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Mass Formulas for Broken SU(4)

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Mass Formulas for Broken SU(4)

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Abstract: We derive mass formulas for broken SU(4) using Okubo's tensor formalism. The mass splitting term is a superposition of an SU(3) scalar operator and the I = 0 component of the SU(3) octet. To accomodate the new resonances, mixing between the SU(4) singlet and fifteenplet is discussed. In addition we consider also the electromagnetic mass shift.

1. Introduction

Recently SU(4) symmetry already considered earlier [1] had a revival

in connection with the discovery of the new mesons $\Psi(3.1)$, $\Psi(3.7)$

and $\Psi(4.1)$ [2]. One way to understand these new resonances is their

interpretation as vector mesons composed predominantly of charmed

quarks. The basic symmetry of strong interaction then becomes SU(4)

instead of SU(3). One considers the Ψ as part of an SU(4) fifteenplet

together with the usual vector mesons ϱ, K^*, ω and ϕ .

Because of the large mass differences inside such a multiplet SU(4)

must be badly broken. In principle the structure of the symmetry

breaking term is unknown. To incorporate the usual well satisfied

SU(3) mass splitting, we assume that the breaking operator H' has

a term H_8 which transforms as the $I = 0$ component of the SU(3)

octet. The mass splitting of the different SU(3) submultiplets inside

an SU(4) multiplet is described by H_{15} which transforms as the

SU(3) scalar component of the SU(4) fifteenplet, so that $H' = H_8 + H_{15}$.

For the tensors of SU(4) we use the Okubo notation. Then it is

easy to derive the Wigner-Eckart theorem for matrix elements

of SU(4) tensor operators within a given SU(4) multiplet. We find that

the matrix elements of H' and also of the electromagnetic current

J_μ can be expressed by the eigenvalues of various operators

which appear in the reduction SU(4) $\supset$ SU(3) $\otimes$ U(1) and

SU(4) $\supset$ SU(2) $\otimes$ SU(2) $\otimes$ U(1). [3]

In section 2 we derive the mass formulas for broken SU(4) neglecting

electromagnetism and apply them to the 15-plet representation of

the vector mesons. In section 3 we consider mixing between singlet

and fifteenplet which leads to a five parameter mass formula. [+]

[+] This problem has been considered also in ref. 4.

This reduces to a three parameter formula if ideal mixing
is imposed. In section 4 we take up the problem of electromagnetic
mass splitting for both, the unmixed and the mixed case.
In section 5 we draw some conclusions.

2. Mass Formulas for Broken SU(4)

In this section we shall derive mass formulas for SU(4) multiplets
which are the generalization to SU(4) of the well-known Okubo
mass formula in SU(3). We shall consider the mass splitting inside
the full SU(4) multiplet and not just the mass splitting between
SU(3) submultiplets [5]. In SU(3) the mass operator H is

$$H = H_o + H_1 \quad , \tag{2.1}$$

where H_o is the SU(3) symmetric part and H_1 transforms like
the isoscalar generator of SU(3), the hypercharge operator $Y_8 \sim F_8$.
In analogy in SU(4) the mass operator is

$$H = H_o + H_1 + H_2 \quad , \tag{2.2}$$

where H_o is now a SU(4) scalar and H_1 and H_2 transform as
F_8 and F_{15} . F_8 is the isoscalar generator of the SU(3) octet
and F_{15} is the SU(3) - scalar generator of SU(4).
For the derivation we need the properties of the SU(4) tensor
operators. We shall use the Okubo notation for these. Then it is a
simple matter to derive the Wigner-Eckart theorem for matrix elements
of a SU(4) tensor operator within a given SU(4) multiplet.

Usually one denotes the SU(4) generators by F_i (i = 1,2,....., 15) which obey

$$\left[F_i, F_j \right] = i f_{ijk} F_k \qquad (2.3)$$

and which can be represented in terms of a set of standard hermitean matrices λ_i by $F_i = \frac{1}{2} \lambda_i$ when acting on the quadruplet representation of SU(4). In the Okubo notation the SU(4) generators are given by a set of sixteen operators F^i_j (i,j = 1,2,3,4)

$$F^i_j = \sum_{n=1}^{15} F_n \left(\lambda_n \right)_{ij} \qquad (2.4)$$

which satisfy

$$\left[F^i_j , F^k_1 \right] = \delta^i_\ell F^k_j - \delta^k_j F^i_\ell \qquad (2.5)$$

$$\sum_{i=1}^{4} F^i_i = 0 \qquad (2.6)$$

$$(F^i_j) = (F^j_i)^+ \qquad (2.7)$$

The F^i_j transform against charge conjugation $\mathscr{C}$ as:

$$\mathscr{C} F^i_j \mathscr{C}^\dagger = - F^j_i . \qquad (2.8)$$

Now, for any regular tensor operator T^i_j we have within a given

irreducible representation R of SU(4) [6]

$$T^i_{\ j} = a\ F^i_{\ j}\ + b \sum_{k=1}^{4} F^i_{\ k}\ F^k_{\ j}\ +\ c \sum_{k,\ell=1}^{4} F^i_{\ k}\ F^k_{\ 1}\ F^1_{\ j}\ , \qquad (2.9)$$

where a, b and c are reduced matrix elements. In the special case

that $R \otimes 15$ contains R at most twice, which covers all the cases

of physical interest, we drop the last term ($c = o$), so that we have

only two reduced matrix elements.

In accordance with (2.2) the mass operator H is:

$$H = m_o T_o\ +\ m_{33}\ T^3_{\ 3}\ +\ m_{44}\ T^4_{\ 4}\ , \qquad (2.10)$$

where T_o is the SU(4) symmetric part, while $T^3_{\ 3}$ transforms

like a linear combination of F_8 and F_{15}, and $T^4_{\ 4}$ is proportional

to F_{15}. The exact relation will be given below.

As we show later $T^4_{\ 4}$ can be expressed in terms of those operators

whose eigenvalues label the states of the SU(3) submultiplets. This is no

longer true for the second term $T^3_{\ 3}$ which describes the SU(3) splitting.

The SU(3) states are characterized by the eigenvalues of I^2, I_3 and Y_8 .

