alpha\beta}\, \sigma(\alpha,\beta)\, \tau(a_1,\alpha)\, \tau(a_1,\beta)\, \tau(a_2,\alpha)\, \tau(a_2,\beta)\, [\mathcal{W}_G^{a_1}]_0^{(a_2\alpha|a_2\beta)}$$

Now 3.1.1. applied to the bracket of the r.h.side of the identity in 3.1. gives:

$$r.h.s. = \tilde{\mathcal{U}}_G \sum_{(\alpha,\beta)} \varepsilon_{\alpha\beta}\, \sigma(\alpha^{a_1a_2},\beta^{a_1a_2})\, \tau(a_2^{a_1a_2},\alpha^{a_1a_2})\, \tau(a_2^{a_1a_2},\beta^{a_1a_2})\, \left[\mathcal{W}_{G(a_1a_2)}^{a_2^{a_1a_2}}\right]_0^{(\alpha^{a_1a_2}|\beta^{a_1a_2})}$$

If the order of lines (s.a.) in $\mathcal{W}_G^{a_1}$ is transferred to $\mathcal{W}_{G(a_1a_2)}^{a_2^{a_1a_2}}$ it is easily seen that

$$\sigma(\alpha^{a_1a_2},\beta^{a_1a_2})\, \tau(a_2^{a_1a_2},\alpha^{a_1a_2})\, \tau(a_2^{a_1a_2},\beta^{a_1a_2})\, [\mathcal{W}_{G(a_1a_2)}^{a_2^{a_1a_2}}]_0^{(\alpha^{a_1a_2}|\beta^{a_1a_2})}$$

$$= \sigma(\alpha,\beta)\, \tau(a_1,\alpha)\, \tau(a_1,\beta)\, \tau(a_2,\alpha)\, \tau(a_2,\beta)\, [\mathcal{W}_G^{a_1}]_0^{(a_2\alpha|a_2\beta)}$$

This and 2.3. immediately prove proposition 3.1..

<u>Proposition 3.2.</u> Let $a \neq b$ be two vertices and $l \in \mathcal{L}(G)$ a line. Denote by a_1, a_2 the endpoints of l. Then the identity holds:

$$\sum_{P \in \mathbb{P}_G(ab)} [P:l]\, \mathcal{U}_{G/P} = \frac{[a_2:l]}{d_l}\left[W_G^{(aa_1|ba_2)} - W_G^{(aa_2|ba_1)} \right]$$

<u>Remark:</u> This is a modification of a formula in [2,3].

<u>Proof:</u> It is sufficient to prove

$$\sum_{P\in \mathcal{P}_G(ab)} \prod_{P} \rho_K \; [P:\ell] \sum_{T_1^P \in \mathcal{T}_{G/P}} \prod_{T_1^P} \rho_K = \rho_\ell [a_2:\ell] \left\{ \sum_{T_2\in \mathcal{T}_2(aa_1|ba_2)} \prod^{T_2} \rho_K - \sum_{T_2\in \mathcal{T}_2(aa_2|ba_1)} \prod_{T_2} \rho_K \right\}$$

On both sides of the equality the summation goes over disjoint sets - each term contributes just once. Thus it has to be shown that each term of l.h.s. is included in r.h.s. and vice versa.

Let (P, T_1^P) be an element of l.h.s.. P is a path connecting a and b and containing l. T_1^P is a tree in G/P. Thus $T_1^P \cup P$ is a tree T_1 in G and $T_1 \setminus 1$ is a 2-tree in G. Let T_a and T_b resp., denote the connected components of $T_1 \setminus 1$ with $a \in v\,(T_a)$ and $b \in v\,(T_b)$. Then one and only one of the following statements is true

(a) $a_1 \in v\,(T_a)$ and $a_2 \in v\,(T_b)$

(b) $a_1 \in v\,(T_b)$ and $a_2 \in v\,(T_a)$

(a) implies $T_1 \setminus 1 \in \mathcal{T}_2\,(aa_1 \mid ba_2)$ and $[P:1] = [a_2:\ell]$
while (b) implies $T_1 \setminus \ell \in \mathcal{T}_2(aa_2 \mid ba_1)$, $[P:\ell] = [a_1:\ell] = -[a_2:\ell]$
Therefore each term of l.h.s. contributes to r.h.s.- (uniquely) with the right sign.

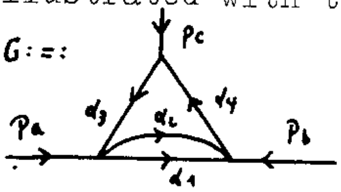
Let T_2 be a 2-tree in G separating (aa_1) and $(a_2 b)$. Then $T_2 \cup 1$ is a tree which includes a unique path P joining a and b. $T_2 \cup \ell / P$ is a tree in G/P. If P is oriented from a to b it follows $[P:1] = [a_2:1]$. Thus all the terms in the first sum of r.h.s. are contained in l.h.s. The same argument applies to the second sum.

Q.E.D.

4. Example

In order to give the idea of the diagonalization without the burden of technicalities the results will be illustrated with the following simple graph.

