DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 75/24
August 1975

Renormalization Problem in a Class of Nonrenormalizable Theories

by

K. Symanzik
Deutsches Elektronen-Synchrotron DESY, Hamburg, Germany

2 HAMBURG 52 · NOTKESTIEG 1

Renormalization problem in a class of nonrenormalizable theories

by

K. Symanzik

Deutsches Elektronen-Synchrotron DESY, Hamburg, Germany

presented at the VI GIFT Seminar in Theoretical Physics,

Universidad de Zaragoza, Jaca, Huesca, 2-6 June 1975

You are certainly aware of the great enthusiasm for renormalizable theories the gauge theories have engendered, and the essentials of renormalization theory have been presented by Prof. Osterwalder at this Seminar. What about nonrenormalizable theories, are they necessarily dead forever? It would be very shortsighted to believe so, and would be merely a reflection of the failure so far to escape, for realistic theories, from the unreasonable and unnatural confines of perturbation theory.

I shall first describe a possible way to approach a certain class of nonrenormalizable theories. For this technical exposition, I choose the simplest (though obviously unrealistic) theory of this class with a chance to exist, massless ϕ^4 theory in more than four space-time dimensions. I shall mention the problems of extension to other nonrenormalizable theories in the class considered and compare the conclusions reached so far with the corresponding ones for renormalizable theories at the end. - Almost all the material presented here is taken from a recent paper of mine /1/, which should be consulted for more details.

1. General outline

Nonrenormalizable theory is here dealt with by using an explicit cutoff Λ in the Lagrangean, studying the Λ-dependence of the Green's functions, deriving sufficient conditions for the existence of the $\Lambda \to \infty$ limit for these, and obtaining the consequences for the limit Green's functions if these conditions are fulfilled.

These tasks are facilitated if one starts with a theory that has a polynomial interaction and is renormalizable in some space-time dimension, and then increases the number of space-time dimensions from that integer to a generic (nonrational) real, or even complex, number. The gain is that hereby (dimensional) degeneracies are lifted that would otherwise lead to logarithms in almost all formulae, and these logarithms are much more complicated to handle than powers, which appear in the generic-dimension case. The restriction to integer dimension is of course the central issue, and we shall discuss what is involved hereby. We shall also find that for the $\Lambda \to \infty$ limit to exist for generic dimension, the sufficient conditions mentioned before are also necessary.

The cutoff that allows most easily to study the cutoff dependence to arbitrary accuracy (as will turn out to be necessary) is the one employing higher derivatives /2/ in the kinetic part of the Lagrangean. It leads necessarily to indefinite metric in the "state space", but this does not affect our analysis that rests on perturbation theory and Euclidean integration, though it does render the problem of unitarity nontrivial as we shall discuss. The restriction to a zero-(physical-) mass is chosen since this diminishes the number of parameters and thus simplifies the analysis.

For the ϕ^4 model, we are led to consider the Lagrangean

$$\mathcal{L} = -\tfrac{1}{2}\phi_B \Box(1+\Lambda^{-2}\Box)\phi_B - \tfrac{1}{4!} g_B \phi_B^4 - \tfrac{1}{2} m_{B0}^2 \phi_B^2 \qquad (1.1a)$$

in $4+\varepsilon$ dimensions, the subscript B meaning bare. The (mass) dimension of ϕ_B is $1+\frac{1}{2}$, such that g_B has dimension $-\varepsilon$ and $g_B \Lambda^{\varepsilon}$ is the only dimensionless parameter in L, such that the bare-mass squared of the zero-mass theory has the form

(1.1b) $$m_{B0}^2 = \Lambda^2 \sum_{k=1}^{\infty} a_k(\varepsilon)\,(g_B \Lambda^{\varepsilon})^k$$

with computable meromorphic coefficients $a_k(\varepsilon)$. Our main objects of study are the vertex functions (VFs) $\Gamma_{\Lambda B}(p_1 \dots p_{2n}; g_B, \varepsilon) \equiv \Gamma_{\Lambda B}((2n); g_B, \varepsilon)$, the Fourier transforms (with a factor $(2\pi)^{4+\varepsilon}\delta(\Sigma p)$ taken out) of the connected amputated one-particle-irreducible parts of the Green's functions $\langle(\phi_B(x_1) \dots \phi_B(x_{2n}))_+\rangle$, and the corresponding "renormalized" functions to be introduced later.

2. Large - Λ expansion of bare vertex functions

It can be shown that the $\Gamma_{\Lambda B}$ deriving from Lagrangean (1.1) admit the following large - Λ expansion:

(2.1) $$\Gamma_{\Lambda B}((2n); g_B, \varepsilon) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \Lambda^{-2j+\varepsilon k} f_{jk}((2n); g_B, \varepsilon)$$

where f_{jk} are certain power series in g_B. The proof /1/, which uses Zimmermann-type "oversubtractions" and is slightly technical, will not be reproduced here. Its essential point is, however, to show that the $\Gamma_{\Lambda B}$ can also be computed from an effective Lagrangean

(2.2) $$L_\Lambda = -\tfrac{1}{2}\phi_B \Box \phi_B - \tfrac{1}{4!} g_B \phi_B^4 - \tfrac{1}{2}\Lambda^{-2}\phi_B \Box^2 \phi_B + \sum_{\substack{r=0 \\ r+s\geq 2}}^{\infty} \sum_{s=1}^{\infty} \sum_{\nu=1}^{n_{rs}} f_{rs\nu}(g_B \Lambda^{\varepsilon}, \varepsilon)\,("D^{2r}\phi_B^{2s}")_\nu\, g_B^{s-1} \Lambda^{4-2r-2s}.$$

This formula needs some explanation:

1) "Analytic integration" /3/ is to be used, with Feynman rules as follow from the first two terms on the r.h.s. of (2.1), all other terms on the r.h.s. being treated by (repeated) insertion into the Green's functions. Analytic integration here means to evaluate

$$\int dk_1 \dots dk_{\mathcal{L}} \left[-k_i A_{ij} k_j + 2B_i k_i + C \right]^{-\alpha} =$$

$$(2.3) \quad = \pi^{(2+\frac{1}{2}\varepsilon)\mathcal{L}} \, i^{\mathcal{L}} \, \Gamma(\alpha)^{-1} \, \Gamma(\alpha - (2+\tfrac{1}{2}\varepsilon)\mathcal{L}) \cdot$$
$$\cdot \left[\mathrm{Det} A \right]^{-2-\frac{1}{2}\varepsilon} \left[C - B_i (A^{-1})_{ij} B_j \right]^{-\alpha + (2+\frac{1}{2}\varepsilon)\mathcal{L}}$$

and to set this zero if the last square bracket is zero. The use of generic ε here prevents the second Γ-function from having a nonpositive-integer argument since α is integer and $\mathcal{L}$ positive integer.

2) The triple sum goes over all n_{rs} scalar monomials, involving 2r derivatives and $2s$ factors ϕ_B, that are linearly independent at zero momentum transfer. For r + s = 2, these are $\phi \Box \phi$ and ϕ^4, for r + s = 3, these are $\phi \Box^2 \phi$, $\phi^3 \Box \phi$, and ϕ^6, in view of, e.g.

$$\phi^2 \partial_\mu \phi \partial^\mu \phi = -\tfrac{1}{3} \phi^3 \Box \phi + \tfrac{1}{12} \Box \phi^4 = -\tfrac{1}{4} \phi^2 \Box \phi^2 + \tfrac{1}{16} \Box \phi^4$$

3) The coefficient functions $f_{rs\nu}(g_B \Lambda^{\varepsilon}, \varepsilon)$ are obtained as power series

$$(2.4) \quad f_{rs\nu}(g_B \Lambda^{\varepsilon}, \varepsilon) = \sum_{\mathcal{L}=1}^{\infty} f_{rs\nu\mathcal{L}}(\varepsilon) \, (g_B \Lambda^{\varepsilon})^{\mathcal{L}}$$

where $\mathcal{L}$ is the number of loops of the (regularized) graphs entering the computation and $2s$ the one of external lines. The $f_{rs\nu\mathcal{L}}(\varepsilon)$ are meromorphic in ε, real for ε real nonrational.

