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Analytic Representations of Simple Lie Groups and Their Continuation to Contractive Representations of Holomorphic Lie Semigroups

by

M. Lüscher
II. Institut für Theoretische Physik der Universität Hamburg

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Analytic representations of simple Lie groups and their continuation to contractive representations of holomorphic Lie semigroups.*)

M. Lüscher

II. Institut für theoretische Physik

Hamburg

Abstract: A unitary representation of a Lie group G is called analytic, if one of the infinitesimal generators of G is represented by $\sqrt{-1}$ times a positive selfadjoint operator. A simple Lie group G with Lie algebra g has analytic representations iff any maximal compactly embedded subalgebra k of g has nonzero center. Irreducible analytic representations have a highest weight with respect to a compact Cartan-subalgebra of g. In case G has analytic representations it is also a boundary group of a holomorphic Lie semigroup S_G whose complex dimension equals the (real) dimension of G. These semigroups are investigated in some detail. It is finally proven that an analytic representation of G can always be analytically continued to a holomorphic, contractive representation of S_G.

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I. A catalogue of problems analysed in this paper

Let begin with a simple example: given a continuous one-parameter group of unitary operators $U(t)$ in a Hilbert space $\mathcal{H}$, then there exists a selfadjoint operator H in $\mathcal{H}$ such that $U(t) = e^{itH}$ $(\forall t \in \mathbb{R})$. Now suppose in addition that $H \geq 0$ i.e. $(\phi, H\phi) \geq 0$ for all ϕ in the domain of definition of H. Then there are operators $T(z)$, $z \in \mathbb{C}$, $\mathrm{Im}\, z \geq 0$, satisfying the three requirements:

(i) $\|T(z)\| \leq 1$

(ii) for any $\phi \in \mathcal{H}$, the vectorvalued function $z \to T(z)\phi$ is holomorphic for $\mathrm{Im}\, z > 0$ and continuous for $\mathrm{Im}\, z \geq 0$.

(iii) if $z = t \in \mathbb{R}$, then $T(z) = U(t)$

Clearly the above conditions determine $T(z)$ uniquely and in fact $T(z) = e^{izH}$. The vectorvalued functions $t \to U(t)\phi$ have thus the remarkable property that they admit of an analytic continuation to the upper half plane.
From (i), (ii) and (iii) it follows immediately that:

(iv) $T(z_1)T(z_2) = T(z_1+z_2)$

(v) $T(z)^+ = T(-z^*)$

($T(z)^+$ is the adjoint of $T(z)$ and z^* denotes the complex conjugate of z).
Thus the operators $T(z)$ form a holomorphic semigroup with boundary group $U(t)$. This is clearly an oversophisticated way of looking at things, however from the above formulations one is naturally lead to the following questions:

a) Let $U(\cdot)$ a unitary continuous representation of a Lie group G and assume that some of the selfadjoint generators of $U(\cdot)$ are positive. This being so, what can be said about the analytic properties of the functions $U(g)\phi$ ($g \in G$, ϕ any vector in Hilbert space)?

b) If G is a Lie group, what additional requirements are needed to assure the existence of a holomorphic Lie semigroup S such that G is some real boundary of S?

The notion of a Lie semigroup will be discussed in some detail in section V. For the present (heuristic) purposes one may well think of a Lie semigroup as some open submanifold of a Lie group which is closed under multiplication.
Of course the questions a) and b) will be rather trivial if G is abelian. I will thus restrict the discussion to Lie groups G which are the "least abelian", in other words: to simple Lie groups. Now, define:

Definition:*) An analytic representation of G is a nontrivial, unitary, continuous representation of G having some nonzero selfadjoint generators positive.

It is then not at all clear that there exist analytic representations of G. Thus:

c) Which real, simple Lie groups G have analytic representations?

Assume for a moment, that G is a real form of some complex Lie group G_c. Then, anticipating a result proved later we may say that question c) has an affirmative answer iff G is a boundary of a holomorphic Lie semigroup S contained in G_c. Remembering question a) it is then tempting to ask:

d) Is it always possible to analytically continue an analytic representation of G to a holomorphic, contractive representation of S?

Having found the class of simple, real Lie groups G admitting analytic representations, the problem arises

e) to classify the irreducible, analytic representations of G and to give explicit realizations of them.

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*) I hope not to confuse representation theory experts who may think of an irreducible analytic representation of G as a representation carried by a space of holomorphic functions defined on a homogeneous space of G. In fact there are strong indications that the two definitions are equivalent (see sec. IV).

(Notice that an analytic representation always decomposes into a direct integral of irreducible representations almost all of which are analytic or trivial.)

Question e) has been answered to a large extent. In fact the holomorphic discrete series representations of G are analytic. The remaining irreducible, analytic representations can be reached from the discrete series representations by analytic continuation in the representation parameters ([1] and references cited therein).

Finally I would like to mention on what earth the present work grows. The earth is relativistic quantum field theory especially the countryside called conformal quantum field theory [10] . In fact there one has to deal with analytic representations of the universal covering $\widetilde{SU(2,2)}$ of the conformal group $SO_0(4,2)/Z_2$. The positively represented generator is the energy operator. The associated contractive representation of the corresponding semigroup S then helps to construct the conformal invariant analyticity domains of the two point vectors $A(x)B(y)|0\rangle$ (A,B any local fields, $|0\rangle$ the vacuum state) [2] . Moreover, for this case problem e) has been completely solved by Mack [3] and leads to convergent globally valid (Wilson-)expansions of operator products applied to the vacuum state. More explicit explanations will be given in [2] .

This mathematical plant was also watered by some nuclear physicists seeking information on how to construct contractive representations of diverse semigroups [4].