$G ::=$



$$(p_a, p_b, p_c) \in R^{4\cdot 3} \qquad p_a + p_b + p_c = 0$$

$$U_G = \alpha_1\alpha_2 + (\alpha_1+\alpha_2)(\alpha_3+\alpha_4)$$

$$V_G = \frac{1}{U_G}\left[\alpha_1\alpha_2\alpha_3 p_a^2 + \alpha_1\alpha_2\alpha_4 p_b^2 + (\alpha_1+\alpha_2)\alpha_3\alpha_4 p_c^2\right]$$

Define momenta $q_c = p_c$, $q_a = p_a + p_c$, $q_b = p_a + p_b + p_c \equiv 0$
and the sets $\emptyset \subset h_c = \{c\}$, $h_a = \{a, c\}$, $h_b = \{a, b, c\}$

It should be noted that these sets are nonoverlapping and are (totally) ordered by inclusion such that - apart from h - each set possesses a <u>unique</u> predecessor. The predecessor of each set includes just one vertex more than the union of its successors. Both the sets and the new momenta can be labelled by these unique vertices. The order of the sets defines an order of the external vertices in a natural way.

<u>Remark:4.1</u>. The transformation $(p_a, p_b, p_c)_{\Sigma p_a = 0} \to (q_c, q_a, q_b)$ is a nonsingular linear mapping on the space of external momenta.

In terms of the new momenta one has

$$U_G \cdot V_G = \alpha_3(\alpha_1\alpha_2 + \alpha_1\alpha_4 + \alpha_2\alpha_4) q_c^2 + \alpha_1\alpha_2 q_a^2 + 2 q_a q_c \{-\alpha_1\alpha_2\alpha_3\}$$

<u>Remark:</u> <u>4.2.1</u>. The coefficient of q_c^2 equals $W_G^{(c|a)}$, the co - 2-tree product sum separating c and its predecessor $\hat{c} = a$; the coefficient of q_a^2 equals $W_G^{(a|b)}$, the co-2-tree product sum separating a and its predecessor $\hat{a} = b$; the coefficient of $2 q_a \cdot q_c$ is equal to $W_G^{(ca|\hat{c}\hat{a})} - W_G^{(c\hat{a}|\hat{c}a)}$
This is a simple example of 6.1..

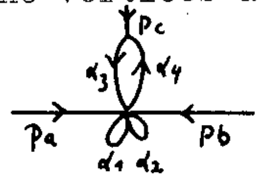
Straightforward calculation proves the identity

$$V_G = \frac{\alpha_1 \alpha_2 (\alpha_3+\alpha_4)}{U_G} \left[q_a - \frac{\alpha_3}{\alpha_3+\alpha_4} q_c \right]^2 + \frac{\alpha_3 \alpha_4}{\alpha_3+\alpha_4} q_c^2$$

$$= \frac{W_G^{(a|\hat{a})}}{U_G} \left[q_a + q_c \frac{W_G^{(ac|\hat{a}\hat{c})} - W_G^{(a\hat{c}|\hat{a}c)}}{W_G^{(a|\hat{a})}} \right]^2 + V_{G(a\hat{a})}$$

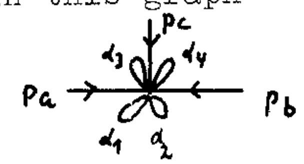
$$V_{G(a\hat{a})} = \frac{U_{G(a\hat{a}|c\hat{c})}}{U_{G(a\hat{a})}} q_c^2$$

<u>Remark: 4.3.1.</u> $W_G^{(a|\hat{a})} = U_{G(a\hat{a})}$ is the co-tree product sum of the graph obtained by identifying the vertices a and $\hat{a}$ = b



<u>4.3.2.</u> The second term in the equalities is equal to the V-function of the graph $G(a\hat{a})$ (s. 4.3.1.)

<u>4.3.3.</u> $U_{G(a\hat{a}|c\hat{c})}$ is the co -tree product sum of the graph G ($a\hat{a}|c\hat{c}$). All the (external) vertices are identified in this graph



<u>4.3.4.</u> The transformation $(p_a, p_b, p_c)_{\Sigma \vec{p_a} = 0}$
(q'_c, q'_a, q'_b)
with $q'_c = q_c$, $q'_b = q_b$, $q'_a = q_a - \frac{\alpha_3}{\alpha_3+\alpha_4} q_c$
is a nonsingular linear mapping of R^8 onto R^8 . It depends continously on $\underline{\alpha}$ in the range $\alpha_i \geq 0$, i = 1,2,3,4.
In terms of the new momenta (q'_c , q'_a)
V_G can be written as follows

$$V_G = \frac{U_{G(a\hat{a})}}{U_G} q_a'^2 + \frac{U_{G(a\hat{a}k\hat{c})}}{U_{G(a\hat{a})}} q_c'^2$$

This is the desired diagonalization for this example.
The generalization of these remarks is given in 6.2, 6.3.