But $T^3_{\ 3}$ cannot be expressed by them . On the contrary $T^3_{\ 3}$ connects

states of the same I^2, I_3 and Y_8 within different SU(3) submultiplets

of an SU(4) multiplet. This is similar to the situation occurring in

SU(3) in connection with the electromagnetic mass splitting where

different I-spin multiplets are mixed. It is wellknown that the

electromagnetic mass splitting in SU(3) is simplified considerably if

another SU(2) subgroup of SU(3) , namely U-spin, is used to label the

states. The situation is quite similar in connection with T^3_3 where part

of it can be expressed by operators of another SU(2) subgroup of SU(4),

called Z-spin.

Therefore we express T^3_3 and T^4_4 in (2.10) through the generators

of the different SU(2) - subgroups of SU(4)[3]. Whereas in SU(3) we

had only three SU(2) - subgroups defining the I-, U- and V-spin,

respectively, in SU(4) we have six different SU(2) -subgroups which

define six spins: I,U,V,W,X and Z:

$$I_- = F^1_2 = F_1 - iF_2 \ , \qquad\qquad I_+ = F^2_1 = F_1 + iF_2 \ ,$$
$$I_3 = \frac{1}{2}(F^1_1 - F^2_2) = F_3 \ ;$$

$$U_- = F^2_3 = F_6 - iF_7 \ , \qquad\qquad U_+ = F^3_2 = F_6 + iF_7 \ ,$$
$$U_3 = \frac{1}{2}(F^2_2 - F^3_3) = -\frac{1}{2}F_3 + \frac{\sqrt{3}}{2}F_8 \ ;$$

$$V_- = F^1_3 = F_4 - iF_5 \ , \qquad\qquad V_+ = F^3_1 = F_4 + iF_5 \ , \qquad\qquad (2.11)$$
$$V_3 = \frac{1}{2}(F^1_1 - F^3_3) = \frac{1}{2}F_3 + \frac{\sqrt{3}}{2}F_8 \ ;$$

$$W_- = F^1_4 = F_9 - iF_{10} \ , \qquad\qquad W_+ = F^4_1 = F_9 + iF_{10} \ ,$$
$$W_3 = \frac{1}{2}(F^1_1 - F^4_4) = \frac{1}{2}F_3 + \frac{1}{2\sqrt{3}}F_8 + \sqrt{\frac{2}{3}}F_{15} \ ;$$

$$X_- = F^2_4 = F_{11} - iF_{12} \ , \qquad\qquad X_+ = F^4_2 = F_{11} + iF_{12} \ ,$$
$$X_3 = \frac{1}{2}(F^2_2 - F^4_4) = -\frac{1}{2}F_3 + \frac{1}{2\sqrt{3}}F_8 + \sqrt{\frac{2}{3}}F_{15} \ ;$$

$$Z_- = F^3_4 = F_{13} - iF_{14} \ , \qquad\qquad Z_+ = F^4_3 = F_{13} + iF_{14} \ ,$$
$$Z_3 = \frac{1}{2}(F^3_3 - F^4_4) = -\frac{1}{\sqrt{3}}F_8 + \sqrt{\frac{2}{3}}F_{15} \ ;$$

$$F_8 = -\frac{1}{2\sqrt{3}}(3F^3_3 + F^4_4), \qquad\qquad F_{15} = -\sqrt{\frac{2}{3}}F^4_4 \ .$$

It is now easy to obtain from (2.9) for c = o:

$$T^3_{\ 3} = (a-2b)\left(-Y_8 + \tfrac{1}{3} Y_{15}\right) + \tfrac{b}{3}\left[\, 3(\vec{Z}^2 - \vec{I}^2) + (\mathscr{C}_3 - Y_{15}^{\ 2})\,\right], \qquad (2.12)$$

$$T^4_{\ 4} = b\,\mathscr{C}_4 - (a-2b)\,Y_{15} - \tfrac{b}{3}\left(\mathscr{C}_3 - Y_{15}^{\ 2}\right). \qquad (2.13)$$

In (2.12) and (2.13) $\mathscr{C}_4$ is one of the Casimir operators of SU(4)

$$\mathscr{C}_4 = \sum_{i=1}^{15} \left(\,F_i\,\right)^2, \qquad (2.14)$$

and $\mathscr{C}_3$ is one of SU(3),

$$\mathscr{C}_3 = 3\sum_{i=1}^{8} \left(\,F_i\,\right)^2, \qquad (2.15)$$

whereas Y_8 and Y_{15} stand for

$$Y_8 = \frac{2}{\sqrt{3}}\,F_8 \quad , \qquad Y_{15} = \sqrt{\frac{3}{2}}\,F_{15} \; . \qquad (2.16)$$

According to (2.10) the mass operator is a linear combination of $T^3_{\ 3}$ and $T^4_{\ 4}$ together with an invariant term.

Finally we get the following four-parameter formula for the mass splitting:

$$H = m + (m_1 - m_2)(\mathscr{C}_3 - Y_{15}^2) + 3m_1(\vec{Z}^2 - \vec{I}^2) \qquad (2.17)$$
$$+ 3f m_1 Y_8 - f(m_1 - 3m_2) Y_{15} \ .$$

Here the contribution of $\mathscr{C}_4$ has been lumped together with that from T_0, and we have introduced $m_1 = m_{33} b/3$, $m_2 = m_{44} b/3$, $f = (2b-a)/b$. For mesons (2.17) stands for the squares of the masses. Since mesons are assigned to self conjugate multiplets the C odd terms in (2.17) must vanish, i.e. $f = 0$. For baryons we take (2.17) as a formula for the masses themselves where C even and odd terms contribute.

To apply the mass formula to actual cases we need the classification of the SU(4) states in terms of the eigenvalues for the operators appearing in (2.17). According to the usual reduction scheme $SU(4) \supset SU(3) \otimes U(1)$ we can choose for the classification of states the irreducible SU(3) representations (p,q) (in the "highest weight" notation) contained in a given SU(4) multiplet. The states within the SU(3) representations are classified as usual by I, I_3 and Y_8 . For the additional U(1) we have then Y_{15} , which commutes with the SU(3) generators F_1, F_2 ... F_8. The Casimir operator $\mathscr{C}_3$ when acting on states of an SU(3) multiplet (p,q) has the eigenvalues [3]

$$\mathscr{C}_3(p,q) = p^2 + q^2 + pq + 3p + 3q \ . \qquad (2.18)$$

For the evaluation of the third term in (2.17) we need the classification of SU(4) states according to the reduction SU(4) $\supset$ SU$_I$(2) $\otimes$ SU$_Z$(2) $\otimes$ U$_P$(1), where $P = Y_8 + \frac{2}{3} Y_{15}$ commutes with all components of $\vec{I}$ and $\vec{Z}$. Since Y_8 and Y_{15} do not commute with the $\vec{Z}$ operators the formula (2.17) automatically leads to mixing between states belonging to different SU(3) multiplets.