II Necessary conditions for a simple Lie group G to have analytic representations.

Let $U(x)$, $x \in G$, an analytic representation of a simple, connected Lie group G and $X_o \neq 0$ an element of g, the Lie algebra of G, with the property that $\frac{1}{i} U(X_o) \geq 0$ (I use the convention: $U(e^X) = e^{U(X)}$). Then, of course, for any $x \in G$, $X_1, X_2 \in \mathfrak{g}$ such that $\frac{1}{i} U(X_j) \geq 0$ $(j = 1,2)$ and $\lambda_j \in \mathbb{R}$, $\lambda_j \geq 0$ $(j = 1,2)$ we have

$$\frac{1}{i} U(\lambda_1 \, Adx \cdot X_1 + \lambda_2 \, Adx \cdot X_2) \geq 0$$

(For, this relation clearly holds on the dense set $\mathcal{H}_\omega$ of analytic vectors for $U(\cdot)$ [5] and extends by essential selfadjointness of $\frac{1}{i} U(\lambda_1 Adx \cdot X_1 + \lambda_2 Adx \cdot X_2)$ on $\mathcal{H}_\omega$ to all vectors in the domain of this operator.)

Thus, taking convex combinations of elements $\lambda \cdot Adx \cdot X_o$, $x \in G$, $\lambda > 0$, one obtains a G-invariant, convex cone, whose interior V will be refered to as <u>the cone generated by X_o.</u> For any $X \in V$ the associated selfadjoint operator $\frac{1}{i} U(X)$ is positive.

<u>Lemma 2.1:</u> V is an open, G-invariant cone in g. Furthermore V is nontrivial, i.e. $V \neq \emptyset$ and $V \neq \mathfrak{g}$.

<u>Proof:</u> Let $W = \{X \in \mathfrak{g} \mid \exists\, x \in G : X = Adx \cdot X_o\}$ and LW the linear span of W. Clearly LW is a G-invariant linear subspace of g and hence an ideal of g. Now we have assumed that $X_o \neq 0$ and that g is simple. Hence LW = g. But then dim V = dim g and $V \neq \emptyset$. If on the other hand V = g, then $\frac{1}{i} U(X) \geq 0$ and $\frac{1}{i} U(-X) \geq 0$ for any $X \in \mathfrak{g}$ from which one easily concludes, that $U(x) = 1$ for all $x \in G$. But the trivial representation of G is not analytic, hence $V \neq g$. ∎

A necessary condition for G to have analytic representations is thus the existence of an open, G-invariant and nontrivial cone contained in g.
Let $\mathfrak{g} = k \oplus p$ a Cartan decomposition, $\sigma : \mathfrak{g} \to \mathfrak{g}$ the associated Cartaninvolution and $K \subset G$ the connected Lie subgroup of G with Lie algebra k.

<u>Theorem 2.2:</u> The Lie algebra g contains an open, G-invariant nontrivial cone V iff k has nonzero center c. In that case, c is one-dimensional and $k = k' \oplus c$ with k' compact. Moreover V contains either H or -H where $c = \mathbb{R}\cdot H$.

<u>Proof:</u> I will first prove that the existence of V implies that k has nontrivial center. The center $\mathfrak{z}$ of G is contained in K [6]. For any $X \in V$ the function $K \ni h \longrightarrow \mathrm{Ad}h\cdot X$ is in fact a function on $K/\mathfrak{z}$ which is known to be a compact Lie group. Let $d\mu(h)$ denote the invariant normalized measure on $K/\mathfrak{z}$. Then the vector

$$Z \doteq \int_{K/\mathfrak{z}} d\mu(h)\ \mathrm{Ad}h\cdot X$$

is welldefined and K-invariant. In order to show, that $Z \neq 0$ we may take a hyperplane E in g passing through th origin such that V lies on one side of E. Relative to an arbitrary euclidean scalar product $\langle .,. \rangle$ on g this means that there exists a vector N satisfying: $\langle N, Y \rangle > 0$ for all $Y \in V$. Since $\mathrm{Ad}h\cdot X \in V$ for all $h \in K/\mathfrak{z}$ it follows that

$$\langle N, Z \rangle = \int_{K/\mathfrak{z}} d\mu(h)\ \langle N, \mathrm{Ad}h\cdot X \rangle \quad > 0$$

hence $Z \neq 0$.

Now $Z = Z_k + Z_p$, $Z_k \in k$, $Z_p \in p$, i.e. $\sigma(Z_k) = Z_k$ and $\sigma(Z_p) = -Z_p$. But σ commutes with $\mathrm{Ad}h$, $h \in K$, thus Z_k and Z_p are separately K-invariant. Since $[p,p] \subset k$ we have $\mathrm{Ad}Z_p \cdot g \subset k$ and so $(\mathrm{Ad}Z_p)^2 = 0$. But $\mathrm{Ad}Z_p$ is diagonalizable implying $\mathrm{Ad}Z_p = 0$ and hence $Z_p = 0$. Thus Z is contained in k and commutes with all of k. In other words, k has nontrivial center. Furthermore it is easily seen that by construction $Z \in V$.