Consider the spin-polynomial Y_G^3 for line 3.

$$Y_G^3 = \frac{1}{U_G}\left[-(\alpha_1\alpha_2 + \alpha_1\alpha_4 + \alpha_2\alpha_4)\, p_a + (\alpha_1 + \alpha_2)\,\alpha_4\, p_b\right]$$

$$= \frac{1}{U_G}\left[-(\alpha_1\alpha_2 + \alpha_1\alpha_2 + \alpha_2\alpha_4)\, q_c + \alpha_1\alpha_2\, q_a\right]$$

<u>Remark: 4.4.</u> The coefficient of momentum q_α (α = c, a) is equal to

$$\sum_{P \in \mathbb{P}_G(\hat{a}\alpha)} [P:3]\; U_{G/P}$$

This result will be extended in 6.4. to some line of an arbitrary graph

A simple calculation shows the effect of the transformation $(q_a, q_c) \rightarrow (q_a', q_c')$ on Y_G^3

$$Y_G^3 = \frac{1}{U_G}\left[q_a - \frac{\alpha_3}{\alpha_3+\alpha_4} q_c\right] - \frac{\alpha_4}{\alpha_3+\alpha_4} q_c$$

$$= \sum_{P \in \mathbb{P}_G(\hat{a}a)} [P:3]\,\frac{U_{G/P}}{U_G}\cdot q_a' + Y^3_{G(\hat{a}a)}$$

$$Y^3_{G(\hat{a}a)} = \sum_{P \in \mathbb{P}_{G(\hat{a}a)}(\hat{c}^{a\hat{a}}c^{a\hat{a}})} [P:3]\,\frac{U_{G(a\hat{a})/P}}{U_{G(a\hat{a})}}\cdot q_c'$$

<u>Remark:</u> <u>4.5.1</u>. The coefficient of q_a' is equal to that of q_a.

<u>4.5.2</u>. The second term in the equation is the spin-polynomial of line 3 in the graph $G(a\hat{a})$

obtained by identifying the vertices a and $\hat{a} = b$ in G. This result will be generalized in 6.5. and 6.6.

5. Choice of momenta

A family $\mathfrak{h}_G$ of subsets h of $\mathfrak{g}_G$ - the set of the external vertices - is called a momenta (m-) -family if it satisfies

$$(\alpha) \quad h_1, h_2 \in \mathfrak{h}_G \Rightarrow \begin{cases} h_1 \subset h_2 & \text{or} \\ h_1 \supset h_2 & \text{or} \\ h_1 \cap h_2 = \emptyset \end{cases}$$

$$(\beta) \quad \emptyset \notin \mathfrak{h}_G \,; \quad \mathfrak{f} \subset \mathfrak{h}_G \Rightarrow \mathfrak{g}_G \setminus \Big(\bigcup_{h \in \mathfrak{f}} h \Big) \notin \mathfrak{h}_G$$

(γ) If $h \in \mathfrak{h}_G$ and $\mathfrak{M}(\mathfrak{h}_G, h) = \{ h_1 \in \mathfrak{h}_G \mid h_1 \subsetneqq h \}$, then

$$|h| = \Big| \bigcup_{h_1 \in \mathfrak{M}(\cdot|\cdot)} h_1 \Big| + 1 \qquad \Big(\Big| \bigcup_{h_1 \in \mathfrak{M}(\cdot|\cdot)} h_1 \Big| = 0 \text{ if } \mathfrak{M}(\mathfrak{h}_G|h) = \emptyset \Big)$$

(δ) $\mathfrak{h}_G$ is maximal

Remarks: 5.1.1. (α) implies that $\mathfrak{h}_G$ is a family of nonoverlapping subsets $h \subset \mathfrak{g}_G$. The elements of $\mathfrak{h}_G$ can partially be ordered by inclusion.

5.1.2. (β) corresponds to the fact that due to momentum conservation the sets $h \subset \mathfrak{g}_G$ and $\mathfrak{g}_G \setminus h$ are equivalent in some sense.

5.1.3.

(γ) indicates that each $h \in \mathfrak{H}_G$ can be labelled by a unique vertex $a_h \in g_G$, i.e. by the unique vertex $a \in h$ not contained in any $h_1 \in \mathfrak{M}(\mathfrak{H}_G | h)$.

5.1.4. $g_G \in \mathfrak{H}_G$

Assume $g_G \notin \mathfrak{H}_G$. Let $h_1, \ldots, h_K$ be the maximal elements of $\mathfrak{H}_G$ satisfying:

$$h_i \cap h_j = \emptyset \quad , \; i \neq j \; , \; i,j = 1,\ldots,k$$

Now (β) implies $\bigcup_1^K h_i \neq g_G$. Thus there exists a vertex $a \in g_G$ with $a \notin \bigcup_1^K h_i$. Define h_a to be the union $\bigcup_1^K h_i \cup \{a\}$. The family $\mathfrak{H}_G \cup \{h_a\} \neq \mathfrak{H}_G$ satisfies (α),(β), (γ) - thus contradicting (δ). This shows $g_G \in \mathfrak{H}_G$.

5.1.5. $|\mathfrak{H}_G| = |g_G|$

In view of 5.1.3. it suffices to prove: "To each $a \in g_G$ corresponds a (necessarily unique) element $h \in \mathfrak{H}_G$ such that $a_h = a$". Assume that $a \in g_G$ does not possess this property. Then, $g_G \in \mathfrak{H}_G$ and (γ) imply the existence of a unique smallest $h_0 \in \mathfrak{H}_G$ satisfying

$$a \in h_0 \quad \text{and} \quad (h \subsetneq h_0 \Rightarrow h \notin \mathfrak{H}_G)$$

If $a, a_1, \ldots, a_K$ are the elements of h_0, the assumption gives $|K| \geq 2$. Define the sets $h_1 = \{a_1\}$, $h_2 = \{a_1, a_2\}$, $h_K = \{a_1, \ldots, a_K\}$. The family $\mathfrak{H}_G \cup \{h_1, \ldots, h_K\}$ fulfils (α),(β),(γ) - contradicting (δ).

5.1.6.