In the following we shall apply (2.17) to problems of current interest. There our SU(4) operators Y_8 and Y_{15} are connected in a specific form to the physical operators for hypercharge Y, charm C and baryon number B:

$$B = \frac{2\sqrt{2}}{3} F_o \qquad \left(F_o = \frac{1}{2} \lambda_o = \frac{1}{2\sqrt{2}} \mathbb{1} \right)$$

$$Y = \frac{1}{4} B + Y_8 - \frac{1}{3} Y_{15} \, , \qquad\qquad (2.19)$$

$$C = \frac{3}{4} B - Y_{15} \, .$$

The charge Q is then

$$Q = I_3 + \frac{1}{2}(Y + C) = I_3 + \frac{1}{2}(B + Y_8) - \frac{2}{3} Y_{15} \, . \qquad (2.20)$$

The definition (2.19) for Y and C corresponds to the usual assignment of the fundamental quark representation (u,d,s,c) with $3Y$ = diag (1,1,-2,1) , C = diag (0,0,0,1) and $3Q$ = diag (2,-1,-1,2).

As an example let us work out the mass matrix for the 15-plet of the

vector mesons ϱ, K^*, ω, ϕ, D^* and F^*.

The classification of these states as SU(3) multiplets and their SU(2) quantum numbers for I, U, W and Z spin together with Y, C and $P \approx Y - C$ are given in table 1 . We listed also the U and W spins. They will be used later in connection with the electromagnetic mass splitting.

As is well known the SU(4) fifteenplet consists of an SU(3) singlet, octet, triplet and antitriplet. The triplet has C = 1 and consists of D^{*^+}, D^{*^0} and F^{*^+}. The I = 0 member of the octet is denoted by ω_8, whereas ω_{15} is the SU(3) singlet component . The physical ω and ϕ will be the mixings of ω_8 and ω_{15} .

In terms of these states the I-, Z-, U- and, W- multiplets are:

I-spin: singlets ω_8; ω_{15}; F^{*+}; F^{*-};

doublets (K^{*+}, K^{*0}); $(\overline{K^{*0}}, - K^{*-})$;

$(\overline{D^{*0}}, D^{*-})$; $(D^{*+}, - D^{*0})$;

triplet $(-\varrho^+, \varrho^0, \varrho^-)$. $\qquad$ (2.21)

Z-spin: singlets ϱ^+; ϱ^0; ϱ^-; $\sqrt{\tfrac{2}{3}}\, \omega_8 + \tfrac{1}{\sqrt{3}}\, \omega_{15}$;

doublets (K^{*-}, D^{*0}); $(\overline{D^{*0}}, - K^{*+})$;

$(\overline{K^{*0}}, D^{*+})$; $(D^{*-}, - K^{*0})$;

triplet $\left(-F^{*-}, -\tfrac{1}{\sqrt{3}}\, \omega_8 + \sqrt{\tfrac{2}{3}}\, \omega_{15}, F^{*+}\right)$. $\qquad$ (2.22)

U-spin: singlets ω_{15}; D^{*0}; $\overline{D^{*0}}$; $\tfrac{1}{2}\left(\omega_8 + \sqrt{3}\, \varrho^0\right)$;

doublets (ϱ^-, K^{*-}); $(K^{*+}, -\varrho^+)$;

(D^{*-}, F^{*-}); $(F^{*+}, - D^{*+})$;

triplet $\left(- K^{*0}, \tfrac{1}{2}(\sqrt{3}\, \omega_8 - \varrho^0), \overline{K^{*0}}\right)$. $\qquad$ (2.23)

W-spin: singlets K^{*0}; $\overline{K}^{*0}$; $\frac{1}{2}(\sqrt{3}\,\omega_8 - \varrho^0)$; $\frac{1}{\sqrt{6}}(\sqrt{3}\,\varrho^0 + \omega_8 - \sqrt{2}\,\omega_{15})$;

          doublets (ϱ^+, D^{*+}); $(D^{*-}, -\varrho^-)$;

                      $(K^{*+}, \mp^{*+})$; $(\mp^{*-}, -K^{*-})$;

          triplet $\left(-\overline{D}^{*0}, \frac{1}{\sqrt{12}}(\sqrt{3}\,\varrho^0 + \omega_8 + 2\sqrt{2}\,\omega_{15}), D^{*0}\right)$ (2.24)

From (2.21) we see that the particles belonging to the same I-spin multiplet have the same quantum numbers for Y and C, according to the reduction $SU(4) \supset SU_I(2) \otimes U_Y(1) \otimes U_C(1)$.

Similarly the other $SU(2)$ reductions are:

$SU(4) \supset SU_Z(2) \otimes U_{Q-C}(1) \otimes U_{Q-Y}(1)$, $SU(4) \supset SU_U(2) \otimes U_Q(1) \otimes U_C(1)$ and $SU(4) \supset SU_W(2) \otimes U_Q(1) \otimes U_Y(1)$.

Of course $Y - C$ is equivalent to $P = Y_8 + \frac{2}{3} Y_{15}$.

Then for U- and W-spin the reduction is $SU(4) \supset SU_U(2) \otimes SU_W(2) \otimes U_Q(1)$.

All these relations are intimately connected with our quantum number assignment and classification of the fundamental quark representation $4 = (1,0,0)$ of $SU(4)$. The I-, Z-, U- and W-spin content of the quark representation is exhibited in table 2. Then the application of the operators I_+, Z_+, U_+ and W_+ on the quark representation has the effect: $I_+ d = u$, $Z_+ c = s$, $U_+ s = d$, $W_+ c = u$ for the quarks. Similarly for the antiquarks: $I_+ \bar{u} = -\bar{d}$, $Z_+ \bar{s} = -\bar{c}$, $U_+ \bar{d} = -\bar{s}$, $W_+ \bar{u} = -\bar{c}$.