Conversely assume that k has nontrivial center c. Let $H \in c$, $H \neq 0$ and V the open, G-invariant cone generated by H. As in the proof of Lemma 2.1 one can show that $V \neq \emptyset$ and I will prove now that $V \nsubseteq g$. Denote by $B(\cdot,\cdot)$ the Killingform on g. Then for any $X \in V$ we have $B(X,H) < 0$. Indeed, if $X = \mathrm{Ad}x\cdot H$, $x \in G$, then $x = h\cdot e^Y$, $h \in K$, $Y \in p$ and hence

$$B(X,H) = B(\mathrm{Ad}h\cdot \mathrm{Ad}\,e^Y\cdot H, H) = B(e^{\mathrm{Ad}Y}\cdot H, \mathrm{Ad}h^{-1}\cdot H) = B(e^{\mathrm{Ad}Y}\cdot H, H) =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\, B((\mathrm{Ad}Y)^n\cdot H, H)$$

Now $(\mathrm{Ad}Y)^{2m} \cdot H \in k$ and $(\mathrm{Ad}Y)^{2m+1} \cdot H \in p$. k and p being orthogonal, it follows that

$$B(X,H) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} B((\mathrm{Ad}Y)^{2n}H, H) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} B((\mathrm{Ad}Y)^n H, (\mathrm{Ad}Y)^n H) =$$

$$= - \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left| B((\mathrm{Ad}Y)^n \cdot H, (\mathrm{Ad}Y)^n \cdot H) \right| < 0$$

By taking convex combinations of vectors $\lambda \mathrm{Ad}x \cdot H$, $\lambda > 0, x \in G$, the above inequality still holds and thus $B(X,H) < 0$ for all $X \in V$. Especially $-H \notin V$, hence $V \neq g$.

Finally it remains to be shown that, if $c \neq \{0\}$, then c is onedimensional. Let $H \in c$, $H \neq 0$. With respect to the scalar product B(.,.) on p, AdH is skew symmetric and $(\mathrm{AdH})^2$ symmetric, hence diagonalizable. It is well known that for simple g k acts irreducibly on p [7]. Hence $(\mathrm{Ad}H)^2|_p = \mu \cdot \mathrm{Id}_p$, $\mu \in \mathbb{R}$ (Id_p denotes the identity mapping of p onto p). AdH has purely imaginary eigenvalues implying $\mu < 0$. By choosing an appropriate normalization of H we may write $(\mathrm{AdH})^2|_p = -\mathrm{Id}_p$.
Next, the real vectorspace p is transformed into a complex vectorspace without changing its real dimension: for $\lambda = \alpha + i\beta$, $\alpha, \beta \in \mathbb{R}$ and $X \in p$ one defines:

$$\lambda \cdot X \doteq \alpha X + \beta \, \mathrm{Ad}H \cdot X \qquad (\in p)$$

and one readily verifies that p has become a complex vectorspace. Moreover k acts complex linear and irreducibly on p, thus by Schurslemma $\mathrm{Ad}Z|_p = \lambda \mathrm{Id}_p$, $\lambda \in \mathbb{C}$, for any Z in c. But as above $(\mathrm{Ad}Z)^2|_p = \lambda^2 \mathrm{Id}_p = -|\lambda|^2 \mathrm{Id}_p$ thus $\mathrm{Ad}Z|_p = \pm|\lambda| \mathrm{Ad}H|_p$. Hence $Z = \pm|\lambda| H$ which means that c consists precisely of the (real) multiples of H.
Concluding the proof of the theorem we remark that for any compact Lie group with Lie algebra k one has $k = [k,k] \oplus c$, $[k,k]$ compact ([6] 3.6.1/2). ∎

Consulting Cartan's list of real simple Lie algebras theorem 2.2 selects the following algebras (in Helgason's [7] notation):

$\mathfrak{su}(p,q) \qquad p \geq 1\ ,\ q \geq 1$

$\mathfrak{so}^*(2n) \qquad n \geq 3$

$\mathfrak{so}(p,2) \qquad p \geq 3$

$\mathfrak{sp}(n,\mathbb{R}) \qquad n \geq 1$

$\mathfrak{e}_{6(-14)}\ ;\ \mathfrak{e}_{7(-25)}$

These are exactly the real simple Lie algebras whose maximal compactly embedded subalgebras have nontrivial center. To save words I will use for them the name "hermitian Lie algebra". This notion arises from the following fact: given a Lie group G with Lie algebra g and maximal compact subgroup K then the coset space G/K is an irreducible hermitian symmetric space iff g is hermitian [7]. Let me briefly recall the definition of an irreducible hermitian symmetric space M [7]:

(i) M is a complex manifold together with an infinitely differentiable hermitian (positive definite) metric.
(ii) Each $x \in M$ is an isolated fixed point of an involutive, holomorphic isometry of M.
(iii) The connected component of the group of all holomorphic isometries containing the identity is simple.

Most remarkable is property (i), namely the statement that the coset space G/K of a Lie group G having analytic representations has a complex structure. This indicates the possibility of realizing analytic representations on spaces of holomorphic functions on G/K. In fact such a construction is standard for the socalled holomorphic discrete series of representations of G.

III. Search for holomorphic Lie semigroups

Let G_c a complex connected Lie group and G a connected real form of G_c. The problem is to find a subsemigroup S^o of G_c that is also an open submanifold of G_c and such that G is contained in the set ∂S of all boundary points of S^o.