Remarks 5.1.3. and 5.1.5. prove the existence of a one-to-one correspondence between the external vertices and the elements of $\mathfrak{H}_G$ such that:

$$a \in g_G \text{ is related to } h \in \mathfrak{H}_G \quad \text{iff } a \in h \text{ and } a \notin h_1 \;\; \forall h_1 \in \mathfrak{M}(\mathfrak{H}_G | h)$$

Moreover each $a = a_h$, $h \in \mathfrak{h}_G$, $h \neq g_G$, possesses a unique predecessor $a_{\hat{h}}$ such that:

$$h \subsetneq \hat{h} \in \mathfrak{h}_G \quad \text{and } h \text{ is minimal .}$$

It is the pair $(a_h , a_{\hat{h}})$, $h \in \mathfrak{h}_G \setminus \{g_G\}$ that plays a dominant rôle in the diagonalization of V_G (see below).

Let $(p_a)_{a \in g_G}$ be the external momenta of G. Define new momenta

$$q_h = \sum_{a \in h} p_a \qquad , \; h \in \mathfrak{h}_G$$

Remark: 5.1.6. shows that the transformation

$$(p_a)_{a \in g_G} \to (q_h)_{h \in \mathfrak{h}_G} \quad , \sum_{a \in g} p_a = 0$$

is a nonsingular linear transformation of $R^{4|g|-4}$ onto $R^{4|g|-4}$.

The momenta p_a, $a \in g_G$ can be expressed in terms of the new momenta

$$p_a = q_h - \sum_{h_1 \in \hat{\mathfrak{A}}(\mathfrak{h}_G|h)} q_{h_1} \quad .$$

Here it is $a_h = a$ and

$$\hat{\mathfrak{A}}(\mathfrak{h}_G|h) = \{ h_1 \in \mathfrak{h}_G \mid \hat{h}_1 = h \} \qquad h \in \mathfrak{h}_G \quad .$$

Convention: $a_{\hat{g}}$ should mean a certain element with the property $a_{\hat{g}} \notin g_G$.

Let k be a nonempty proper subset of g_G and define the families

$$\mathfrak{A}(\mathfrak{h}_G|K) = \{ h \in \mathfrak{h}_G \mid a_h \in K, \; a_{\hat{h}} \notin K \}$$

$$\mathfrak{L}(\mathfrak{h}_G|K|h) = \{ h_1 \in \mathfrak{h}_G \mid a_{h_1} \notin K, \; a_{\hat{h}_1} \in K, \; h_1 \subset h, \; h_1 \text{ maximal} \} \quad h \in \mathfrak{A}(\mathfrak{h}_G|K)$$

Remark: 5.2.1.

By the maximality condition imposed on $h_1 \in \mathfrak{L}(\mathfrak{h}_G|K|h)$ all the predecessors $a_{\hat{h}_1}, a_{\hat{\hat{h}}_1}, \ldots, a_h$ of a_{h_1} are contained in h.

5.2.2. It is easily seen that:

$$K = \bigcup_{h\in\mathfrak{A}(\mathfrak{h}_0|K)} \Big\{ h \setminus \bigcup_{h_1\in\mathfrak{L}(\cdot|K|h)} h_1 \Big\}$$

5.2.3. The sets $\{h \setminus \bigcup_{\mathfrak{L}(\cdot|K|h)} h_1\}$, $h \in \mathfrak{A}(\mathfrak{h}_0|K)$ are mutually disjoint.
The same is true for the sets $\mathfrak{L}(\cdot|K|h)$, $h\in\mathfrak{A}(\mathfrak{h}_0|K)$.

5.2.4. A simple argument shows:

$$\bigcup_{h\in\mathfrak{A}(\mathfrak{h}_0|K)} \mathfrak{L}(\mathfrak{h}_0|K|h) = \mathfrak{A}(\mathfrak{h}_0|K')$$

5.2.5. The following formula is valid:

$$\sum_{a\in K} p_a = \sum_{h\in\mathfrak{A}(\mathfrak{h}_0|K)} \Big\{ q_h - \sum_{h_1\in\mathfrak{L}(\cdot|K|h)} q_{h_1} \Big\}$$

For, in view of the remarks 2.2., 2.3. one has

$$\sum_{a\in K} p_a = \sum_{h\in\mathfrak{A}(\mathfrak{h}_0|K)} \; \sum_{a\in h\setminus(\bigcup_{h_1\in\mathfrak{L}(\cdot|K|h)} h_1)} p_a$$

Thus it is sufficient to consider the equality

$$\sum_{a\in h\setminus(\bigcup_{h_1\in\mathfrak{L}(\cdot|K|h)} h_1)} p_a = \sum_{a\in h\setminus(\bigcup_{h_1\in\mathfrak{L}(\cdot|K|h)} h_1)} \Big\{ q_{h_a} - \sum_{h_2\in\hat{\mathfrak{A}}(\mathfrak{h}_0|h_a)} q_{h_2} \Big\}$$

Let $a \in h\setminus(\bigcup h_1)$ be such that $h \supsetneqq h_a > h_1$.
Due to the construction of $\mathfrak{L}(\cdot|K|h)$ a unique vertex $a_{\hat{h}_a}$ exists with $h_a \in \hat{\mathfrak{A}}(\mathfrak{h}_0|\hat{h}_a)$. Thus the vector q_{h_a} appears just twice - with different sign however. Therefore, only the terms

$$q_h - \sum_{h_1\in\mathfrak{L}(\cdot|K|h)} q_{h_1}$$

survive.