4) (2.2) yields in view of (2.4) the expansion (2.1). Since an $\mathcal{L}$-loop graph contributes in (2.1) only with $k \leq \mathcal{L}$, the expansion (2.1) becomes for finite-order graphs an ordinary asymptotic expansion for large Λ. L_Λ is termed effective Lagrangean since it must be evaluated according to the special prescription given under 1), and then yields directly the asymptotic expansion (for finite-order graphs) (2.1). In contrast, the true Lagrangean (1.1) yields (in principle) the exact (i.e., nonexpanded) VFs on the l.h.s. of (2.1).

Using only the first two terms on the r.h.s. of (2.2) and integrating analytically yields $\hat{f}_{00}$ in (2.1). Then the ordinary UV divergences appear for positive-rational ε. Since the $T'_{\Lambda B}$ are actually finite order by order in g_B, identically in Λ, for $0 \leq \varepsilon < 3$, on the r.h.s. of (2.1) there must be cancellations of all ε-singularities in this range, between terms that, in (2.1), obtain the same Λ-dependence at these ε. (Inversely, this shows that in the f_{jk} there can be ε-singularities only at rational ε). At such ε, $\ln\Lambda$-factors will arise. In particular, for $\varepsilon = 0$ there arise terms $\Lambda^{-2j}(\ln\Lambda)^k$ and the removal of the $j = 0, k > 0$ terms hereof is the subject of ordinary renormalization theory.

3. T'_Λ-definition and properties

In (2.2) we combine the terms proportional to the ones that define the Feynman rules. We introduce

$$1 - 2 f_{111}(g_B \Lambda^\varepsilon, \varepsilon) \equiv Z_3^{-1}(g_B \Lambda^\varepsilon, \varepsilon) \tag{3.1}$$

$$g_B [1 - 4! f_{021}(g_B \Lambda^\varepsilon, \varepsilon)] Z_3^2 (g_B \Lambda^\varepsilon, \varepsilon) = g \mu^{-\varepsilon} \tag{3.2a}$$

where μ is some unit of mass and g is dimensionless. (3.2a) can be solved for $g_B \Lambda^\varepsilon$:

$$g_B \Lambda^\varepsilon = \bar{g}(g \mu^{-\varepsilon} \Lambda^\varepsilon, \varepsilon) \tag{3.2b}$$

where the function $\bar{g}(\cdot, \varepsilon)$ will later (in sect. 4) be characterized in a different and more informative manner. Using (3.2b) we can introduce functions (for $r + s \geqslant 3$)

$$\begin{aligned} c'_{rs\nu}(g\mu^{-\varepsilon}\Lambda^\varepsilon, \varepsilon) &= -\tfrac{1}{2}\delta_{r2}\delta_{s1} Z_3(g_B\Lambda^\varepsilon, \varepsilon) + \\ &+ f_{rs\nu}(g_B\Lambda^\varepsilon, \varepsilon) Z_3^s(g_B\Lambda^\varepsilon, \varepsilon)(g_B\Lambda^\varepsilon)^{s-1}(g\mu^{-\varepsilon}\Lambda^\varepsilon)^{1-s} \end{aligned} \tag{3.3}$$

by substituting for $g_B \Lambda^\varepsilon$ on the r.h.s. Then defining

$$\phi = Z_3^{-\frac{1}{2}}(g_B \Lambda^\varepsilon, \varepsilon)\, \phi_B \tag{3.4}$$

(2.2) takes the form

$$\begin{aligned} L_\Lambda &= -\tfrac{1}{2}\phi \Box \phi - \tfrac{1}{4!} g \mu^{-\varepsilon} \phi^4 + \\ &+ \sum_{\substack{r=0 \\ r+s \geqslant 3}}^{\infty} \sum_{s=1}^{\infty} \sum_{\nu=1}^{n_{rs}} c'_{rs\nu}(g\mu^{-\varepsilon}\Lambda^\varepsilon, \varepsilon)(\text{"}\partial^{2r}\phi^{2s}\text{"})_\nu \,\cdot \\ &\cdot (g\mu^{-\varepsilon})^{s-1} \Lambda^{4-2r-2s} \end{aligned} \tag{3.5}$$

where the $C'_{rs\nu}$ are power series in their first argument beginning with the first order, except C'_{211} which begins with one half. The coefficients in all power series are meromorphic in ε .

(3.5) is to be evaluated by treating the terms of the triple sum as (repeated) insertions into Green's functions defined by the Feynman rules deriving from the first two terms on the r.h.s., and integrating analytically. This yields for "renormalized" vertex functions the expansions

$$\Gamma_\Lambda((2n);g\mu^{-\varepsilon},\varepsilon) = h_{oo}((2n);g\mu^{-\varepsilon},\varepsilon) + \\ + \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\Lambda^{-2j+\varepsilon k}\, h_{jk}((2n);g\mu^{-\varepsilon},\varepsilon) \tag{3.6}$$

where h_{oo} is the unregularized function identical to f_{oo} of (2.1) with g_B there replaced by $g\mu^{-\varepsilon}$.

Simple dimension counting gives

$$\Gamma_\Lambda(00;g\mu^{-\varepsilon},\varepsilon) = 0 \tag{3.7a}$$

$$[\partial/\partial p^2]\,\Gamma_\Lambda(p(-p);g\mu^{-\varepsilon},\varepsilon)\Big|_{p=0} = i \tag{3.7b}$$

$$\Gamma_\Lambda(0000;g\mu^{-\varepsilon},\varepsilon) = -ig\mu^{-\varepsilon} \tag{3.7c}$$

whereby the r.h. sides derive from h_{oo} only, and in fact only from the Born terms of this function. The relations (3.7) can be taken as renormalization conditions for direct construction of the Γ_Λ from the Feynman rules deriving from the Lagrangean

$$(3.8) \quad \mathcal{L}_{\Lambda} = -\tfrac{1}{2}\phi\,\Box\,(1+\Lambda^{-2}\Box)\phi - \tfrac{1}{4!}\,g\mu^{-\varepsilon}\phi^{4} +$$

$$+ \text{ counter terms.}$$

Subtractions are to be made as one would in four dimensions to enforce the renormalization conditions (3.7), such that the counter terms (apart from the now necessary mass counter term) are in form proportional to the original terms in the Lagrangean, provided the second subtraction on self energy parts implementing (3.7b) is by subtracting const $p^2(1-\Lambda^{-2}p^2)$ rather than const p^2 as one would usually. (3.8) is obtained from (1.1) by using (3.4) and (3.2b), and noting that

$$\bar{g}(z,\varepsilon) = z + O(z^2)$$

as follows from (3.2a). Again, (3.5) is the effective Lagrangean to the ("renormalized") Lagrangean (3.8).

From the Feynman rules embodied in (3.5) follows that the functions Γ_{Λ} are free of ε-singularities for $0<\varepsilon<4$, and we shall prove this also in the next section in an entirely different, and suggestive, manner. For $\varepsilon = 0$, the Γ_{Λ} are singular in contrast to the $\Gamma_{\Lambda B}$; these are infrared singularities brought about by the conditions (3.7b-c) which are, for $\Lambda=\infty$ and $\Lambda<\infty$, inappropriate for the massless theory in four dimensions /4/. This feature, and a related infrared difficulty with (3.7b-c) in $4+\varepsilon$ dimensions to be seen later, are the reasons for introducing a set of differently "renormalized" VFs in the next section.