Assume that S^o exists. Then the closure $\overline{S}$ of S^o is also a semigroup. If $x \in \partial S$ then $x \cdot S^o \subset S^o$. Denote by W the set of all tangentvectors X at 1, the group identity, which are tangent to some curve $\gamma(t) \subset G_c$, $t \in [0,\infty)$, with $\gamma(0) = 1$, $\gamma(t) \in S^o$ for all $t > 0$ *). For any $x \in G$, the curve $x \cdot \gamma(t) x^{-1}$ is of the same type. This means that W is G-invariant. Furthermore, if $X_1, X_2 \in W$ corresponding to the curves $\gamma_1(t)$ resp. $\gamma_2(t)$ then $\lambda_1 X_1 + \lambda_2 X_2 \in W$ $(\lambda_1, \lambda_2 > 0)$ too, for this vector is tangent to the curve $\gamma_1(\lambda_1 t) \cdot \gamma_2(\lambda_2 t)$. Hence W is a G-invariant cone in the tangentspace at 1. This space can be identified with the Lie algebra g_c of G_c. Since $G \subset \partial S$ we have $W + X = W$ for all $X \in g$ (g is the Lie algebra of G). Writing $g_c = g \oplus ig$ and correspondingly $W = g \oplus iV$ it is readily verified, that V is a nontrivial, G-invariant cone contained in g. According to theorem 2.2 the real simple Lie algebras g containing such a cone are precisely the hermitian Lie algebras.

I will thus assume for the rest of the section that g is hermitian. Now fix an element $H \neq 0$ in c, the center of a maximal compactly embedded subalgebra k of g and let V be the cone generated by H. Consider the sets

$$(3.1) \qquad S^o \doteq \{ x \in G_c \mid x = u \cdot e^{iX_1} \cdots e^{iX_k} \; ; \; u \in G \; ; \; X_j \in V \; (j = 1, \dots, k) \; ; \; k \geq 1 \}$$

$$S \doteq S^o \cup G$$

<u>Lemma 3.1:</u> S and S^o are semigroups and S^o is an open submanifold of G_c.

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*) Assume that such curves exist; this is clearly the case if S^o has a sufficiently regular boundary.

For a proof, see [8].
I will show later that S^o is nontrivial, i.e. $S^o \neq G_c$.
Because any element of G is a limit point of elements of S^o (and $G \cap S^o = \emptyset$) we have $G \subset \partial S$.
Clearly one could also start with -H and -V instead of H and V respectively. The semigroups obtained in this way are the complex conjugates of S^o and S and will be refered to as S^{o*} and S^*. Moreover if $s \in S$ then $s^{-1} \in S^*$ and vice versa.
The semigroups S^o and S^{o*} ar minimal in the sense that any holomorphic subsemigroup of G_c having G at the (sufficiently regular) boundary contains either S^o or S^{o*} (in general, e.g. for G = SU(2,2), there actually exist nontrivial semigroups which are truly larger than S^o). A proof of this statement can be found at the end of the proof of Th. 3.3. Next the relationship between the semigroups S, S^* and the cosetspace G/K is investigated. This necessitates a detailed description of G/K. Following Helgason's [7] treatment we choose the normalization of H such that $(\mathrm{Ad}H)^2|_p = -\mathrm{Id}_p$. Then the complexification $p_c = p \oplus ip$ of p splits into eigenspaces p_+, p_- corresponding to the eigenvalues +i, -i of $\mathrm{Ad}H|_{p_c}$ respectively. The following Lemma summarizes a few properties of $p_\pm$:

<u>Lemma 3.2:</u> [7] p_+ and p_- are abelian subalgebras of g_c and

$$[k, p_+] \subset p_+ \;;\; [k, p_-] \subset p_- \;;\; p_c = p_- \oplus p_+$$

Let P_+, P_- and K_c denote the connected Lie subgroups of G_c with Lie algebras p_+, p_- and $k_c = k \oplus ik$. Then exp is a diffeomorphism of $p_+ (p_-)$ onto $P_+ (P_-)$ and the mapping $(x_-, h, x_+) \to x_- \cdot h \cdot x_+$ is a diffeomorphism of $P_- \times K_c \times P_+$ onto an open submanifold of G_c containing G.

Since $K_c \cdot P_+$ is a closed complex subgroup of G_c, the cosetspace $G_c/K_c \cdot P_+$ has a natural complex structure such that the action $G_c \times G_c/K_cP_+ \to G_c/K_cP_+$ of G_c on G_c/K_cP_+ is holomorphic. Let $\pi: G_c \to G_c/K_cP_+$ the canonical projection i.e. $\pi(x) = x \cdot K_cP_+$. Then it turns out, that G/K is diffeomorphic to an open subset of G_c/K_cP_+ namely $\pi(G)$. According to the above Lemma $\pi(G) \subset \pi(P_-)$ which is diffeomorphic to p_-. Furthermore it has been shown, that $\pi(G)$ equals $\pi(\exp D)$ where D is a bounded open set in the complex vectorspace p_-. This makes the complex structure of G/K manifest. The G-invariant , hermitian metric on G/K is an assignment of a hermitian scalarproduct $Q_x(.,.)$ on the tangentspace at each point $x \in G/K$ in a differentiable manner. G-invariance means that for any $h \in G, x \in G/K$ and X, Y in the tangentspace at x the equality

$$(3.2) \qquad Q_x(X,Y) = Q_{h\cdot x}(dh\cdot X, dh\cdot Y)$$

holds. Here $h \cdot x$ is the picture of the point $x \in G/K$ under the action of h and dh is the differential of the mapping $x \to h\cdot x$. Eq. (3.2) fixes Q_x uniquely up to a real positive constant. For $x = \pi(1) \doteq o$ we may identify the tangentspace at x with p_- and choose

$$(3.3) \qquad Q_o(X,Y) = B(\theta X, Y) \quad ; \quad X, Y \in p_-$$

Here, $\theta : g_c \to g_c$ is the complex conjugation with respect to g, i.e. $\theta(X_1 + iX_2) = X_1 - iX_2$ for all $X_1, X_2 \in g$. Given a differentiable curve $\gamma(t)$, $t \in [0,1]$ in G/K we may calculate its length, namely:

$$(3.4) \qquad L_\gamma = \int_0^1 dt \, [Q_{\gamma(t)}(X(t), X(t))]^{1/2} \quad ; \quad X(t) \doteq \frac{d\gamma}{dt}$$