5.2.6. Application of 5.2.5., 5.2.4., 5.2.3. yields the identity:

$$\sum_{a\in K} p_a = \sum_{h\in\mathfrak{A}(\mathfrak{h}_0|K)} q_h - \sum_{h\in\mathfrak{A}(\mathfrak{h}_0|K')} q_h$$

Notation

Let a_{h_0} be a certain external vertex $h_0 \in \mathcal{H}_G \setminus \{g_G\}$.
For each vertex $a \in v(G)$ a^{h_0} will denote the vertex in the graph $G(a_{h_0} a_{\hat{h}_0})$ which results from the vertex a in the graph G. The external momenta in $G(a_{h_0} a_{\hat{h}_0})$ will be chosen as follows.

$$p_{a^{h_0}} = p_a \qquad a \neq a_{h_0}$$
$$p_{a^{h_0}} = p_{a_{\hat{h}_0}} + p_{a_{h_0}} \qquad a^{h_0} = a^{h_0}_{h_0} \equiv a^{h_0}_{\hat{h}_0}$$

The family $\mathcal{H}_{G(a_{h_0} a_{\hat{h}_0})}$ is defined to be the set of all subsets:

$$h^{h_0} = \{ a^{h_0} \in g_{G(a_{h_0} a_{\hat{h}_0})} \mid a \in h,\ a \neq a_{h_0} \} \subset g_{G(a_{h_0} a_{\hat{h}_0})}$$
$$h \in \mathcal{H}_G \setminus \{h_0\}$$

Remark: 5.3.1. $\mathcal{H}_{G(a_{h_0} a_{\hat{h}_0})}$ is a m-family in the graph $G(a_{h_0}\ a_{\hat{h}_0})$. Thus the above remarks concerning $\mathcal{H}_G$ remain valid for $\mathcal{H}_{G(a_{h_0} a_{\hat{h}_0})}$. It should be noted that for

$$h^{h_0} \in \mathcal{H}_{G(a_{h_0} a_{\hat{h}_0})} \qquad \widehat{h^{h_0}} = \hat{h}^{h_0}.$$

5.3.2. The momenta $q_{h^{h_0}}$, $h^{h_0} \in \mathcal{H}_{G(a_{h_0} a_{\hat{h}_0})}$ satisfy:

$$q_{h^{h_0}} = q_h \qquad h \in \mathcal{H}_G \setminus \{h_0\}$$

5.3.3. Let $a^{h_0}_{h_1}$ be an element of $g_{G(a_{h_0} a_{\hat{h}_0})}$.
The above construction applies to $a^{h_0}_{h_1}$, $a^{h_0}_{\hat{h}_1}$ in $G(a_{h_0} a_{\hat{h}_0})$.
The vertices in $G(a_{h_0} a_{\hat{h}_0} | a_{h_1} a_{\hat{h}_1})$ will be denoted by $a^{h_0 h_1}$, a being the corresponding vertex in G.

This procedure can be carried on until all the external vertices are identified.

6. Results

The notation is as in the sections 1. and 5..

Statement 6.1.

$$\mathcal{U}_G \cdot V_G = \sum_{h \in \mathfrak{h}_G} q_h^2 \, W_G^{(a_h | a_{\hat{h}})} + \sum_{\substack{h_1 \neq h_2 \\ h_1, h_2 \in \mathfrak{h}_G}} q_{h_1} \cdot q_{h_2} \left\{ W_G^{(a_{h_1} a_{h_2} | a_{\hat{h}_1} a_{\hat{h}_2})} - W_G^{(a_{h_1} a_{\hat{h}_2} | a_{\hat{h}_1} a_{h_2})} \right\}$$

Remark 6.1.1. This formula describes the effect of the transformation

$$(p_a)_{\substack{a \in g_G \\ \Sigma p_a = 0}} \longrightarrow (q_h)_{h \in \mathfrak{h}_G}$$ on the V_G - function.

Proof: Insertion of 5.2.6. in the definition of V_G shows:

$$\mathcal{U}_G \cdot V_G = \sum_{\substack{\{K, K'\} \\ K \subset g_G}} W_G^{(K|K')} \Big(\sum_{h \in \mathfrak{A}(\mathfrak{h}_G|K)} q_h - \sum_{h \in \mathfrak{A}(\cdot|K')} q_h \Big)^2$$

Let a number $\varepsilon^K_{h_1 h_2}$ be defined by:

$$\varepsilon^K_{h_1 h_2} = \begin{cases} +1 & a_{h_1}, a_{h_2} \in K \text{ and } a_{\hat{h}_1}, a_{\hat{h}_2} \in K' \\ -1 & a_{h_1}, a_{\hat{h}_2} \in K \text{ and } a_{\hat{h}_1}, a_{h_2} \in K' \\ 0 & \text{otherwise} \end{cases} \qquad \begin{matrix} K \subsetneqq g_G \\ h_i \in \mathfrak{h}_G \\ i = 1,2 \end{matrix}$$

From the definition of $\mathfrak{A}(\mathfrak{h}_G|K)$ and $\varepsilon^K_{h_1 h_2}$ it follows immediately that:

$$\mathcal{U}_G \cdot V_G = \sum_{h_1, h_2 \in \mathfrak{h}_G} (q_{h_1} \cdot q_{h_2}) \sum_{\substack{\{K, K'\} \\ K \subset g_G}} \varepsilon^K_{h_1 h_2} \, W_G^{(K|K')}$$

$$= \sum_{h_1, h_2 \in \mathfrak{h}_G} (q_{h_1} \cdot q_{h_2}) \left\{ \sum_{\substack{K \supset \{a_{h_1}, a_{h_2}\} \\ K' \supset \{a_{\hat{h}_1}, a_{\hat{h}_2}\}}} W_G^{(K|K')} - \sum_{\substack{K \supset \{a_{h_1}, a_{\hat{h}_2}\} \\ K' \supset \{a_{\hat{h}_1}, a_{h_2}\}}} W_G^{(K|K')} \right\}$$

By use of Remark 2.2.2. the statement is proved at once.