We apply now (2.17) to the vector meson 15-plet. From (2.22) we see that all states except ω_8 and ω_{15} are also diagonal in Z-spin. Therefore we have the following equations for the squared masses of ϱ, K^*, D^* and F^* (denoted by the particle symbols):

$$\varrho \quad = \quad m + 3m_1 - 9m_2 \quad ,$$

$$K^* \quad = \quad m + 9m_1 - 9m_2 \quad ,$$

$$D^* \quad = \quad m + 3m_1 - 3m_2 \quad , \qquad (2.25)$$

$$F^* \quad = \quad m + 9m_1 - 3m_2 \quad ,$$

whereas the matrix M for the singlet - octet mixing has the form:

$$M \quad = \quad \begin{pmatrix} m + 11 m_1 - 9 m_2 & -2\sqrt{2}\, m_1 \\ -2\sqrt{2}\, m_1 & m + 4 m_1 \end{pmatrix} . \qquad (2.26)$$

The eigenvalues of this matrix are identified with the squared masses of the physical ω and ϕ meson. (2.25) yields one mass relation through elimination of m_o, m_1 and m_2

$$\varrho + F^* = K^* + D^* . \qquad (2.27)$$

From (2.25) and (2.26) two further relations follow if one uses the trace which is equal to $\omega + \phi$ and the determinant of (2.26) which is equal to $\omega\phi$. These are

$$3F^* + \varrho \quad = \quad 2\left(\omega + \phi\right) , \qquad (2.28)$$

$$\left(\omega - \varrho\right)\left(\phi - \varrho\right) = \frac{2}{3}\left(K^* - \varrho\right)\left(2\omega + 2\phi - \varrho - 3K^*\right). \qquad (2.29)$$

The eigenfunctions of ω and ϕ are then respectively (up to normalization):

$$|\omega\rangle \sim (m + 4m_1 - m_\omega)|\omega_8\rangle + 2\sqrt{2}\, m_1 |\omega_{15}\rangle,$$

$$|\phi\rangle \sim (m + 4m_1 - m_\phi)|\omega_8\rangle + 2\sqrt{2}\, m_1 |\omega_{15}\rangle. \qquad (2.30)$$

The formulas (2.27) - (2.29) have been given before [7]. If the SU(3) splitting term proportional to T^3_3 is neglected we have [5]

$$3M_3 = M_8 + 2M_1 \qquad (2.31)$$

where M_1, M_3 and M_8 are the average mass squared of the SU(3) singlet, triplet and octet, respectively.

Already (2.31) shows that the average mass squared of the charmed triplet (D^{*+}, D^{*0}, F^{*+}) is rather low of the order of 1 GeV^2.

For completeness we give solutions of (2.27) to (2.29) using the ω, ϕ and K^* masses as input. The two solutions are (masses in GeV):

(i) $m_\rho = 0.64$, $m_{D*} = 0.76$, $m_{F*} = 0.98$;

(ii) $m_\rho = 1.07$, $m_{D*} = 1.06$, $m_{F*} = 0.85$.

The SU(4) symmetry breaking has the nice feature to lead to definite SU(3) singlet - octet mixing for ω and ϕ (see 2.30).

On the other hand the results (i) and (ii) are not tenable since the ρ-mass cannot be fitted. The solution of the SU(4) mass breaking for the vector mesons lies in additional SU(4) singlet - fifteenplet mixing which will be discussed in the next section.

Here we still note that the mass formula (2.17) , when applied to the quark representation gives, according to table 2:

$$\frac{1}{2}(m_u + m_d + m_s + m_c) = m , \quad m_s - m_u = -3fm_1,$$

$$m_u = m_d, \qquad\qquad m_c - m_u = -3fm_2, \qquad (2.32)$$

so that $R \equiv (m_c - m_u) / (m_s - m_u) = m_2/m_1$. In deducing (2.32)
we have taken into account the fact that for the fundamental
representation $\underline{4}$ only one reduced matrix element appears in (2.9),
i.e. $b = c = 0$, since $\underline{15}$ is contained in $\underline{4} \times \underline{\bar{4}}$ only onces.

3. Mixing Analysis.

In the last section we have seen that a mass formula based on SU(4)
is contradicted by the empirical masses of the vector mesons. With
the discovery of the new vector meson $\Psi(3095)$ a new situation
arose. There are now three $I = 0$ vector mesons ω, ϕ and Ψ which
in principle could be mixed states. Apparently the Ψ has no SU(4)
partners. It is appropriate then to consider the mixing of the
fifteenplet with a singlet. In this case the mass matrix (2.26)
is replaced by

$$M = \begin{pmatrix} m + 11\,m_1 - 9\,m_2 & -2\sqrt{2}\,m_1 & A \\ -2\sqrt{2}\,m_1 & m + 4m_1 & B \\ A & B & m_o \end{pmatrix}. \qquad (3.1)$$

m_o stands for the SU(4) singlet mass squared and A and B are
the nondiagonal matrix elements which mix the singlet ω_o with ω_8
and ω_{15}, respectively. (3.1) has six free parameters whereas only
five masses are known, ϱ, K^*, ω, ϕ and Ψ, to be used as input.
Therefore (3.1) has no predictive power without further assumptions.

(2.17) shows that F_8 and F_{15} appear in a definite ratio. We assume that the same determines B/A, the ratio of the ω_{15} and ω_8 coupling to the singlet. This gives the ratio

$$\alpha \equiv \frac{B}{A} = -\frac{m_1 - 3m_2}{2\sqrt{2}\, m_1} \quad . \tag{3.2}$$

The five-parameter mass matrix thus obtained agrees with the matrix used recently in ref. [4]. Other authors based their mass mixing on the nonet symmetry for vector mesons [8]. In our notation this means to fix A by

$$A = -2\sqrt{6} \quad m_1 \quad , \tag{3.3}$$

but we shall not use this additional constraint.
In (3.1) two parameters are fixed by the ϱ and K^* masses from (2.25)

$$m_1 = \frac{1}{6}(K^* - \varrho) , \tag{3.4}$$

$$m - 9m_2 = \frac{1}{2}(3\varrho - K^*) . \tag{3.5}$$

As the three unknown in (3.1) we use α, A and m_0. m_2 is determined via (3.2)

$$m_2 = \frac{m_1}{3}(1 + 2\sqrt{2}\,\alpha) . \tag{3.6}$$

Expressing the three tensor invariants of the matrix (3.1) by the eigenvalues (squared masses of ω, ϕ and ψ) gives us three equations for α, A and m. Elimination of A and m leads to a third order equation in α. This equation has three solutions. Only for two of them A is real. In one solution the masses of charmed particles are around 1 GeV. We must take the solution where m_{D*} and m_{F*} are around 2 GeV, otherwise the ψ can copiously decay into these mesons. Then with the input

$$m_\rho = 0.76737 \text{ GeV} , \qquad m_\omega = 0.78266 \text{ GeV} ,$$
$$m_{K*} = 0.89433 \text{ GeV} , \qquad m_\phi = 1.01969 \text{ GeV} , \qquad (3.7)$$
$$m_\psi = 3.095 \quad \text{GeV} ,$$

we obtain

$$\alpha = 21.4184, \qquad m_1 = 0.0352 \text{ GeV}^2,$$
$$m = 6.9786 \text{ GeV}^2, \qquad m_2 = 0.7218 \text{ GeV}^2, \qquad (3.8)$$
$$A = -0.1843, \qquad m_0 = 3.2414 \text{ GeV}^2.$$