4. $\bar{\Gamma}_\Lambda$ -definition and properties

From the unrenormalized VFs $\Gamma_{\Lambda B}$ we construct renormalized VFs $\bar{\Gamma}_\Lambda$ by the formula

$$(4.1)\quad \bar{\Gamma}_\Lambda((2n);\mu,\bar{g},\varepsilon) = $$
$$= Z_3(\bar{g},\Lambda/\mu)^n\,\Gamma_{\Lambda B}((2n);g_B(\bar{g},\mu,\Lambda,\varepsilon),\varepsilon)$$

where we set

$$(4.2a)\quad g_B(\bar{g},\mu,\Lambda,\varepsilon) = \mu^{-\varepsilon}\{\bar{g} + \sum_{k=1}^{\infty}[(\Lambda/\mu)^{\varepsilon k} - 1]\,a_k(\bar{g},\varepsilon)\}$$

$$(4.2b)\quad Z_3(\bar{g},\Lambda/\mu,\varepsilon) = 1 + \sum_{k=1}^{\infty}[(\Lambda/\mu)^{\varepsilon k} - 1]\,z_{3k}(\bar{g},\varepsilon)$$

The functions (power series in $\bar{g}$ with coefficients meromorphic in ε) a_k and z_{3k} can be so chosen that the large -Λ expansion for $\bar{\Gamma}_\Lambda$ becomes

$$(4.3)\quad \bar{\Gamma}_\Lambda((2n);\mu,\bar{g},\varepsilon) = $$
$$= \bar{h}_{00}((2n);\mu,\bar{g},\varepsilon) + \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\Lambda^{-2j+\varepsilon k}\,\bar{h}_{jk}((2n);\mu,\bar{g},\varepsilon)$$

That such a choice is possible follows from sect. 2, and is exhibited in the Λ -independent relation between Γ_Λ in (3.6) and $\bar{\Gamma}_\Lambda$ that we shall establish in (4.15) below.

Applying $\mu[\partial/\partial\mu]|_{g_B,\Lambda}$ to (4.1) yields

$$(4.4a)\quad Op_{2n}\,\bar{\Gamma}_\Lambda((2n);\mu,\bar{g},\varepsilon) = 0$$

with

$$(4.4b) \quad Op_{2n} \equiv \mu[\partial/\partial\mu] + \beta(\bar{g},\varepsilon)[\partial/\partial\bar{g}] - 2n\,\gamma(\bar{g},\varepsilon)$$

where

$$(4.5a) \quad \beta(\bar{g},\varepsilon) = \mu[\partial/\partial\mu]\bar{g}\,|_{g_B,\Lambda} =$$

$$= -\{[\partial/\partial\bar{g}]g_B|_{\Lambda,\mu}\}^{-1}\mu[\partial/\partial\mu]\,g_B|_{\Lambda,\bar{g}} =$$

$$= \varepsilon\bar{g} + b_0(\varepsilon)\bar{g}^2 + b_1(\varepsilon)\bar{g}^3 + \dots$$

$$(4.5b) \quad \gamma(\bar{g},\varepsilon) = \tfrac{1}{2}\mu[\partial/\partial\mu]\ln Z_3\,|_{g_B,\Lambda} =$$

$$= \tfrac{1}{2}Op_{00}\ln Z_3 = c_0(\varepsilon)\bar{g}^2 + c_1(\varepsilon)\bar{g}^3 + \dots$$

with computable functions $b_k(\varepsilon), c_k(\varepsilon)$ meromorphic in ε, and regular for $0 \leqslant \varepsilon < 4$ as we shall prove. The Λ-independence of β and γ follows from (4.2) and (4.3): The first implies that no factors $\Lambda^{-2j+\varepsilon k}$, $j \geqslant 1$ may occur, and the second excludes the occurrence of $\Lambda^{\varepsilon k}$, $k > 0$.

We next introduce the function

$$(4.6) \quad g(\bar{g},\varepsilon) \equiv \bar{g}\exp\{\varepsilon\int_0^{\bar{g}} d\bar{g}'[\beta(\bar{g}',\varepsilon)^{-1} - (\varepsilon\bar{g}')^{-1}]\} =$$

$$= \bar{g} - \varepsilon^{-1}b_0(\varepsilon)\bar{g}^2 + \dots$$

which increases monotonically in the interval $0 < \bar{g} < \bar{g}_\infty(\varepsilon)$ where $\bar{g}_\infty(\varepsilon)$ is the first positive zero of $\beta(\bar{g},\varepsilon)$ if there is one, and we shall only consider $\bar{g}$ in this interval. The inverse function we denote by $\bar{g}(g,\varepsilon)$:

(4.7a) $$\bar{g}(g(\bar{g},\varepsilon),\varepsilon)=\bar{g}$$

(4.7b) $$\bar{g}(g,\varepsilon)=g+\varepsilon^{-1}b_0(\varepsilon)g^2+\cdots.$$

(We shall prove it to be identical with the function in (3.2b).) From (4.5a), (4.2a) and (4.6), (4.7b) follows

(4.8a) $$g_B(\bar{g},\mu,\Lambda,\varepsilon)=\Lambda^{-\varepsilon}\bar{g}(g(\bar{g},\varepsilon)\mu^{-\varepsilon}\Lambda^{\varepsilon},\varepsilon)$$

and from (4.5b) and (4.2b)

(4.8b) $$Z_3(\bar{g},\Lambda/\mu,\varepsilon)=\exp\Big[2\int_{g_B\Lambda^{\varepsilon}}^{\bar{g}}d\bar{g}'\beta(\bar{g}';\varepsilon)^{-1}\gamma(\bar{g}';\varepsilon)\Big].$$

We now consider the function

(4.9) $$[\partial/\partial p^2]\,\Gamma_{1B}(p(-p);g_B,\varepsilon)\big|_{p=0}\equiv i\,F(g_B\Lambda^{\varepsilon};\varepsilon).$$

From (4.1) we have

(4.10) $$F(g_B\Lambda^{\varepsilon};\varepsilon)=-i\,Z_3(\bar{g},\Lambda/\mu,\varepsilon)^{-1}[\partial/\partial p^2]\,\bar{\Gamma}_1(p(-p);\mu,\bar{g},\varepsilon)\big|_{p=0}.$$

The last derivative herein satisfies (4.4a) with n = 1, which implies

(4.11) $$[\partial/\partial p^2]\,\bar{\Gamma}_1(p(-p);\mu,\bar{g},\varepsilon)\big|_{p=0}=$$
$$=i\exp\Big[2\int_0^{\bar{g}}d\bar{g}'\beta(\bar{g}';\varepsilon)^{-1}\gamma(\bar{g}';\varepsilon)\Big]\,\tilde{F}(g(\bar{g},\varepsilon)\mu^{-\varepsilon}\Lambda^{\varepsilon};\varepsilon)$$

where for dimensional reasons Λ can only occur as indicated. For regularity reasons, $\tilde{\tilde{F}}$ must be a power series in its first argument. The l.h.s. in (4.11) is a renormalized and, due to $\varepsilon > 0$, finite (i.e. not IR-divergent) quantity. (4.3) is then applicable and implies that $\tilde{F}(\cdot,\varepsilon)$ must actually be independent of its first argument, such that $\bar{g} \to 0$ yields $\tilde{F}(\cdot,\varepsilon) = 1$. (4.9-10) then yield, with (4.8b)

$$(4.12) \quad [\partial/\partial p^2]\,\Gamma_{1B}(p(-p);g_B,\varepsilon)\Big|_{p=0} = i\,\exp\Big[2\int_0^{g_B\Lambda^\varepsilon} dt\,\beta(t,\varepsilon)^{-1}\gamma(t,\varepsilon)\Big].$$

In the analogous way, one derives

$$(4.13) \quad \Gamma_{1B}(0000;g_B,\varepsilon) = \\ = -i\Lambda^{-\varepsilon} g(g_B\Lambda^\varepsilon;\varepsilon)\,\exp\Big[4\int_0^{g_B\Lambda^\varepsilon} dt\,\beta(t,\varepsilon)^{-1}\gamma(t,\varepsilon)\Big].$$

From these relations with (4.1) and (4.8) follows

$$(4.14a) \quad [\partial/\partial p^2]\Big\{\exp\Big[-2\int_0^{\bar{g}} d\bar{g}'\beta(\bar{g}';\varepsilon)^{-1}\gamma(\bar{g}';\varepsilon)\Big]\,\bar{\Gamma}_1(p(-p);\mu,\bar{g},\varepsilon)\Big\}\Big|_{p=0} = \\ = i$$

$$(4.14b) \quad \exp\Big[-4\int_0^{\bar{g}} d\bar{g}'\beta(\bar{g}';\varepsilon)^{-1}\gamma(\bar{g}';\varepsilon)\Big]\,\bar{\Gamma}_1(0000;\mu,\bar{g},\varepsilon) = \\ = -i\mu^{-\varepsilon} g(\bar{g},\varepsilon).$$