Statement 6.2. Let $h \neq g_G$ be some element of $\mathfrak{g}_G$.
Then an identity holds:

$$V_G = \frac{U_{G(a_h a_{\hat h})}}{U_G}\left[q_h + \sum_{\substack{h_1\neq h\\ h_1\in\mathfrak{g}_G}} q_{h_1} \frac{W_G^{(a_h a_{h_1}|a_{\hat h} a_{\hat h_1})} - W_G^{(a_h a_{\hat h_1}|a_{\hat h} a_{h_1})}}{W_G^{(a_h|a_{\hat h})}}\right]^2 + V_{G(a_h a_{\hat h})}$$

Proof: Let A denote the first term on the r.h. side of the above identity. Then 6.1. gives:

$$V_G = A + \frac{(U_G)^{-1}}{W_G^{(a_h|a_{\hat h})}} \sum_{h_1,h_2\neq h} (q_{h_1}\cdot q_{h_2}) \Big\{ W_G^{(a_h|a_{\hat h})} \Big[W_G^{(a_{h_1} a_{h_2}|a_{\hat h_1} a_{\hat h_2})} - W_G^{(a_{h_1} a_{\hat h_2}|a_{\hat h_1} a_{h_2})}\Big]$$

$$- \Big[W_G^{(a_h a_{h_1}|a_{\hat h} a_{\hat h_1})} - W_G^{(a_h a_{\hat h_1}|a_{\hat h} a_{h_1})}\Big]\Big[h_1 \to h_2\Big]\Big\}$$

Application of the crucial proposition 3.1. shows:

$$V_G = A + \frac{1}{W_G^{(a_h|a_{\hat h})}} \sum_{\substack{h_1,h_2\neq h\\ \in\mathfrak{g}_G}} (q_{h_1}\cdot q_{h_2}) \Big\{ W_{G(a_h a_{\hat h})}^{(a^h_{h_1} a^h_{h_2}|a^h_{\hat h_1} a^h_{\hat h_2})} - W_{G(a_h a_{\hat h})}^{(a^h_{h_1} a^h_{\hat h_2}|a^h_{\hat h_1} a^h_{h_2})}\Big\}$$

Remarks 5.3.1.,5.3.2. together with 6.1. now applied to the graph $G(a_h\ a_{\hat h})$ prove 6.2.. (It should be noted that $W_G^{(a_h|a_{\hat h})} = U_{G(a_h a_{\hat h})}$ (see2.5.))

Statement 6.3. Let the elements of $\mathfrak{g}_G$ h_i , $i=1,..,|g_G|$ $h_{|g_G|} = g_G$ be ordered in some fashion indicated by the indices i. Define $G(a_{h_0}\ a_{\hat h_0})$ to be the graph G and W_{ij} to be the function

$$W_{G(a_{h_1}a_{\hat h_1}|\cdots|a_{h_{i-1}}a_{\hat h_{i-1}})}^{(a_{h_i}^{h_1\cdots h_{i-1}}\, a_{h_j}^{h_1\cdots h_{i-1}}|a_{\hat h_i}^{h_1\cdots h_{i-1}}\, a_{\hat h_j}^{h_1\cdots h_{i-1}})} - W_{G(a_{h_1}a_{\hat h_1}|\cdots|a_{h_{i-1}}a_{\hat h_{i-1}})}^{(a_{h_i}^{h_1\cdots h_{i-1}}\, a_{\hat h_j}^{h_1\cdots h_{i-1}}|a_{\hat h_i}^{h_1\cdots h_{i-1}}\, a_{h_j}^{h_1\cdots h_{i-1}})}$$

Then the following formula is valid:

$$V_G = \sum_{i=1}^{|g_G|-1} \frac{U_{G(a_{h_1}a_{\hat h_1}|\cdots|a_{h_i}a_{\hat h_i})}}{U_{G(a_{h_1}a_{\hat h_1}|\cdots|a_{h_{i-1}}a_{\hat h_{i-1}})}}\left[q_{h_i} + \sum_{j>i} q_{h_j}\frac{W_{ij}}{U_{G(a_{h_1}|\cdots|a_{h_i}a_{\hat h_i})}}\right]^2$$

Proof: Successive application of 6.2. and use of 5.3.3. prove the statement in a trivial manner.