The corresponding masses of the charmed particles are

$$m_{D*} = 2.218 \text{ GeV},$$
$$\qquad\qquad\qquad (3.9)$$
$$m_{F*} = 2.265 \text{ GeV}.$$

The large value of α means that the SU(4) breaking is roughly twenty times larger than the SU(3) breaking which is determined by m_1.

The wave functions for the mixed states ω, ϕ and ψ in terms of the 8, 15 and 0 components are:

$$|\omega\rangle = 0.6595 \ |\omega_0\rangle + 0.6301 \ |\omega_8\rangle + 0.4098 \ |\omega_{15}\rangle,$$

$$|\phi\rangle = 0.5342 \ |\omega_0\rangle - 0.7765 \ |\omega_8\rangle - 0.3342 \ |\omega_{15}\rangle, \quad (3.10)$$

$$|\psi\rangle = 0.5288 \ |\omega_0\rangle - 0.0015 \ |\omega_8\rangle - 0.8488 \ |\omega_{15}\rangle.$$

It is easy to express $|\omega\rangle$, $|\phi\rangle$ and $|\psi\rangle$ in terms of the ideally mixed states $|\omega_\sigma\rangle$, $|\omega_\lambda\rangle$ and $|\omega_c\rangle$ defined by

$$|\omega_\sigma\rangle = \frac{1}{\sqrt{3}} |\omega_8\rangle + \frac{1}{\sqrt{6}} |\omega_{15}\rangle + \frac{1}{\sqrt{2}} |\omega_0\rangle \ ,$$

$$|\omega_\lambda\rangle = -\sqrt{\frac{2}{3}} |\omega_8\rangle + \frac{1}{\sqrt{12}} |\omega_{15}\rangle + \frac{1}{2} |\omega_0\rangle \ , \quad (3.11)$$

$$|\omega_c\rangle = \qquad\qquad -\frac{\sqrt{3}}{2} |\omega_{15}\rangle + \frac{1}{2} |\omega_0\rangle \ .$$

The quark content of these states is $(u\bar{u} + d\bar{d})/\sqrt{2}$, $s\bar{s}$ and $c\bar{c}$, respectively. The physical states then are:

$$|\omega\rangle = 0.9975 \ |\omega_\sigma\rangle - 0.0664 \ |\omega_\lambda\rangle - 0.0251 \ |\omega_c\rangle,$$

$$|\phi\rangle = 0.0659 \ |\omega_\sigma\rangle + 0.9976 \ |\omega_\lambda\rangle - 0.0223 \ |\omega_c\rangle, \quad (3.12)$$

$$|\psi\rangle = 0.0265 \ |\omega_\sigma\rangle + 0.0206 \ |\omega_\lambda\rangle + 0.9994 \ |\omega_c\rangle.$$

We see that the physical ω, ϕ and ψ states deviate very little from the ideally mixed states (3.11). Whether the amount of deviation from the ideally mixed situation is small enough to understand the small total and partial decay widths of the ψ will be taken up in a forthcoming paper. From (3.10) we note that the $\omega-\phi$ mixing comes out reasonable, the mixing angle being 39°.

It is of interest to see what are the mass constraints resulting from a pure ideal mixing. Imposing that the eigenfunctions of (3.1) are exactly given by (3.11) we get the following constraints on the parameters in (3.1)

$$m_o = m + 6 (m_1 - m_2) = m + 4m_1 (1 - \sqrt{2}\alpha) ,$$

$$A = - 2 \sqrt{6} m_1 .$$

(3.13)

As can be read off from (2.17) the quantity $m + 6 (m_1 - m_2)$ is just the mean mass $\overline{m}_{15}$ of the 15-plet. The constraint (3.13) results in the mass formulas for ideal mixing, i.e.

$$\rho = \omega , \qquad\qquad 2 D^* = \omega + \Psi ,$$

$$2 K^* = \omega + \phi , \qquad\qquad 2 F^* = \phi + \Psi .$$

(3.14)

The constraints for ρ and K^* are the well known nonet symmetry constraints which are well satisfied. As is well known for the pseudoscalar mesons the corresponding relations for η , η' , π and K are not satisfied, whereas for the tensor mesons the agreement is similar to that of the vector mesons.

4. Electromagnetic Current in SU(4).

In our scheme where the charges of the four fundamental quarks u, d, s and c are (2/3, -1/3, - 1/3, 2/3) the electric charge operator Q has the form (2,20)

$$Q = \frac{\sqrt{2}}{3} F_0 + F_3 + \frac{1}{\sqrt{3}} F_8 - \sqrt{\frac{2}{3}} F_{15} \; . \tag{4.1}$$

Therefore the electromagnetic current J and any electromagnetic operator related linearly to it is the same superposition of F_0, F_3, F_8 and F_{15} as in (4.1). In Okubo's notation this means that

$$J = e \, T_0 + e'(\, T^1_1 + T^4_4) \; , \tag{4.2}$$

where T_0 is the SU(4) invariant part and T^1_1 and T^4_4 are irreducible tensor operators whose matrix elements for an irreducible SU(4) representation are obtained again from (2.9). We observe that J commutes with all U- and W- spin components. Therefore J is a U -spin scalar as in SU(3) and in addition a W-spin scalar. T^4_4 has been expressed already in (2.13) by Y_{15} and the SU(3)-Casimir operator $\mathcal{C}_3$.
Similarly to T^3_3 we can transform T^1_1 into

$$T^1_1 = (a-2b)(I_3 + \tfrac{1}{2} Y_8 + \tfrac{1}{3} Y_{15}) + \frac{b}{3} \left[3(\vec{W}^2 - \vec{U}^2) + \mathcal{C}_3 - Y^2_{15} \right]. \tag{4.3}$$

We see that the breaking caused by T^1_1 is controlled by

U- and W-spin in analogy to I- and Z-spin which essentially

determine T^3_3.