Then (3.2) implies that for any $h \in G$: $L_\gamma = L_{h\cdot\gamma}$. Of course, for an arbitrary differentiable mapping $s: G/K \to G/K$ this relation may not hold and a natural measure for length invariance breaking is:

$$(3.5) \qquad \|s\| = \sup_{\substack{\gamma \\ L_\gamma \neq 0}} L_{s(\gamma)}/L_\gamma$$

(if the supremum does not exist, set $\|s\| = \infty$)
Obviously, for composite mappings $s_1 \circ s_2$ we have

$$(3.6) \qquad \|s_1 \circ s_2\| \leq \|s_1\| \cdot \|s_2\| \qquad (\|s_{1,2}\| < \infty)$$

A mapping $s: G/K \to G/K$ will be called contractive if (1) the closure of $s(G/K)$ is compact and (2) $\|s\| < 1$.

The semigroup $S^* \subset G_c$ acts naturally on G_c/K_cP_+ . A general element of G_c does not leave invariant the open subset $G/K \cong \pi(G)$ of G_c/K_cP_+ . However:

<u>Theorem 3.3:</u> Under the natural action of S^* on G_c/K_cP_+ the open subset $G/K \cong \pi(G)$ is left invariant, i.e. $s \cdot \pi(G) \subset \pi(G)$ for all $s \in S^*$. Furthermore, for $s \in S^{o*}$ the mapping $x \to s \cdot x$ of G/K into G/K is contractive.

I would like to make three remarks:

a) The theorem states in particular, that the action of G on G/K can be analytically continued to a holomorphic action of S^{o*} on G/K.

b) By choosing the conjugate complex structure on G/K one would obtain by analytic continuation an action of S on G/K. The present formulation will however lead more naturally to the construction of analytic representations U(·) of G with $\frac{1}{i}U(H) \geq 0$ (sec. IV/V)

c) The theorem shows also that S and S^* are nontrivial.

<u>Proof:</u> Let S^o_{contr} the set of all $s \in G_c$ such that $s \cdot \pi(G) \subset \pi(G)$ and such that the mapping $x \to s \cdot x$ of $G/K \cong \pi(G)$ into itself is contractive.

First, I will prove that for $t > 0$ $e^{-itH} \in S^o_{contr}$. To this end consider a Cartansubalgebra $\mathfrak{h}$ of $\mathfrak{g}_c$ containing H. Because H does not commute with any $X \in \mathfrak{p}_c$ we have $\mathfrak{h} \subset \mathfrak{k}_c$. Let Δ_p^- the set of roots α (with respect to $\mathfrak{h}$) with $\alpha(H) = +i$ and X_α the corresponding root vectors. Set $X_{-\alpha} = \theta(X_\alpha)$, the complex conjugate of X_α. Then:

$$\mathfrak{p}_+ = \sum_{\alpha \in \Delta_p^-} \mathbb{C} X_\alpha \quad ; \quad \mathfrak{p}_- = \sum_{\alpha \in \Delta_p^-} \mathbb{C} X_{-\alpha} \tag{3.7}$$

$$\mathfrak{p} = \sum_{\alpha \in \Delta_p^-} \mathbb{R}(X_\alpha + X_{-\alpha})$$

<u>Lemma 3.4 [7]:</u> There exists a subset $\{\alpha_1, \ldots, \alpha_n\}$ of Δ_p^- such that the subspace

$$\mathfrak{a} \doteq \sum_{j=1}^{n} \mathbb{R}(X_{\alpha_j} + X_{-\alpha_j})$$

is a maximal abelian subalgebra of $\mathfrak{p}$. Moreover for $Z = \sum_{j=1}^{n} t_j (X_{\alpha_j} + X_{-\alpha_j})$ we have :

(3.8) $$e^{Z} = e^{X_-} \cdot e^{Y} \cdot e^{X_+}$$

where $X_{\pm} = \sum_{j=1}^{n} \mathrm{tgh}(t_j) X_{\pm\alpha_j}$; $Y = \sum_{j=1}^{n} \ln(\mathrm{ch}\, t_j)[X_{\alpha_j}, X_{-\alpha_j}]$

Observe now that $x \in \tilde{\pi}(G)$ iff there is an $X \in p$ with $x = \exp X \cdot K_c P_+$. According to a well-known theorem ([6] 7.5.5) there is ℓ in K so that $\mathrm{Ad}\,\ell \cdot X \in \mathfrak{a}$. Hence the decomposition:

(3.9) $$x = \ell \cdot e^{X_-} K_c P_+ \quad ; \quad \ell \in K,\ X_- = \sum_{j=1}^{n} \mathrm{tgh}(t_j)\ X_{-\alpha_j} \in p_-$$

Thus

$$e^{-itH} \cdot x = \ell \cdot \exp(e^{-t} X_-)\ K_c P_+ =$$

$$= e^{X'} K_c P_+$$

where $X' = \mathrm{Ad}\,\ell \cdot \sum_{j=1}^{n} t_j' (X_{\alpha_j} + X_{-\alpha_j}) \in p$,