Comparison with (3.7) leads to the identification

$$\exp\Big[-2n\int_0^{\bar{g}} d\bar{g}'\,\beta(\bar{g}',\varepsilon)^{-1}\gamma(\bar{g}',\varepsilon)\Big]\,\bar{\Gamma}_1((2n);\mu,\bar{g},\varepsilon)= \tag{4.15}$$
$$=\Gamma_1((2n);\mu^{-\varepsilon}g(\bar{g},\varepsilon),\varepsilon)$$

and finally, with (4.1) and (4.8), to

$$\Gamma_1((2n);\mu^{-\varepsilon}g,\varepsilon)=Z_3(g_0\Lambda^{\varepsilon},\varepsilon)^n\,\Gamma_{10}((2n);g_0,\varepsilon) \tag{4.16a}$$

where

$$g_0=\Lambda^{-\varepsilon}\,\bar{g}(g\mu^{-\varepsilon}\Lambda^{\varepsilon},\varepsilon) \tag{4.16b}$$

and

$$Z_3(g_0\Lambda^{\varepsilon},\varepsilon)=\exp\Big[-2\int_0^{g_0\Lambda^{\varepsilon}} dt\,\beta(t,\varepsilon)^{-1}\gamma(t,\varepsilon)\Big]. \tag{4.16c}$$

This shows that the $\bar{g}$ -function (4.7) is identical with the one in (3.2b) and also yields an expression for Z_3 in (3.1) and (3.4).

The formulae (4.12) and (4.13) allow the functions $\beta(\cdot,\varepsilon)$ and $\gamma(\cdot,\varepsilon)$ to be introduced without any reference to renormalization, but only to regularization, and may even be useful for computation of these functions. They also show directly the dependence of these functions on the manner of regularization, which means that the $\bar{g}$-parametrization of the renormalized functions is regularization dependent. This dependence will also persist in the $\varepsilon \to 0$ limit. While for $\varepsilon = 0$ the $\Lambda \to \infty$ limit is formally trivial, cp. (3.6) and (4.3), the (in principle, simpler) g-parametrization is not applicable as we noted at the end of the last section; the $\bar{g}$-parametrization, however, is then still applicable, and from (4.6) one finds

$$g(\bar{g}, \varepsilon) = \varepsilon\, b_0^{-1} + O(\varepsilon^2 \ln \varepsilon,\ \varepsilon^2)$$

$$\text{for} \quad \varepsilon \to 0, \ \bar{g} > 0.$$

The definition of a renormalized ϕ^4 theory in $4 + \varepsilon$ dimensions, $\varepsilon \leq 0$, by the ansatz (4.1-3) leading to (4.4) with Λ-independent parametric functions (and to a similar PDE for the $\Gamma_{\Lambda B}$ themselves) is due to Zinn-Justin /5/. While for $\varepsilon < 0$ the construction in perturbation theory is not actually possible /6/, for $\varepsilon = 0$ factors $[\ln(\Lambda/\mu)]^K$ replace the square brackets in (4.2).

5. $\Gamma_{\Lambda i}$-definition and differentiation identities

We shall find it useful to define $\Gamma_{\Lambda i}$ functions whose graphs are obtained from the graphs of the Γ_Λ functions by inserting an extra vertex O_i with zero momentum transfer. Thus the $\Gamma_{\Lambda i}$ are the Fourier transforms of the amputated one-particle-irreducible parts of the connected functions

$$(5.1) \quad \int dx \left\langle \left(O_i(x)\, \phi(x_1) \dots \phi(x_{2n}) \right)_+ \right\rangle_{conn} = G_{\Lambda i}(x_1 \dots x_{2n};\, g\mu^{-\varepsilon}, \varepsilon)$$

defined, following Zimmermann /7/, by renormalization conditions complementing (3.7). Specifically, we consider the set of operators

$i =$	1	2	3	4	5	6
$O_i =$	ϕ^2	$\phi\Box\phi$	ϕ^4	$\phi\Box^2\phi$	$\phi^3\Box\phi$	ϕ^6
$\dim O_i =$	$2+\varepsilon$	$4+\varepsilon$	$4+2\varepsilon$	$6+\varepsilon$	$6+2\varepsilon$	$6+3\varepsilon$

Here

$$(5.2) \qquad \dim O_i = 2\alpha_i + \varepsilon k_i \quad , \quad 1 \leq k_i \leq \alpha_i$$

is the ordinary dimension; in the notation for the O_i, we have suppressed, besides counter terms, only dimensionless factors. We have listed only the operators that are, for zero momentum transfer, linearly independent, note e.g.

$$\phi^2 \partial_\mu \phi \partial^\mu \phi = -\frac{1}{3}\phi^3 \Box \phi + \frac{1}{3}\partial_\mu (\phi^3 \partial^\mu \phi) =$$
$$= -\frac{1}{4}\phi^2 \Box \phi^2 + \frac{1}{4}\partial_\mu (\phi^2 \partial^\mu \phi^2).$$

If we now define the renormalized operators by minimal subtraction for $\varepsilon > 0$ sufficiently small, we may choose the renormalization conditions

$$(5.3a) \quad \Gamma_{1i}(00;\, g\mu^{-\varepsilon}, \varepsilon) = \delta_{i1} \qquad \forall i$$

$$(5.3b) \quad [\partial/\partial p^2]\, \Gamma_{1i}(p(-p);\, g\mu^{-\varepsilon}, \varepsilon)\big|_{p=0} = \delta_{i2} \qquad i > 1$$

$$(5.3c) \quad \Gamma_{1i}(0000;\, g\mu^{-\varepsilon}, \varepsilon) = \delta_{i3} \qquad i > 1$$

$$(5.3d) \quad [\partial/\partial p^2]^2\, \Gamma_{1i}(p(-p);\, g\mu^{-\varepsilon}, \varepsilon)\big|_{p=0} = \delta_{i4} \qquad i > 3$$

$$(5.3e) \quad [\partial/\partial p^2]\, \Gamma_{1i}(p(-p)00;\, g\mu^{-\varepsilon}, \varepsilon)\big|_{p=0} = \delta_{i5} \qquad i > 3$$

$$(5.3f) \quad \Gamma_{1i}(000000;\, g\mu^{-\varepsilon}, \varepsilon) = \delta_{i6} \qquad i > 3$$

which imply that in the construction of O_i as counterterms only operators of the form O_j with $\alpha_j \leq \alpha_i$ are admitted. The conditions (5.3) lead for $0 < \varepsilon < 3$ to unambiguous IR and UV-singularity-free functions $\Gamma_{\Lambda i}$. We shall indicate below how to obtain formally exact expressions for the O_i, i = 1,2,3 in terms of ordinary operator products, and also how functions $\bar{\Gamma}_{\Lambda i}$, analogous to the $\Gamma_{\Lambda i}$ but in the $\bar{g}$ parametrization, are constructed.