This formula is the desired diagonalization of V_G. If new momenta

$$q'_{h_i} = q_{h_i} + \sum_{j>i} q_{h_j} \frac{W_{ij}}{U_{G(a_{h_1}|\cdots|\cdot a_{\hat{h}_i})}} \qquad i,j = 1,\dots,|g_G|$$

are introduced, V_G is equal to

$$V_G = \sum_{i=1}^{|g_G|-1} \frac{U_{G(a_{h_1}a_{\hat{h}_1}|\cdots|a_{h_i}a_{\hat{h}_i})}}{U_{G(a_{h_1}|\cdots|\cdot a_{\hat{h}_{i-1}})}} \left(q'_{h_i}\right)^2$$

Remark 6.3.1. The transformation $q_{h_i} \longrightarrow q'_{h_i}$, $i=1,\dots,|g_G|-1$, is a nonsingular linear mapping of $R^{4|g|-4}$ onto $R^{4|g|-4}$ and depends continously on the Feynman-parameter $\underline{\alpha}$ in the range $\alpha_\ell \geq 0$, $\ell \in \mathcal{L}(G)$.

6.3.2. The coefficient of $(q'_{h_i})^2$ is equal to the quotient of the U-functions corresponding to the graphs $G(a_{h_1} a_{\hat{h}_1}| \dots | a_{h_j} a_{\hat{h}_j})$ $j=i,i-1$. $G(a_{h_1} a_{\hat{h}_1}|\dots|a_{h_\ell} a_{\hat{h}_\ell})$ is obtained by identifying the vertices

$$a_{h_\ell}^{h_1\cdots h_{\ell-1}} \quad \text{and} \quad a_{\hat{h}_\ell}^{h_1\cdots h_{\ell-1}} \qquad \ell = 1,\dots,j$$

in $G(a_{h_1} a_{\hat{h}_1}|\dots|a_{h_{\ell-1}} a_{\hat{h}_{\ell-1}})$. Each coefficient is a continous function of $\underline{\alpha}$ in the same range as above.

6.3.3. The numerator of the coefficient of $(q'_{h|g|-1})^2$ is the U-function of the graph G(g). All the external vertices are identified in this graph. G(g) can replace the graph G_∞ in [1]. The singularity-families $\mathcal{F}_\infty$ of [1] can be constructed out of G(g). The unique external vertex g_o of G(g) plays the rôle of the vertex v_∞ of [1] and the notion " a subgraph $H \subset G^o$ is irreducible in view of infinity " of [1] corresponds to the notion "a subgraph $H(g) \subset G(g)$ is g_o-irreducible [8] ". The graphs G(g) and G_∞ are equivalent in view of the con_ struction of $(b_\infty, \sigma_\infty)$ of [1]. The partial ordering of the Feynman-parameter $\alpha_\ell, \ell \in \mathcal{L}(G)$ induced by the families $(b_\infty, \sigma_\infty)$ gives a simultaneous resolution of singularities for all functions $U_{G(a_{k_1}|\cdots|\cdot a^{\wedge}_{k_{j-1}})}$ $j=1,\ldots,|g_G|$.

The following statements describe the effect of the transformations

$$(p_a)_{a\in g_G} \longrightarrow (q_h)_{h\in \mathfrak{h}_G} \longrightarrow (q_h')_{h\in \mathfrak{h}_G})$$

on the spin-polynomial of an arbitrary line.

Statement 6.4. Let $l \in \mathcal{L}(G)$ be an arbitrary line. Then the following formula is valid:

$$Y_G^\ell = \frac{1}{U_G} \sum_{h\in \mathfrak{h}_G} q_h \cdot \sum_{P\in \mathbb{P}_G(a^{\wedge}_h a_h)} [P:\ell]\, U_{G/P}$$

Proof: The momenta q_h ,$h \in \mathfrak{h}_G$,$h \neq g_G$, can be independently prescribed. Thus it suffices to consider some term $h \in \mathfrak{h}_G \setminus \{g_G\}$ and to impose the condition $q_{h_1} = 0$,$h_1 \in \mathfrak{h}_G \setminus \{h\}$. The construction of $\mathfrak{h}_G$ implies:

$$p_{a_h} = q_h \,, \quad p_{a^{\wedge}_h} = -q_h \,, \quad p_b = 0 \quad \forall b \in g_G \setminus \{a_h, a^{\wedge}_h\}$$

In this case the spin-polynomial for line l is equal to

$$Y_G^\ell = \frac{1}{U_G} \sum_{P\in \mathbb{P}_G(a^{\wedge}_h a_h)} [P:\ell]\, U_{G/P} \cdot p_{a_h}$$

$$= \frac{1}{U_G} \sum_{P\in \mathbb{P}_G(a^{\wedge}_h a_h)} [P:\ell]\, U_{G/P} \cdot q_h$$

Q.E.D.

Statement 6.5. Let $h \neq g_G$ be some element of $\mathfrak{h}_G$. Then

$$Y_G^\ell = \frac{1}{U_G} \sum_{P\in \mathbb{P}_G(a^{\wedge}_h a_h)} [P:\ell]\, U_{G/P} \left\{ q_h + \sum_{\substack{h_1\in \mathfrak{h}_G \\ h_1 \neq h}} q_{h_1} \frac{W_G^{(a_{h_1} a_h | a^{\wedge}_{h_1} a^{\wedge}_h)} - W_G^{(a_{h_1} a^{\wedge}_h | a^{\wedge}_{h_1} a_h)}}{W_G^{(a_h | a^{\wedge}_h)}} \right\} + Y^\ell_{G(a_h a^{\wedge}_h)}$$