(3.10) $$\mathrm{tgh}(t_j') = e^{-t}\ \mathrm{tgh}(t_j) \qquad (t \geqslant 0)$$

Having (3.9) in mind it is clear that the bounded subset $\mathbb{D}$ of p_- consists precisely of all vectors $Z \in p_-$ of the form: $Z = \mathrm{Ad}\,\ell \cdot (\sum_{j=1}^{n} \lambda_j X_{-\alpha_j})$, $\ell \in K, \lambda_j \in \mathbb{R}$, $|\lambda_j| < 1$. Now $\exp(-itH)$ acts on Z simply by multiplication with e^{-t}. Hence $\exp(-itH) \cdot \tilde{\pi}(G)$ is a relatively compact subset of $\tilde{\pi}(G)$ $(t > 0)$.
For any $s \in G_c$ with $s \cdot \tilde{\pi}(G) \subset \tilde{\pi}(G)$ let $\|s\|$ be defined as in (3.5) with $s(\gamma)$ replaced by $s \cdot \gamma$. Then:

(3.11) $$\|s\| = \sup_{\substack{x \in G \\ Z \in p_-,\, \|Z\| = 1}} \| d(x'^{-1} \cdot s \cdot x)\, Z \| \qquad ; \qquad \|Z\| \doteq [Q_0(Z,Z)]^{1/2}$$

$x' \in G$ depends on s and x through $x' \cdot K_c P_+ = s \cdot x \cdot K_c P_+$. Note that $x'^{-1} \cdot s \cdot x \in K_c P_+$ and, as usual, dr is the differential of the mapping $y \longrightarrow r \cdot y$ of $G_c / K_c P_+$ onto $G_c / K_c P_+$.
Especially for $s = \exp(-itH)$ we have

$$x = \ell \cdot e^{X_-} \cdot e^{Y} \cdot e^{X_+} \cdot \ell^{-1} \quad ; \quad x' = \ell \cdot e^{X_-'} \cdot e^{Y'} \cdot e^{X_+'}\, \ell^{-1}$$

with t'_j as in (3.10). Hence

$$\| d(x'^{-1} \cdot s \cdot x) \, \mathrm{Ad}\ell \cdot Z \| = e^{-t} \| d(e^{-X'_+} e^{(Y-Y')} \exp(e^t X_+)) \cdot Z \| =$$

$$= e^{-t} \| d(e^{(Y-Y')}) \cdot dr \cdot Z \| \qquad ; \qquad r \in P_+$$

But $d(r)Z = \mathrm{Ad}r \cdot Z \big|_{\text{projected on } p_-} = Z$ and hence $\|d(x'^{-1}sx)\mathrm{Ad}\ell \cdot Z\| = e^{-t}\|e^{\mathrm{Ad}(Y-Y')}Z\|$.

Now $Y - Y' = \sum_{j=1}^{n} \lambda_j [X_{\alpha_j}, X_{-\alpha_j}]$, $\lambda_j \in \mathbb{R}$, $\lambda_j \geq 0$. The commutators $[X_{\alpha_j}, X_{-\alpha_j}]$ are elements of $\mathbf{h}$, in fact,

$$[X_{\alpha_j}, X_{-\alpha_j}] = B(X_{\alpha_j}, X_{-\alpha_j}) \cdot H_{\alpha_j} \quad ; \quad B(H_{\alpha_j}, Z) = \alpha_j(Z) \text{ for all } Z \in \mathbf{h}$$

From $X_{\alpha_j} = X_j - i\,\mathrm{Ad}H \cdot X_j$, $X_j \in p$, one easily concludes that $B(X_{\alpha_j}, X_{-\alpha_j}) > 0$.

<u>Lemma 3.5:</u> Let $\alpha, \beta \in \Delta_p^-$ and $H_\alpha \in \mathbf{h}$ such that $B(H_\alpha, Z) = \alpha(Z)$ for all $Z \in \mathbf{h}$. Then $\beta(H_\alpha) \geq 0$.

<u>Proof:</u> Consider the α-series of roots containing β ([7] p. 143). This series contains at most $\beta, \beta-\alpha, \beta-2\alpha$. Hence $\beta(H_\alpha)/\alpha(H_\alpha) \geq 0$ and since $\alpha(H_\alpha) = B(H_\alpha, H_\alpha) > 0$ the Lemma follows ∎ (Lemma)

If we now write the quadratic form $\|e^{\mathrm{Ad}(Y-Y')} \cdot Z\|^2$ in terms of the orthogonal basis $\{X_{-\alpha} \, ; \, \alpha \in \Delta_p^-\}$ of p_- it will diagonalize with eigenvalues $(e^{-t}\rho_\alpha)^2$ where

$$\rho_\alpha = \exp\Big\{ -\sum_{j=1}^{n} \lambda_j \, B(X_{\alpha_j}, X_{-\alpha_j}) \, \alpha(H_{\alpha_j}) \Big\} \leq 1$$

Thus: $\|\exp(-itH)\| = e^{-t}$ and consequently $\exp(-itH) \in S^o_{contr}$. In the next step I'm going to show that S^o_{contr} is open. Let $s_o \in S^o_{contr}$ and $\mathcal{O}$ a neighborhood of s_o in G_c such, that $s \cdot \pi(G) \subset C$ for all $s \in \mathcal{O}$, where C is a compact subset of $\pi(G)$. Now

$$\big| \|s\| - \|s_o\| \big| \leq \sup_{\substack{x \in G \\ Z \in p_-, \, \|Z\| = 1}} \big| \| d(x'^{-1}_s \cdot s \cdot x) Z \| - \| d(x'^{-1}_{s_o} \cdot s_o \cdot x) Z \| \big| \leq$$

$$\leq \sup_{\substack{x \in G \\ Z \in p_-,\, \|Z\|=1}} \| \{ d(x_s'^{-1} \cdot s \cdot x) - d(x_{s_o}'^{-1} \cdot s_o \cdot x) \} \cdot Z \|$$

where $x_s' \cdot K_c P_+ = s \cdot x \cdot K_c P_+$, $x_{s_o}' \cdot K_c P_+ = s_o \cdot x \cdot K_c P_+$; x_s' , $x_{s_o}' \in G$.