Thus we have

$$T^1_1 + T^4_4 = b\,\mathscr{C}_4 + (a-2b)\,(Q - \tfrac{1}{2}\,B) + b\,(\vec{W}^2 - \vec{U}^2) \qquad (4.4)$$

The SU(4) invariant contributions in (4.4) can be combined

with T_0 and introducing $f = 3(2b-a)/b$ as in (2.17),

J turns out to be of the form:

$$J = e_0 + 3e_1(\vec{W}^2 - \vec{U}^2) - fe_1 Q \qquad (4.5)$$

J consists of a charge-conjugation even part and a charge-conjugation

odd part (proportional to Q). The $\mathscr{C}$-even part for example

is of interest for radiative transitions $V \to P + \gamma$, where V are

members of the vector and P of the pseudoscalar meson fifteenplet.

Both terms contribute for example to the electromagnetic form

factors of the baryons. The application of (4.5) to the radiative

decays of the meson fifteenplet is easily done with the help of

the decomposition into U- and W-spin submultiplets. This will be

done in a forthcoming paper.

Now we shall apply (4.5) to the problem of electromagnetic mass

splitting. As an example we consider the vector meson 15 plet as

in section 2. Since electromagnetic mass shifts are of second order in the electromagnetic interaction, that is, quadratic in the current, some caution is necessary in connection with the application of (4.5). If we assume that the electromagnetic mass breaking term has nevertheless the transformation properties of $J \sim (T^1_1 + T^4_4)$ we deduce from (4.5) and the classification (2.23) and (2.24) of the particles into U- and W-spin multiplets the following relations for the non-SU(4)-invariant electromagnetic mass shifts $\delta V = \langle V | (J - e_0) | V \rangle$:

$$\delta \varrho^+ = \delta \varrho^0 = \delta K^{*+} = \delta D^{*+} = \delta F^{*+} = 0$$

$$\delta K^{*0} = - \delta D^{*0} = \frac{3}{2} \delta \omega_8 = - \frac{3}{2} \delta \omega_{15}$$

$$= - \sqrt{3} \, \delta \omega_{3,8} = - \sqrt{6} \, \delta \omega_{3,15} = - 3\sqrt{2} \, \delta \omega_{8,15}$$

$$= - 6 \, e_1$$

(4.6)

Because of our simplifying assumption the electromagnetic mass difference between ϱ^+ and ϱ^0 vanishes in the case of no mixing between ϱ^0, ω_8 and ω_{15} components. This agrees with well-known results for SU(3)[9]. To overcome this difficulty and to obtain relations for electromagnetic mass shifts valid to all orders, we go back to the statement that J and any power J^n are U- and W-spin scalars. Then the matrix elements of J^n are related in the following way:

$$\langle \rho^+ | J^n | \rho^+ \rangle = \langle K^{*+} | J^n | K^{*+} \rangle = \langle D^{*+} | J^n | D^{*+} \rangle = \langle F^{*+} | J^n | F^{*+} \rangle \, ,$$

$$2 \langle K^{*0} | J^n | K^{*0} \rangle = 3 \langle \omega_8 | J^n | \omega_8 \rangle - \langle \rho^0 | J^n | \rho^0 \rangle \, ,$$

$$2 \langle D^{*0} | J^n | D^{*0} \rangle = \langle \omega_8 | J^n | \omega_8 \rangle + 4 \langle \omega_{15} | J^n | \omega_{15} \rangle - 3 \langle \rho^0 | J^n | \rho^0 \rangle \, ,$$

$$2 \langle \rho^0 | J^n | \omega_8 \rangle = \sqrt{3} \langle \rho^0 | J^n | \rho^0 \rangle - \sqrt{3} \langle \omega_8 | J^n | \omega_8 \rangle \, ,$$

$$2 \sqrt{2} \langle \omega_{15} | J^n | \omega_8 \rangle = \langle \omega_8 | J^n | \omega_8 \rangle + 2 \langle \omega_{15} | J^n | \omega_{15} \rangle - 3 \langle \rho^0 | J^n | \rho^0 \rangle \, ,$$

$$\langle \rho^0 | J^n | \omega_{15} \rangle = \sqrt{3} \langle \omega_{15} | J^n | \omega_8 \rangle \, , \tag{4.7}$$

for the states of the fifteenplet, and

$$\langle \rho^0 | J^n | \omega_0 \rangle = \sqrt{3} \langle \omega_8 | J^n | \omega_0 \rangle = - \sqrt{\frac{3}{2}} \langle \omega_{15} | J^n | \omega_0 \rangle \tag{4.8}$$

for the matrix elements between 15-plet and singlet.

It follows from (4.7) that the electromagnetic mass splitting of the 15- plet is determined by four parameters:

$$\langle \rho^+ | J^n | \rho^+ \rangle \, , \quad \langle \rho^0 | J^n | \rho^0 \rangle \, , \quad \langle \omega_8 | J^n | \omega_8 \rangle \quad and \quad \langle \omega_{15} | J^n | \omega_{15} \rangle .$$

One further constraint is obtained from (4.5) by evaluating it between I = 1/2 and I = 0 states. In this case only that part of the operator J^n contributes which has the same structure as J . This constraint is

$$2 \langle \rho^0 | J^n | \rho^0 \rangle = \langle \omega_{15} | J^n | \omega_{15} \rangle + \langle \omega_8 | J^n | \omega_8 \rangle \, . \tag{4.9}$$

So we are left with three parameters for the electromagnetic mass shifts to all orders in the electromagnetic coupling which we

denote by $\quad \Delta_\rho = \langle \rho^+ | J^n | \rho^+ \rangle$,

$\Delta_8 = \langle \omega_8 | J^n | \omega_8 \rangle$, $\qquad \Delta_{15} = \langle \omega_{15} | J^n | \omega_{15} \rangle$.