Proof: Define B to be the first term of the r.h. side and let b_1, b_2 be the endpoints of line l. Combination of the propositions 3.2. and 6.4. gives:

$$\mathcal{U}_G \cdot Y_G^{\ell} = B \cdot \mathcal{U}_G + \frac{1}{\alpha_\ell} \sum_{h_1 \neq h} q_{h_1} \frac{[b_2 : \ell]}{W_G^{(a_h | a_{\hat{h}})}} \Big\{ W_G^{(a_h | a_{\hat{h}})} \Big[W_G^{(a_{\hat{h}_1} b_1 | a_{h_1} b_2)} - W_G^{(a_{\hat{h}_1} b_2 | a_{h_2} b_1)} \Big] -$$

$$- \Big[W_G^{(a_h b_2 | a_{\hat{h}} b_1)} - W_G^{(a_h b_1 | a_{\hat{h}} b_2)} \Big] \Big[W_G^{(a_{h_1} a_h | a_{\hat{h}_1} a_{\hat{h}})} - W_G^{(a_{h_1} a_{\hat{h}} | a_{\hat{h}_1} a_h)} \Big]$$

3.1. shows

$$Y_G^{\ell} = B + \frac{1}{\alpha_\ell} \sum_{h_1 \neq h} q_{h_1} \frac{[b_2 : \ell]}{W_G^{(a_h | a_{\hat{h}})}} \Big\{ W_{G(a_h a_{\hat{h}})}^{(a^h_{\hat{h}_1} a^h_{b_1} | a^h_{h_1} a^h_{b_2})} - W_{G(a_h a_{\hat{h}})}^{(a^h_{\hat{h}_1} a^h_{b_2} | a^h_{h_1} a^h_{b_1})} \Big\}$$

Comparison of 5.3.1., 5.3.2. and 3.2, 6.4. - now applied to the graph G ($a_h\, a_{\hat{h}}$)- establishes the statement if the equality

$$[b_2 : \ell] \{ \cdots \} = [b_2^h : \ell] \{ \cdots \}$$

can be proved.

First, consider the case in which $\{b_1, b_2\} = \{a_h, a_{\hat{h}}\}$. This implies that the bracket $\{\cdots\}$ vanishes as well as $[b_2^h : \ell] = [b_2 : \ell] + [b_1 : \ell]$ .Thus the equality holds for this special configuration of $b_1, b_2, a_h, a_{\hat{h}}$. In all other cases it may be assumed that $b_2 \notin \{a_h, a_{\hat{h}}\}$. Then $[b_2^h : \ell] = [b_2 : \ell]$ (see 2.4). Therefore the equality is valid.

Q.E.D.

<u>Statement 6.6.</u> Let h_i i = $1, -, |g_G|$ and q'_{h_i} be defined as in 6.4. Successive application of 6.5. to the sequence of graphs

G , G ($a_{h_1}\, a_{\hat{h}_1}$), ..., G ($a_{h_1} a_{\hat{h}_1} | \cdots | a_{\hat{h}_{|g|-1}}$)= G (g) yields the identity:

$$Y_G^{\ell} = \sum_{i=1}^{|g|-1} q'_{h_i} \frac{1}{\mathcal{U}_{G(a_{h_1} | \cdots | a_{\hat{h}_{i-1}})}} \Big\{ \sum_{P \in \mathbb{P}_{G(a_{h_1} | \cdots | a_{\hat{h}_{i-1}})}(a_{\hat{h}_i}^{h_1 \cdots h_{i-1}}\, a_{h_i}^{h_1 \cdots h_{i-1}})}^{P} [P : \ell]\, \mathcal{U}_{G(a_{h_1} | \cdots | a_{\hat{h}_{i-1}}) / P} \Big\}$$

Remark:

The similarity between 6.6. and 6.4. should be noted. It is the sequence of graphs

$$G\ ,\ G(a_{h_1}\, a_{\hat{h}_1})\ ,\ldots\ldots,\ G(a_{h_1}\, a_{\hat{h}_1} | \ldots\ldots | a_{h_{|g|_G-1}}\, a_{\hat{h}_{|g|_G-1}})$$

which characterizes both formula. The U - functions of these graphs contain all the information needed to give a satisfactory treatment of the analytic and dimensional renormalization of massless field theories along lines similar to [3,4] .

7. References and Footnotes

[1] Pohlmeyer, K.: Desy - Preprint 74/36

[2] Nakanishi, N.: "Graph Theory and...."; Gordon and Breach, Science Publ., Inc., New York 197o

[3] Speer, E.R.: "Generalized Feynman Amplitudes", Princeton University Press, Princeton 1969

[4] Speer, E.R.: Comm. math. phys 37, 83 (1974)

[5] Gröbner, W.: "Matrizenrechnung", B.I. Hochschultaschenbücher 1o3/1o3a, Mannheim 1966

[6] Poenaru, V.: "Analyse Différentielle", Lecture notes in mathematics 371, Springer-Verlag (1974)

Hironaka, N.: Ann. of Math. 79, 1o9 (1964)

[7] Trute, H. : Forthcoming Desy - preprint

[8] A subgraph H(g) of G(g) is called g_0-irreducible if it satisfies:

a) For each $l \in \mathcal{L}(G(g))$ the graph $G(g)/(\mathcal{L}(G) \setminus \{l\})$ is connected.

b) For each $v \in v(G(g)) \setminus \{g_0\}$ the graph

$$\left(v(G(g)) \setminus \{v\},\ \mathcal{L}(G(g)) \setminus S(v),\ \varphi_{G(g)}\big|_{\mathcal{L}(G) \setminus S(v)}\right)$$

is connected.