First we derive the "counting identities" /8/. The generating functional /9/ of Green's functions is

$$
\begin{aligned}
(5.4)\quad & G\{J\} = \\
& = \ln\Big[\int \mathcal{D}(\phi) \exp\Big\{i\int dx\big[L(\phi,\partial\phi,\Box\phi) + J\phi\big]\Big\}\Big] \\
& \quad - \text{(same with J = 0)}.
\end{aligned}
$$

We perform the variable replacement $\phi \to (1+\delta)\phi$, δ infinitesimal. This yields

$$
\begin{aligned}
(5.5)\quad 0 = & \Big[\int \mathcal{D}(\phi) \exp\Big\{i\int dx\big[L(\phi,\partial\phi,\Box\phi) + J\phi\big]\Big\}\Big]^{-1} \cdot \\
& \cdot i\int \mathcal{D}(\phi) \int dx\Big[\phi\frac{\partial L}{\partial\phi} + \partial\phi\frac{\partial L}{\partial\partial\phi} + \Box\phi\frac{\partial L}{\partial\Box\phi} + J\phi\Big]\exp\{\ldots\} - \\
& - \text{(same with J = 0)}
\end{aligned}
$$

whereby the effect of the infinitesimal change on $\mathcal{D}(\phi)$ has cancelled out. (5.5) can be rewritten

$$
(5.6)\quad \Delta G\{J\} + J[\delta/\delta J]\, G\{J\} - \Delta G\{0\} = 0
$$

whereby $\Delta G\{J\}$ is the connected part induced by a certain infinitesimal change of the Lagrange function, for fixed J, evaluated by Schwinger's action principle. Going over to one-particle-irreducible functions by standard formulae /9/ transforms (5.6) into the relation for the generating functional of VFs

$$\Delta \Gamma\{\alpha\} - \alpha[\delta/\delta\alpha]\,\Gamma\{\alpha\} = 0$$

or, for the VFs themselves, using (1.1) and (3.4),

$$n\Gamma_1((2n); \mu^{-\varepsilon}, g, \varepsilon) =$$

$$(5.7)\qquad = i\left\langle \left(\int dx\left[-\tfrac{1}{2} Z_3\, \phi\Box(1+\Lambda^{-2}\Box)\phi - 2\tfrac{1}{4!} Z_3^2 g_B\, \phi^4 - \tfrac{1}{2} Z_3\, m_{B0}^2\, \phi^2\right] \tilde{\phi}(p_1)\cdots\tilde{\phi}(p_{2n})\right)_+ \right\rangle^{proper}$$

$$= \sum_{i=1}^{6} e_i(g\mu^{-\varepsilon}, \Lambda, \varepsilon)\, \Gamma_{\Lambda i}((2n); g\mu^{-\varepsilon}, \varepsilon).$$

Note that the sum herein goes over all six O_i rather than only the first four, since the renormalized operators involve all six formal expressions O_i as counter terms. Dimension counting gives

$$(5.8)\qquad e_i(g\mu^{-\varepsilon}, \Lambda, \varepsilon) = \Lambda^{4+\varepsilon-\dim O_i}\, e_i(g\mu^{-\varepsilon}\Lambda^{\varepsilon}, \varepsilon).$$

From (3.7), (5.2-3) and (5.7) follows

$$(5.9)\qquad e_1 = 0\;, \quad e_2 = i\;, \quad e_3(z,\varepsilon) = -2iz.$$

The expressions for the e_i, i = 4,5,6, obtained from (5.7-9) and (5.2-3), in terms of higher Γ_Λ functions and their derivatives at zero momentum are unilluminating and involve, for $\varepsilon \leq 2$, delicate cancellations of IR divergences. That all e_i are IR and UV divergence-free for $0 < \varepsilon < 3$ follows, however, directly from (5.7) by solving it for the e_i at six sets of generic momenta.

The Schwinger action principle also gives

$$[\partial/\partial(g\mu^{-\varepsilon})]\,\Gamma_\Lambda((2n);\, g\mu^{-\varepsilon},\varepsilon) =$$

$$(5.10)\qquad = i\,\Big\langle\Big(\int dx\,\big[-\tfrac{1}{2}\phi\Box(1+\Lambda^{-2}\Box)\phi\;\partial Z_3/\partial(g\mu^{-\varepsilon}) -$$

$$-\tfrac{1}{4!}\phi^4\,\partial(g_0 Z_3^2)/\partial(g\mu^{-\varepsilon}) - \tfrac{1}{2}\phi^2\,\partial(m_{B0}^2 Z_3)/\partial(g\mu^{-\varepsilon})\big]\cdot$$

$$\cdot\,\tilde\phi(p_1)\ldots\tilde\phi(p_{2n})\Big)_+\Big\rangle^{proper} =$$

$$= \sum_{i=1}^{6} d_i(g\mu^{-\varepsilon},\Lambda,\varepsilon)\,\Gamma_{\Lambda i}((2n);\,\mu^{-\varepsilon}g,\varepsilon)$$

with

$$(5.11)\qquad d_i(g\mu^{-\varepsilon},\Lambda,\varepsilon) = \Lambda^{4+2\varepsilon-\dim O_i}\, d_i(g\mu^{-\varepsilon}\Lambda^{\varepsilon},\varepsilon).$$

From (3.7) and (5.2-3) follows

$$(5.12)\qquad d_1 = d_2 = 0,\; d_3 = -i.$$

Since d_i, i = 4,5,6 and e_i, i = 4,5,6, both obtain in (5.10) respectively (5.7) only contributions from the $\phi\Box^2\phi$ term, we must have

$$d_i(g\mu^{-\varepsilon}\Lambda^{\varepsilon},\varepsilon)\,/\,e_i(g\mu^{-\varepsilon}\Lambda^{\varepsilon},\varepsilon) =$$
$$= \Lambda^{-\varepsilon}\,\partial(\ln Z_3)/\partial(g\mu^{-\varepsilon}) \qquad i=4,5,6$$

and evaluating the r.h.s. using (4.16b-c) and the definition (4.7a) of $\bar{g}(\cdot,\varepsilon)$ yields

$$d_i(z,\varepsilon)\,/\,e_i(z,\varepsilon) = -2\varepsilon^{-1}z^{-1}\gamma(\bar{g}(z,\varepsilon),\varepsilon) \qquad i=4,5,6. \tag{5.13}$$

A third insertion identity is obtained by computing

$$\Lambda[\partial/\partial\Lambda]\,\Gamma_\Lambda((2n);g\mu^{-\varepsilon},\varepsilon) =$$
$$= i\langle\Big(\int dx\,\big[-\tfrac{1}{2}\phi\Box(1+\Lambda^{-2}\Box)\phi\,\Lambda\,\partial Z_3/\partial\Lambda +$$
$$+\Lambda^{-2}\phi\Box^2\phi\, Z_3 - \tfrac{1}{4!}\phi^4\,\Lambda\,\partial(g_0 Z_3^2)/\partial\Lambda -$$
$$-\tfrac{1}{2}\phi^2\,\Lambda\,\partial(m_{0o}^2 Z_3)/\partial\Lambda\big]\,\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_{2n})\Big)_+\rangle^{proper} = \tag{5.14}$$
$$= \sum_{i=1}^{6} c_i(g\mu^{-\varepsilon},\Lambda,\varepsilon)\,\Gamma_{\Lambda i}((2n);g\mu^{-\varepsilon},\varepsilon)$$

with

$$c_i(g\mu^{-\varepsilon},\Lambda,\varepsilon) = \Lambda^{4+\varepsilon-\dim O_i}\, c_i(g\mu^{-\varepsilon}\Lambda^{\varepsilon},\varepsilon). \tag{5.15}$$

(3.7) and (5.2-3) yield

(5.16) $c_1 = c_2 = c_3 = 0$

and

(5.17a) $$c_4(g\mu^{-\varepsilon}\Lambda^{\varepsilon}, \varepsilon) = \\ = \frac{1}{2}\Lambda^3[\partial/\partial\Lambda][\partial/\partial p^2]^2 \Gamma_1(p(-p);\mu^{-\varepsilon}g, \varepsilon)\Big|_{p=0}$$

(5.17b) $$c_5(g\mu^{-\varepsilon}\Lambda^{\varepsilon}, \varepsilon) = \\ = \Lambda^{3+\varepsilon}[\partial/\partial\Lambda][\partial/\partial p^2]\Gamma_1(p(-p)00; g\mu^{-\varepsilon}, \varepsilon)\Big|_{p=0}$$

(5.17c) $$c_6(g\mu^{-\varepsilon}\Lambda^{\varepsilon}, \varepsilon) = \\ = \Lambda^{3+2\varepsilon}[\partial/\partial\Lambda]\Gamma_1(000000; g\mu^{-\varepsilon}, \varepsilon).$$

On the other hand, from (5.14) and (5.7) follows

$$c_i(g\mu^{-\varepsilon}\Lambda^{\varepsilon}, \varepsilon)/e_i(g\mu^{-\varepsilon}\Lambda^{\varepsilon}, \varepsilon) = \\ = \Lambda[\partial/\partial\Lambda]\ln Z_3 - 2 \qquad i = 4, 5, 6$$

and evaluating the r.h.s.