Then we have the following mass formulas instead of (2.25):

$$
\begin{aligned}
\rho^+ &= m + 3m_1 - 9m_2 + \Delta_\rho \quad , \\
K^{*+} &= m + 9m_1 - 9m_2 + \Delta_\rho \quad , \\
K^{*0} &= m + 9m_1 - 9m_2 + \frac{5}{4}\Delta_8 - \frac{1}{4}\Delta_{15} \quad , \\
D^{*+} &= m + 3m_1 - 3m_2 + \Delta_\rho \quad , \\
D^{*0} &= m + 3m_1 - 3m_2 + \frac{5}{4}\Delta_{15} - \frac{1}{4}\Delta_8 \quad , \\
F^{*+} &= m + 9m_1 - 3m_2 + \Delta_\rho \quad .
\end{aligned}
\tag{4.10}
$$

We see that we have still the mass relation (2.27), but only for the charged components

$$
\rho^+ + F^{*+} = K^{*+} + D^{*+} \quad .
\tag{4.11}
$$

This is a consequence of the first line in (4.7) that the electromagnetic mass shifts of all charged members of the 15-plet are equal. The masses of the physical ρ^0, ω and ϕ are obtained by diagonalizing the symmetric mass matrix M

$$
M = \begin{pmatrix}
m + 3m_1 - 9m_2 + \frac{1}{2}(\Delta_8 + \Delta_{15}) & \frac{\sqrt{3}}{4}(-\Delta_8 + \Delta_{15}) & \frac{\sqrt{6}}{8}(-\Delta_8 + \Delta_{15}) \\[2mm]
\frac{\sqrt{3}}{4}(-\Delta_8 + \Delta_{15}) & m + 11m_1 - 9m_2 + \Delta_8 & -2\sqrt{2}\,m_1 + \frac{\sqrt{2}}{8}(-\Delta_8 + \Delta_{15}) \\[2mm]
\frac{\sqrt{6}}{8}(-\Delta_8 + \Delta_{15}) & -2\sqrt{2}\,m_1 + \frac{\sqrt{2}}{8}(-\Delta_8 + \Delta_{15}) & m + 4m_1 + \Delta_{15}
\end{pmatrix}
$$

$$
\tag{4.12}
$$

which replaces (2.26). The electromagnetic mass shifts in (4.12) reduce to the relations (4.6), obtained in first order, if $\Delta_8 = -\Delta_{15}$. In (4.12) the parameters m_1 and m can be eliminated by:

$$m_1 = \frac{1}{6}\left(K^{*+} - \varrho^+\right) ,$$ (4.13)

$$m - 9m_2 = \frac{3}{2}\left(\varrho^+ - K^{*+}\right) + K^{*0} - \frac{5}{4}\Delta_8 + \frac{1}{4}\Delta_{15} ,$$

so that only three parameters are left over to be determined by the input masses of ϱ^0, ω and ϕ.

Then we have only two unknowns

$$\Delta = \frac{1}{4}\left(\Delta_{15} - \Delta_8\right) ,$$ (4.14)

$$\bar{m} = m + \Delta_{15} ,$$

and M has the following form:

$$M = \begin{pmatrix} 3m_1 + k + 3\Delta & \sqrt{3}\,\Delta & \sqrt{\frac{3}{2}}\,\Delta \\ \sqrt{3}\,\Delta & 11m_1 + k + \Delta & -2\sqrt{2}\,m_1 + \frac{1}{\sqrt{2}}\Delta \\ \sqrt{\frac{3}{2}}\,\Delta & -2\sqrt{2}\,m_1 + \frac{1}{\sqrt{2}}\Delta & 4m_1 + \bar{m} \end{pmatrix} ,$$ (4.15)

with $k = \frac{3}{2}\left(\varrho^+ - K^{*+}\right) + K^{*0}$

The tensor invariants of the matrix M give us three equations for the two unknowns in terms of m_1, k and the masses of ϱ^0, ω and ϕ. By eliminating Δ and $\bar{m}$ we obtain a very complicated mass formula. We do not expect useful solutions from this mass formula similar

to the case without electromagnetic mass splitting. The masses of
the charmed mesons are determined from (4.11) and

$$\rho^0 + \omega + \phi = \frac{3}{2} D^{*0} + \frac{3}{2} K^{*0} ,$$

$$D^{*+} - K^{*+} = \frac{2}{3} (\rho^0 + \omega + \phi) - 18 m_1 - 2k - 6\Delta \qquad (4.16)$$

In order to incorporate the ψ meson we consider the case of singlet
and 15-plet mixing together with electromagnetic mass shifts.
Then, compared to (4.15), we have three more matrix elements
Δ_{03}, Δ_{08} and Δ_{015}, which describe the contribution of the
electromagnetic mass shift in the nondiagonal elements between the
singlet and the 3, 8 and 15 component of the 15-plet. In analogy
to the procedure in section 3 we fix the ratios of these nondiagonal
elements by the ratios of the corresponding components in the
electromagnetic current (see (4.5) :

$$\Delta_{015} : \Delta_{08} : \Delta_{03} = -\sqrt{\frac{2}{3}} : \frac{1}{\sqrt{3}} : 1 \qquad (4.17)$$

Then the mass matrix in the 3, 8, 15, 0 representation has the
following form:

$$M = \begin{pmatrix} 3m_1 + k + 3\Delta & \sqrt{3}\,\Delta & \sqrt{\frac{3}{2}}\,\Delta & \Delta_{03} \\ \sqrt{3}\,\Delta & 11 m_1 + k + \Delta & -2\sqrt{2} m_1 + \frac{1}{\sqrt{2}}\Delta & A + \frac{1}{\sqrt{3}}\Delta_{03} \\ \sqrt{\frac{3}{2}}\,\Delta & -2\sqrt{2} m_1 + \frac{1}{\sqrt{2}}\Delta & 4m_1 + \overline{m} & \alpha A - \sqrt{\frac{2}{3}}\Delta_{03} \\ \Delta_{03} & A + \frac{1}{\sqrt{3}}\Delta_{03} & \alpha A - \sqrt{\frac{2}{3}}\Delta_{03} & m_0 \end{pmatrix} \qquad (4.18)$$

Since $\overline{m} = 3m_1 (1 + 2\sqrt{2}\alpha) + k + 5\Delta$ the matrix (4.18) has still

five unknowns α, A, m_o, Δ and Δ_{03} , so that the mass

spectrum cannot be predicted from the masses of ρ^o, ω, ϕ

and ψ. We eliminate Δ_{03} by the constraint

$$\Delta_{03} = - \frac{\sqrt{3}\,A}{4\,m_1}\,\Delta \ . \tag{4.19}$$

(4.19) follows from the assumption that

$$m_{30} : m_{80} = m_{3,15} : m_{8,15} \ . \tag{4.20}$$

(4.19) agrees for $A = - 2\sqrt{6}\ m_1$ (see (3.3)) with the relation

which follows from the assumption that the electromagnetic mass

breaking term has the transformation properties of J, so that

it obeys (4.5).