As above: $x = e^{X_-} e^{Y} e^{X_+}$, $X_- \in p_-$, $X_+ \in p_+$, $Y \in ik$

and:

$$\sup_{\substack{x \in G \\ Z \in p_-,\, \|Z\|=1}} \| d(e^{Y} \cdot e^{X_+}) Z \| \leq 1$$

Hence

$$| \|s\| - \|s_o\| | \leq \sup_{\substack{x \in G \\ Z \in p_-,\, \|Z\|=1}} \| \{ d(x_s'^{-1} \cdot s \cdot e^{X_-}) - d(x_{s_o}'^{-1} \cdot s_o \cdot e^{X_-}) \} \cdot Z \|$$

The mappings $x \cdot K_c P_+ \longrightarrow x_s' \cdot K_c P_+ (s \in \Theta)$ may be looked at as mappings from $\mathbb{D}$ into $\exp(p) \subset G$. Since $x_s' \cdot K_c P_+$ varies over a relatively compact subset of $\tilde{\pi}(G)$ as X_- varies through $\mathbb{D}$ one may extend the relation $X_- \to x_s'$ by continuity to all of the compact closure $\overline{\mathbb{D}}$ of $\mathbb{D}$. Thus

$$| \|s\| - \|s_o\| | \leq \sup_{\substack{X_- \in \overline{\mathbb{D}} \\ Z \in p_-,\, \|Z\|=1}} \| \{ d(x_s' \cdot s \cdot e^{X_-}) - d(x_{s_o}'^{-1} \cdot s_o \cdot e^{X_-}) \} Z \|$$

By standard arguments it now follows that the norm $||s||$ varies continuously ($s \in \Theta$). Thus S^o_{contr} is an open subset of G_c. Finally observe that S^o_{contr} is an open subsemigroup of G_c having G at the boundary. Thus the construction carried out at the beginning of this section yields a cone $W_{contr} = \mathfrak{g} \oplus iV_{contr}$. Since $(-i)H \in W_{contr}$ we have $-V \subset V_{contr}$ (V is the cone generated by H). Now one easily shows that $\exp X$, $X \in W_{contr}$, is contained in the closure $\overline{S^o_{contr}}$ of S^o_{contr}. Let $\mathcal{N} \subset \mathfrak{g}$ an open neighborhood of zero such that $\exp' : \mathcal{N} \times \mathcal{N} \to G_c$ according to $\exp'(X,Y) = \exp X \exp iY$ is a diffeomorphism. Then if $X \in -V \cap \mathcal{N}$ there is some $t>0$ such that $\exp itH \cdot \exp X = \exp'(X',Y')$, $X' \in -V \cap \mathcal{N}$, $Y' \in \mathcal{N}$. Hence $\exp X$ is the product of an element of S^o_{contr} with an element of $\overline{S^o_{contr}}$ and thus contained in S^o_{contr}.

This proves that $S^{o*} \subset S^{o}_{contr}$. ∎ (theorem)

To give the reader a feeling of how the semigroups S^o may look like, I will briefly describe three examples (see table). They are all constructed along the same scheme: G_c is a holomorphic, pseudoalgebraic [7] subgroup of $Gl(n,\mathbb{C})$. The group G then consists of all elements $x \in G_c$ leaving some hermitian, nondegenerate sesquilinear form (.,.) on $\mathbb{C}^n$ invariant. With the help of an $n \times n$-matrix J one may write: $(z_1, z_2) = z_1^{*T} \cdot J \cdot z_2$, $z_1, z_2 \in \mathbb{C}^n$ (z^T denotes the transpose of the columnvector z). The cone V generated by H (as listed in the table) then consists precisely of all matrices $\Sigma \in \mathfrak{g}$ with

$$(3.12) \qquad (z, i\Sigma z) < (z,z) \qquad \text{for all} \quad z \in \mathbb{C}^n \;, \; z \neq 0$$

and S^o is exactly the set of all $\Lambda \in G_c$ satisfying

$$(3.13) \qquad (\Lambda z, \Lambda z) < (z,z) \qquad \text{for all} \quad z \in \mathbb{C}^n \;, \; z \neq 0$$

Shortly, while the elements of G leave the hermitian form (.,.), invariant those of S^o contract it.
The three types of semigroups S^o described above have a remarkable property: they admit of a polar decomposition. By definition this means that

(i) the map $(u, X) \to u \cdot \exp iX$ of GxV into S^o is a surjective diffeomorphism and

(ii) there exists a neighborhood $N \subset \mathfrak{g}$ of zero such that $\exp'$: $N \times N \to G_c$, $\exp'(X,Y) = e^X \cdot e^{iY}$ is diffeomorphic and $\exp'(X,Y) \in S^o$ iff $Y \in V$.

The proof of polar decomposability of the semigroups discussed here is carried out in appendix A.

G_c		G	J	H
$Sl(n+m, \mathbb{C})$	; $n, m \geq 1$	$SU(n,m)$	$\begin{pmatrix} I_n & 0 \\ 0 & -I_m \end{pmatrix}$	$i \begin{pmatrix} \frac{1}{n} I_n & 0 \\ 0 & -\frac{1}{m} I_m \end{pmatrix}$
$SO(2n, \mathbb{C})$	; $n \geq 3$	$SO^*(2n)$	$\frac{1}{i} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$	$i\,J$
$Sp(n, \mathbb{C})$	; $n \geq 1$	$Sp(n, \mathbb{R})$	$\frac{1}{i} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$	$i\,J$

(I_n is the nxn unit matrix, other notations used are those of Helgason [7])

IV. Existence and classification of irreducible analytic representations of simple Lie groups.

Let G a connected Lie group with hermitian Lie algebra g. I will not assume that G has a faithful finite dimensional representation. Denote by K the connected subgroup of G with Lie algebra k, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ being a Cartandecomposition of g.