(5.18) $$c_i(z,\varepsilon)/e_i(z,\varepsilon) = -2[1+\gamma(\bar{g}(z,\varepsilon), \varepsilon)] \qquad i = 4, 5, 6.$$

The coefficients $C'_{rs\nu}(g\mu^{-\varepsilon}\Lambda^{\varepsilon}, \varepsilon)$ in (3.5) are closely related to expressions of the variety (5.17). To show this, we form, using again the Schwinger action principle,

$$\Lambda[\partial/\partial\Lambda]\, \Gamma_{\Lambda}((2n); g\mu^{-\varepsilon}, \varepsilon) =$$

(5.19) $$= i \langle \Big(\int dx \Big\{ \sum_{\substack{r=0\\ r+s\geq 3}}^{\infty} \sum_{s=1}^{\infty} \sum_{\nu=1}^{n_{rs}} ({}^{\shortparallel}D^{2r}\phi^{2s\,\shortparallel})_{\nu}\, (g\mu^{-\varepsilon})^{s-1} \cdot$$

$$\cdot \Lambda[\partial/\partial\Lambda][C'_{rs\nu}(g\mu^{-\varepsilon}\Lambda^{\varepsilon}, \varepsilon)\, \Lambda^{4-2r-2s}] \Big\} \tilde{\phi}(p_1) \ldots \tilde{\phi}(p_{2n}) \Big)_{+} \rangle^{proper}$$

where the r.h.s. is to be evaluated by analytic integration using the Lagrangean (3.5) again. Consider, e.g., the term r = 0, s = 3. This term is isolated by forming

$$\Lambda[\partial/\partial\Lambda]\, \Gamma_{\Lambda}(000000; g\mu^{-\varepsilon}, \varepsilon) =$$

(5.20) $$= i\, 6!\, (g\mu^{-\varepsilon})^{2}\, \Lambda[\partial/\partial\Lambda][C'_{031}(g\mu^{-\varepsilon}\Lambda^{\varepsilon}, \varepsilon)\, \Lambda^{-2}]$$

since for these momenta, all other terms in (5.19) <u>and</u> all not-lowest--order terms in the evaluation of the matrix element do not contribute for dimensional reasons, in view of the homogeneous propagator i/p^2 in the analytic integrations and a simple momentum-power counting. Thus, using (5.17c) on the l.h.s. of (5.20) we have

$$\varepsilon z[\partial/\partial z][C'_{031}(z, \varepsilon)\, z^{-2/\varepsilon}] = -i\, (6!)^{-1} z^{-2-2/\varepsilon}\, c_6(z, \varepsilon)$$

and finally

(5.21) $C'_{031}(z,\varepsilon) =$

$$= -i\,(6.!)^{-1}\varepsilon^{-1} z^{2/\varepsilon} \underset{\text{from } \varepsilon>2}{\text{anal. cont.}} \int_0^z dt\, t^{-3-2/\varepsilon} C_6(t,\varepsilon)$$

whereby the prescription for evaluation of the integral follows from (2.4), (3.3), and the property $C_6(z,\varepsilon) \sim \mathcal{O}(z^3)$ implied by (5.17c). Using (5.17c) again, (5.21) becomes

(5.22) $C'_{031}(g\mu^{-\varepsilon}\Lambda^{\varepsilon},\varepsilon) =$

$$= -i\,(6.!)^{-1}(g\mu^{-\varepsilon})^{-2}\Lambda^{2} \underset{\text{from } \varepsilon>2}{\text{anal. cont.}} \left[\Gamma_1(000000; g\mu^{-\varepsilon}; \varepsilon) - \Gamma_0(000000; g\mu^{-\varepsilon}; \varepsilon)\right]$$

where the prescription now relates to the extra IR divergences of the Γ_0 function not cancelled out by those of the Γ_1-function .- Similar "exact" formulae to (5.22) can be derived for all coefficient functions in (3.5).

The relations (5.7), (5.10) and (5.14) can be written as

(5.23a) $$a_e\phi^2 + b_e\,\phi\Box\phi + c_e\phi^4 + d_e\,\phi\Box^2\phi = \sum_{i=2}^{6} e_i\,O_i$$

(5.23b) $$a_d\phi^2 + b_d\,\phi\Box\phi + c_d\phi^4 + d_d\,\phi\Box^2\phi = \sum_{i=3}^{6} d_i\,O_i$$

(5.23c) $$a_c\phi^2 + b_c\,\phi\Box\phi + c_c\phi^4 + d_c\,\phi\Box^2\phi = \sum_{i=4}^{6} c_i\,O_i\,.$$

Using (5.13) and (5.18) we can obtain herefrom expressions for O_2 and O_3 as linear combinations of ϕ^2, $\phi\Box\phi$, and ϕ^4 (we neglect c-number terms), with coefficients expressible in terms of $m^2_{\beta_0}$ of (1.1b) and the parametric functions β and γ. O_1 is equal to $\frac{1}{2} Z_2 \phi^2$ with Z_2 a renormalization factor involving a parametric function /1/ not introduced here. As to the higher operators, only the linear combination $e_4 O_4 + e_5 O_5 + e_6 O_6$ can be expressed directly in terms of the ordinary operator products ϕ^2, $\phi \Box \phi$, ϕ^4, and $\phi \Box^2 \phi$.

There are analoga $\bar{O}_1 \ldots \bar{O}_6$ to the operators $O_1 \ldots O_6$, adapted to the Zinn-Justin parametrization using $\bar{g}$. To define these, one starts from an Ansatz extending (4.1) and (4.2) to VFs containing these operator product insertions in a natural manner. The operators so introduced are defined also for $\varepsilon = 0$, and they have particularly simple construction rules in perturbation theory. (For $\varepsilon = 0$ and $\Lambda = \infty$, they are linear combinations of massless- ϕ^4-theory compositeoperators as one may introduce /10/ using 't Hooft's dimensional renormalization /11/. The $\bar{g}$-parametrization is in this case not identical to the 't Hooft one of ϕ^4 theory, in fact, as mentioned in section 4, it remains regularization-manner dependent.)

6. The $\Lambda \to \infty$ limit

It should not be surprising that we cannot prove that the functions Γ_Λ, $\Gamma_{\Lambda i}$ or equivalently, $\bar{\Gamma}_\Lambda$ etc., have a limit for $\Lambda \to \infty$. Even for $\varepsilon = 0$, the renormalizable case, one can so far not prove removability of the cutoff (in the $\bar{\Gamma}_\Lambda$ functions) except termwise in perturbation theory. The characteristic feature of nonrenormalizable theory is that the cutoff cannot be removed even in perturbation theory; we can only speculate on what the resulting theory might look like assuming the cutoff could be removed - and this assumption necessarily leads to a description of the theory that is not (ordinary) perturbation theory. This we shall show in this section, and will also point out some further consequences.