With the constraint (4.19) we have four unknowns which can be

determined from the masses of ρ^o, ω, ϕ and ψ . The explicit

solution of this mixing problem is rather complicated and will

be considered elsewhere.

5. Conclusions

In this paper we derived formulas for mass breaking in SU(4)

multiplets using Okubo's tensor formalism. These formulas

are applied to mass breaking of the vector meson 15-plet

with and without singlet - 15 -plet mixing. Without mixing

no sensible solutions are obtained. For the case with mixing

we obtain two solutions. In only one of them the masses of the charmed vector mesons are around 2 GeV. For this solutions the Ψ (3.1) meson is dominantly a $c\bar{c}$ system. We emphazise that a parameter-free solution for the masses of the charmed mesons comes out only if the nondiagonal matrix elements between singlet ω_0 and ω_8 and ω_{15} respectively are constrained by (3.2).

The electromagnetic mass splitting is considered also in some detail. The mass matrix for the 15-plet is derived including singlet - 15-plet mixing. To obtain parameter free solutions for the charmed mesons one needs three further constraints for the nondiagonal elements of the electromagnetic mass breaking matrix.

Table 1: Quantum numbers of the 15-plet.

	Q	$Y=S$	C	$Y-C$	$(p,q)_{SU_3}$	(I,I_3) *)	(Z,Z_3) *)	(U,U_3) *)	(W,W_3) *)
ρ^+	$+1$	0	0	0	$8=(1,1)$	$(1,+1)$	$(0,0)$	$(\frac{1}{2},-\frac{1}{2})_a$	$(\frac{1}{2},+\frac{1}{2})_a$
ρ°	0	0	0	0	$8=(1,1)$	$(1,0)$	$(0,0)$	$(0,0)+(1,0)$	$(0,0)+(0,0)'+(1,0)$
ρ^-	-1	0	0	0	$8=(1,1)$	$(1,-1)$	$(0,0)$	$(\frac{1}{2},+\frac{1}{2})_b$	$(\frac{1}{2},-\frac{1}{2})_b$
K^{*+}	$+1$	$+1$	0	$+1$	$8=(1,1)$	$(\frac{1}{2},+\frac{1}{2})_a$	$(\frac{1}{2},-\frac{1}{2})_a$	$(\frac{1}{2},+\frac{1}{2})_a$	$(\frac{1}{2},+\frac{1}{2})_c$
$K^{*\circ}$	0	$+1$	0	$+1$	$8=(1,1)$	$(\frac{1}{2},-\frac{1}{2})_a$	$(\frac{1}{2},-\frac{1}{2})_b$	$(1,+1)$	$(0,0)$
$\bar{K}^{*\circ}$	0	-1	0	-1	$8=(1,1)$	$(\frac{1}{2},+\frac{1}{2})_b$	$(\frac{1}{2},+\frac{1}{2})_c$	$(1,-1)$	$(0,0)$
K^{*-}	-1	-1	0	-1	$8=(1,1)$	$(\frac{1}{2},-\frac{1}{2})_b$	$(\frac{1}{2},+\frac{1}{2})_d$	$(\frac{1}{2},-\frac{1}{2})_b$	$(\frac{1}{2},-\frac{1}{2})_d$
D^{*+}	$+1$	0	$+1$	-1	$3=(1,0)$	$(\frac{1}{2},+\frac{1}{2})_c$	$(\frac{1}{2},-\frac{1}{2})_c$	$(\frac{1}{2},-\frac{1}{2})_c$	$(\frac{1}{2},-\frac{1}{2})_a$
$D^{*\circ}$	0	0	$+1$	-1	$3=(1,0)$	$(\frac{1}{2},-\frac{1}{2})_c$	$(\frac{1}{2},-\frac{1}{2})_d$	$(0,0)$	$(1,-1)$
$\bar{D}^{*\circ}$	0	0	-1	$+1$	$\bar{3}=(0,1)$	$(\frac{1}{2},+\frac{1}{2})_d$	$(\frac{1}{2},+\frac{1}{2})_a$	$(0,0)$	$(1,+1)$
D^{*-}	-1	0	-1	$+1$	$\bar{3}=(0,1)$	$(\frac{1}{2},-\frac{1}{2})_d$	$(\frac{1}{2},+\frac{1}{2})_b$	$(\frac{1}{2},+\frac{1}{2})_d$	$(\frac{1}{2},+\frac{1}{2})_b$
F^{*+}	$+1$	$+1$	$+1$	0	$3=(1,0)$	$(0,0)$	$(1,-1)$	$(1,+1)$	$(\frac{1}{2},-\frac{1}{2})_c$
F^{*-}	-1	-1	-1	0	$\bar{3}=(0,1)$	$(0,0)$	$(1,+1)$	$(1,-1)$	$(\frac{1}{2},+\frac{1}{2})_d$
ω_8	0	0	0	0	$8=(1,1)$	$(0,0)$	$(0,0)+(1,0)$	$(0,0)+(1,0)$	$(0,0)+(0,0)'+(1,0)$
ω_{15}	0	0	0	0	$1=(0,0)$	$(0,0)$	$(0,0)+(1,0)$	$(0,0)$	$(0,0)'+(1,0)$

*) Particles belonging to the same doublet $(\frac{1}{2},\pm\frac{1}{2})_i$ are indicated by the same index $i = a,b,c$ and d.

Table 2: Quantum numbers of the fundamental 4-plet.

	Q	Y	B	S	C	$(p,q)_{SU_3}$	(I,I_3)	(Z,Z_3)	(U,U_3)	(W,W_3)
u	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$3=(1,0)$	$(\frac{1}{2},+\frac{1}{2})$	$(0,0)$	$(0,0)$	$(\frac{1}{2},+\frac{1}{2})$
d	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$3=(1,0)$	$(\frac{1}{2},-\frac{1}{2})$	$(0,0)$	$(\frac{1}{2},+\frac{1}{2})$	$(0,0)$
s	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	-1	0	$3=(1,0)$	$(0,0)$	$(\frac{1}{2},+\frac{1}{2})$	$(\frac{1}{2},-\frac{1}{2})$	$(0,0)$
c	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	1	$1=(0,0)$	$(0,0)$	$(\frac{1}{2},-\frac{1}{2})$	$(0,0)$	$(\frac{1}{2},-\frac{1}{2})$

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