Consider first the Lagrangean $\mathcal{L}_\Lambda$ of (3.5). We can write it as

$$(6.1a)\qquad \mathcal{L}_\Lambda = -\tfrac{1}{2}\phi\Box\phi - \tfrac{1}{4!}\, g\mu^{-\varepsilon}\phi^4 +$$

$$+\sum_{\substack{r=0\\ r+s\geq 3}}^{\infty}\sum_{s=1}^{\infty}\sum_{\nu=1}^{n_{rs}} c_{rs\nu}(g\mu^{-\varepsilon}\Lambda^{\varepsilon},\varepsilon)\,(\text{"}D^{2r}\phi^{2s}\text{"})_\nu\,(g\mu^{-\varepsilon})^{s-1+2(r+s-2)/\varepsilon}$$

with

$$(6.1b)\qquad c'_{rs\nu}(z,\varepsilon)\, z^{2(2-r-s)/\varepsilon} \equiv c_{rs\nu}(z,\varepsilon).$$

Since the Feynman rules described by the first two terms of $\mathcal{L}_\Lambda$ do not involve Λ , a sufficient condition for $\Gamma_\Lambda \to \Gamma_\infty$ at least in a certain expansion is that $\mathcal{L}_\Lambda \to \mathcal{L}_\infty$ in the sense

$$(6.2)\qquad \lim_{\Lambda\to\infty} c_{rs\nu}(g\mu^{-\varepsilon}\Lambda^{\varepsilon},\varepsilon) = \bar{c}_{rs\nu}(\varepsilon)$$

which, if satisfied, would yield for the limit of (3.6)

$$(6.3)\qquad \Gamma_\infty((2n);\, g\mu^{-\varepsilon},\varepsilon) = \sum_{j=0}^{\infty}(g\mu^{-\varepsilon})^{2j/\varepsilon}\, h_j((2n);\, g\mu^{-\varepsilon},\varepsilon)$$

since the integer powers of $g\mu^{-\varepsilon}$ in (6.1a) can be absorbed in the h_j, which are power series in $g\mu^{-\varepsilon}$. These functions, however, involve the unknown constants $\bar{c}_{rs\nu}(\varepsilon)$, with the exception of $h_o \equiv h_{oo}$ of (3.6).

From (5.22) and (6.1b) we have

$$(6.4)\qquad \bar{c}_{031}(\varepsilon) = -i\,(6.1)^{-1}\,(g\mu^{-\varepsilon})^{-2-2/\varepsilon}\,.$$

$$\cdot \underset{\text{from } \varepsilon>2}{\text{anal. cont.}}\ \lim_{\Lambda\to\infty}\left[T_\Lambda(000000;g\mu^{-\varepsilon},\varepsilon) - T_0(000000;g\mu^{-\varepsilon},\varepsilon)\right]$$

and similar formulae can be derived for the other $\bar{c}_{rs\nu}(\varepsilon)$ functions. These functions have ε-singularities by virtue of which the ε-singularities at rational ε in $0<\varepsilon<3$ of the h_o term in (6.3) are cancelled, as they are in (3.6) for finite Λ. Indeed, omitting the lim-sign, the ε-singularities on the r.h.s. of (6.4) are Λ-independent since there are none in the function $c_6(z,\varepsilon)$ as emphasized earlier.

To make the singularities in $\bar{c}_{031}(\varepsilon)$ more explicit, one uses the expansion

$$(6.5)\qquad c_6(z,\varepsilon) = \sum_{k=3}^{\infty} c_{6k}(\varepsilon)\, z^k$$

and splits the integration region in (5.21) into 0 to 1 and 1 to z, obtaining

$$(6.6)\qquad \bar{c}_{031}(\varepsilon) = \sum_{k=3}^{\infty}\left[-2+(k-2)\varepsilon\right]^{-1} c_{6k}(\varepsilon) +$$

$$+ (g\mu^{-\varepsilon})^{-2-2/\varepsilon}\ \underset{\text{from } \varepsilon>2}{\text{anal. cont.}}\ \lim_{\Lambda\to\infty}\left[T_\Lambda(000000;g\mu^{-\varepsilon},\varepsilon) - \right.$$

$$\left. - T_{\mu g^{-1/\varepsilon}}(000000;g\mu^{-\varepsilon},\varepsilon)\right].$$

The $C_{6k}(\varepsilon)$ are regular in $0<\varepsilon<3$ and so is the Λ-limitand in (6.6); the explicit ε-singularities in (6.6) are the ones needed for cancellation of singularities in h_o or h_{oo} and are indeed Λ-independent. Since h_o is independent of the manner of regularization (cp. sect. 7 below), the values $C_{6k}(2(k-2)^{-1}),\ k\geq 3$ are regularization-manner independent though the functions $C_{6k}(\varepsilon)$ are not.

In (6.6) the two terms in the square bracket have separately singularities at $\varepsilon=2/\mathcal{L}$, $\mathcal{L}$ the number of loops, but these cancel. Thus, for $\varepsilon>2$, which secures that the integration leading to the first term on the r.h.s. of (6.6) is allowed, existence of $\lim_{\Lambda\to\infty}\Gamma_\Lambda(000000;g\mu^{-\varepsilon},\varepsilon)$, the limitand being there free of ε-singularities, implies existence of $\bar{C}_{031}(\varepsilon)$. Hereby we disregard the convergence problem of the sum in (6.6) on which we cannot say much except that at least for $\varepsilon>2$ there is no reason for nonconvergence. Since similar considerations apply to all the $\bar{C}_{rs\nu}(\varepsilon)$, the existence of the $\bar{C}_{rs\nu}(\varepsilon)$ in (6.2) is not only a sufficient condition for the existence of the Γ_∞ at least in the form (6.3), but also a necessary one.

There is another way to arrive at the expansion (6.3), namely by reintegrating (5.14), hereby using partial integrations and the properties of the $C_i(z,\varepsilon)$. Though in this way one in principle avoids perturbation expansions, the conclusions concerning Γ_∞ are best summarized in the quasi-perturbation expansion (6.3). For details hereto I refer to the original paper /1/.

It is obvious that for $\varepsilon = 1$ or 2, (6.3) will reduce to the expansion

(6.7) $$\Gamma_\infty((2n);\mu^{-\varepsilon}g,\varepsilon) =$$
$$= \sum_{k=0}^{\infty}[\ln(g\mu^{-\varepsilon})]^k H_k((2n);g\mu^{-\varepsilon},\varepsilon) \qquad \varepsilon = 1 \text{ or } 2$$

where again infinitely many constants definable by limits similar to the one in (6.2) enter. One is really only interested in positive integer ε, and it is possible to derive (6.7) directly from e.g. the integer-ε form of L_Λ which does involve $\ln\Lambda$-terms directly. This direct derivation is desirable since the condition of $(\phi^4)_{4+\varepsilon}$ existence with generic ε is for integer ε not directly sufficient and perhaps unnecessary.

We will derive some consequences of the hypothesis that Γ_∞ does exist. From (4.15) we learn that $\bar{\Gamma}_\infty$ also exists, and it will satisfy the PDE (4.4). In the familiar way (the quite nontrivial assumptions needed hereby are listed e.g. in /12/) one argues that, if $\beta(\bar{g},\varepsilon)$ has a first zero for positive $\bar{g}_\infty(\varepsilon)$, the family of theories with $\bar{g} < \bar{g}_\infty(\varepsilon)$ obeys asymptotic scale (and conformal /12/) invariance with the anomalous dimension for the operator ϕ $1+\gamma(\bar{g}_\infty(\varepsilon),\varepsilon)$. Positivity reasons (rigorous only for integer ε) give $\gamma(\bar{g}_\infty(\varepsilon),\varepsilon) \geq 0$. (That $\bar{g} \to \bar{g}_\infty(\varepsilon)$ implies $g \to \infty$ reflects that (3.7b-c) are unadmissible for a scale invariant theory.) In (5.14), the l.h.s. will vanish for $\Lambda \to \infty$. Thus, because of (5.18) and (5.13), the contributions from the i = 4,5,6 terms on the r.h.s. of (5.7) and (5.10) will vanish for $\Lambda \to \infty$ if only $\gamma(\bar{g}(z,\varepsilon),\varepsilon)$ is for large z bounded away from -1. It follows that $\Gamma_{\infty 2}$ and $\Gamma_{\infty 3}$ will exist, and (5.7) and (5.10) will hold with the i = 4,5,6 terms omitted i.e. as if the theory were unregularized.

7. Discussion

Renormalizable theories, defined by perturbation theory, have the following features

a) they proceed with a finite number of expansion parameters

b) all terms in the expansion have, within logarithms, the same large-momenta behaviour (on and off the mass shell, for gauge theories in renormalizable gauges)

c) it is not possible, for given "particle species", to increase the number of parameters without violating b)

d) unitarity and causality are formally satisfied (in terms of validity of e.g. GLZ /14/ or Epstein-Glaser /15/ systems)

e) these theories can be obtained from local Lagrangeans using a cutoff and letting $\Lambda \to \infty$ termwise in the perturbation expansions.

In the construction of Γ_∞ in the form (6.3) a), d), and e) in a modified way are satisfied, while b) is violated and c), therefore, empty. The same will hold for integer-ε theory in the form (6.7).
Ordinary BPHZ renormalization theory (see Osterwalder's lecture for references) or the equivalent Epstein-Glaser construction /15/ will lead to theories satisfying d) above but neither e) nor the, in our approach closely related, a).

One may impose a) also by demanding that in the BPHZ construction, $g\mu^{-\varepsilon}$ remain the only expansion parameter. This is equivalent to replace the $\bar{C}_{rs\nu}(\varepsilon)$ that appear in the $\Lambda \to \infty$ limit of (6.1a) by functions of ε that have the ε-singularities needed for cancellation of such singularities in the Γ_∞ but with arbitrary finite parts (e.g., for $\bar{C}_{031}(\varepsilon)$ this means keeping the singularities on the r.h.s. of (6.6) but changing the nonsingular part arbitrarily), and is effectively the proposal of Blokhintsev, Efremov, and Shirkov /16/. What is being gained by satisfying besides a) also e), as we assume to be possible?

Loss of b), as is almost certain for expansions (6.3) and (6.7), will lead to results of no direct use, since higher-order terms will increase, by powers, more and more strongly in momentum space, and it is only at large momenta $\gtrsim \mu\, g^{-1/\varepsilon}$ that the characterizing features of the theory can show themselves at all. Thus, a resummation will be necessary to bring that increase under control. The less there are undetermined coefficients in expansions such as (6.3) or (6.7), the better are the chances for imaginable resummation procedures to be successful.

A particular case in point is unitarity. While a general nonrenormalizable BPHZ or Epstein-Glaser theory has no reason to be free of indefinite metrik (if one resums such as to obtain something meaningful at all), we believe the theory gotten by the $\Lambda \to \infty$ process has a better chance since formally it then has an ordinary (i.e. unregularized) Lagrangean. This latter claim is somewhat substantiated by the differential relations discussed at the end of section 6, which for Γ_∞ take the form they would for an unregularized $(\phi^4)_{4+\varepsilon}$ Lagrangean.

The big problem is, of course, resummation, and it has been attacked by Parisi /17/ from an entirely different starting point. His proposals, which I shall not discuss here, rest on anomalous dimensions; they are mainly made for theories of the class considered here, but with the condition that the renormalizable ($\varepsilon = 0$) theory be "asymptotically free". It is indeed likely that for theories of this type, Parisi's method (cp. also /18/) might lead more easily to physically meaningful approximations than the detour we are taking, of which the "resummation" part is nonexisting still. It ought to lead also to anomalous dimensions, as discussed already in section 6. - The most radical way to proceed is, of course, to start directly from anomalous dimensions and to construct directly the scale - and conformal invariant theory (cp. /19/ and references therein); the $\bar{g} < \bar{g}_\infty(\varepsilon)$ theory is then the preasymptotic one /12/ hereto. In contrast, the present approach is relatively old-fashioned and elucidates why nonrenormalizable theories are not renormalizable in ordinary perturbation theory; it is akin to T.D. Lee's methods /20/ of "summing leading terms", which it confirms as far as these have been developed.

There is the question of ambiguities in our method. The sole ambiguity so far lies in the choice of regularization in (1.1a). For instance, replacing $1+\Lambda^{-2}\Box$ by $\prod_{i=1}^{R}(1+a_i^2\Lambda^{-2}\Box)$ would lead for $\Lambda<\infty$ to a dependence of all functions on R and the ratios of the a_i, and this dependence might persist for $\Lambda\to\infty$. (E.g., for $\varepsilon = 0$ it leads to a different $\bar{g}$-parametrizations of the same family of theories as mentioned in sects. 4 and 5, the g-parametrization not being available). At present we have no clue as to whether this dependence would lead to different theories satisfying the conditions (3.7).

The class of theories to which our method is, in principle, applicable was described in sect. 1. A condition is that the continuous increase of dimension does not pose difficulties, as e.g. the well-known γ_5 one in spinor theories. In that case, one could still treat the theory directly at positive-integer ε as mentioned in sect. 6, whereby e.g. the γ_5 problem is avoided. (This was pointed out to the author by P.K. Mitter.) Theories of interest are mainly four-fermion theories in four dimension, and perhaps, for study purposes, ϕ^6 theory, renormalizable in three dimensions, instead in four dimensions. In any case, the next, and still formal, step will be to develop the method directly for integer space-time dimension.

References

/1/ Symanzik, K.: Comm. math. Phys. (DESY 75/12)

/2/ Pauli, W., Villars, F.: Rev. Mod. Phys. 21, 434 (1949)
Pais, A., Uhlenbeck, G.: Phys. Rev. 79, 145 (1950)

/3/ Wilson, K.G.: Phys. Rev. D7, 2911 (1973)
't Hooft, G., Veltman, M.: in: Particle Interactions at Very High Energies, part B, Eds. D. Speiser, F. Halzen, J. Weyers, New York: Plenum Press 1974

/4/ Symanzik, K.: Comm. math. Phys. 34, 7 (1973);
Acta Physica Austriaca, Suppl. XI, 199 (1973)

/5/ Zinn-Justin, J.: in: Cargèse Lectures in Physics 1973 (Ed., E. Brézin)
Zinn-Justin, J.: TH.1943-CERN, Oct. 1974
Brézin, E., Le Guillou, J.C., Zinn-Justin, J.: DPh-T/74/100, Saclay, Dec. 1974.

/6/ Symanzik, K.: Lett. Nuovo Cimento 8, 771 (1973);
Cargèse Lectures in Physics 1973 (Ed. E. Brézin) (DESY 73/58)

/7/ Zimmermann, W.: Ann. Phys. (N.Y.) 77, 536, 570 (1973)

/8/ Lowenstein, J.L.: Comm. math. Phys. 24, 1 (1971)

/9/ Symanzik, K.: Comm. math. Phys. 16, 48 (1970); in:
Cargèse Lectures in Physics 1970, Ed. D. Bessis, New York: Gordon and Breach 1972

/10/ Collins, J.C.: Nucl. Physics B92, 477 (1975)
Breitenlohner, P., Maison, D.: MPI-PAE/pTh 15/75

/11/ 't Hooft, G.: Nucl. Physics B61, 455 (1973); in: Renormalization and Invariance in Quantum Field Theory, Ed. R. Caianiello, New York: Plenum Press 1974

/12/ Symanzik, K.: Comm. math. Phys. 23, 49 (1971); in: Particles, Quantum Fields and Statistical Mechanics, Eds. M. Alexanian, A. Zepeda, Lecture Notes in Physics v. 32, Berlin: Springer 1975

/13/ Schroer, B.: Lett. Nuovo Cimento 2, 867 (1971)

/14/ Glaser, V., Lehmann, H., Zimmermann, W.: Nuovo Cimento 6, 1122 (1957) Steinmann, O.: Perturbation Expansion in Axiomatic Field Theory; Berlin: Springer 1971

/15/ Epstein, H., Glaser, V.G.: in: Mechanique statistique et théorie quantique des champs, Eds. De Witt, C., Stora, R., New York: Gordon and Breach 1971, and TH. 1344-CERN (1971)

/16/ Blokhintsev, D.I., Efremov, A.V., Shirkov, D.V.: JINR E 2-8027 (1974)

/17/ Parisi, G.: Nota Interna, p. 573, Istituto di Fisica "G. Marconi", I.N.F.N.-Roma

/18/ Sugar, R.L., White, A.R.: Phys. Rev. D10, 4063 (1974)

/19/ Mack, G.: in: Renormalization and Invariance in Quantum Field Theory, Ed. E.R. Caianiello, New York: Plenum Press 1974

/20/ Lee, T.D.: Phys. Rev. 128, 899 (1962); Nuovo Cimento 52A, 579 (1969) Cp. also: Feinberg, C., Pais, A.: Phys. Rev. 132, 2724 (1963)