<u>Lemma 4.1:</u> Any irreducible, unitary, continuous representation U(·) of G in a Hilbertspace $\mathcal{H}$ is K-finite.

This Lemma is wellknown when the center of G is finite. However the hermitian Lie algebras are precisely those simple Lie algebras allowing infinite centers. A proof of Lemma 4.1 is given in Appendix B.
For the reader's convenience let me briefly review the notion of K-finiteness. It is defined by: [6]

(i) Under the restriction $U(\cdot)|_K$ of U(.) on K, $\mathcal{H}$ decomposes into $\bigoplus_{\delta\in\hat{K}} \mathcal{H}(\delta)$ (unitary direct sum) where $\mathcal{H}(\delta)$ is a <u>finite</u> sum of irreducible representationspaces of type $\delta \in \hat{K}$. ($\hat{K}$ is the set of equivalence classes of irreducible finitedimensional representations of K)

(ii) If $v \in \mathcal{H}(\delta)$ for some $\delta \in \hat{K}$ then the mapping $x \longrightarrow U(x)v$, $x \in G$, is realanalytic on G (i.e. v is an analytic vector for U(·)).

If U(·) is like in lemma 4.1 set

$$\mathcal{H}_K = \sum_{\delta\in\hat{K}} \mathcal{H}(\delta) \quad \text{(algebraic sum)}$$

$\mathcal{H}_K$ carries an irreducible representation of g_c, the complexification of g (define U(X+iY) = U(X) + iU(Y) for X, $Y \in \mathfrak{g}$).
Let h a maximal abelian subalgebra of k. $\mathfrak{h}$ contains the center $\mathfrak{c} = \mathbb{R}\cdot H$ of k. Define $\mathfrak{h}_c, \mathfrak{k}_c, \mathfrak{p}_c$ to be the complexifications of $\mathfrak{h}, \mathfrak{k}, \mathfrak{p}$ respectively. $\mathfrak{h}_c$ is a Cartansubalgebra of $\mathfrak{g}_c$. The set of nonvanishing roots Δ corresponding to h_c splits into $\Delta_k \cup \Delta_p$ as follows: $\alpha \in \Delta_k$ $(\alpha \in \Delta_p)$

iff the rootvector X_α is an element of $k_c(p_c)$.
We may choose a Weylchamber P ⊂ ih such that $i\alpha(H) \geq 0$ for all positive roots $\alpha \in \Delta^+$ (H as above, see also sec. III). In fact $\alpha(H) = 0$ for $\alpha \in \Delta_k^+ \doteq \Delta_k \cap \Delta^+$ and $i\alpha(H) = 1$ for $\alpha \in \Delta_p^+ \doteq \Delta^+ \cap \Delta_p$.
Let $k = k' \oplus c$, k' compact (theorem 2.2). Then Δ_k is the rootsystem of k_c' with respect to $h_c' = h_c \cap k_c'$ (k_c' the complexification of k'). Define a linear form Λ_H on h_c through $\Lambda_H(H) = i$, $\Lambda_H(Z) = 0$ for all $Z \in h_c'$.

Theorem 4.2: There are two branches of nonequivalent analytic representations U(·) of G, namely those for which $\frac{1}{i}U(H) \geq 0$ and those with $\frac{1}{i}U(H) \leq 0$. There exists an involutive automorphism A of G such that $x \to U(A(x))$ is of the second type if U(·) is of the first type and vice versa.
If U(·) is an irreducible analytic representation of G with $\frac{1}{i}U(H) \geq 0$ then the representation of g_c on $\mathcal{H}_K$ has a highest weight Λ with respect to (h_c, Δ^+). Furthermore $\Lambda = d\cdot\Lambda_H + \Lambda'$ where $d \in \mathbb{R}$, $d \geq 0$ and $\Lambda'(H) = 0$, $\Lambda'|_{h_c'}$ dominant integral with respect to (h_c', Δ_k^+).
Clearly then, two irreducible analytic representations with $\frac{1}{i}U(H) \geq 0$ are equivalent iff their highest weights with respect to (h_c, Δ^+) coincide.

By definition, the representation of g_c on $\mathcal{H}_K$ is said to have highest weight Λ if

(i) there is a vector $\Omega \in \mathcal{H}_K$ such that $U(Z)\Omega = \Lambda(Z)\Omega$ for all $Z \in h_c$ and $U(X_\alpha)\Omega = 0$ for all root vectors X_α, $\alpha \in \Delta^+$.

(ii) $\mathcal{H}_K$ is the linear span of the vectors obtained by iterated application of the operators $U(X)$, $X \in g_c$, on Ω.

Proof: I will first show the existence of A, namely of an involutive automorphism of G such that $A(e^Z) = e^{-Z}$ for $Z \in h$. Set $u = k \oplus ip$.
Then u is a compact real Lie algebra and h a maximal abelian subalgebra of u.
Let $A: u \to u$ be an automorphism of u such that $A(X) = -X$ for all $X \in h$ ([7] p.332). Then A extends by linearity to an automorphism of g_c that will also be denoted by A.

If $\alpha \in \Delta$ we have: $A[Z, X_\alpha] = \alpha(Z) AX_\alpha = -[Z, AX_\alpha]$ $(Z \in h_c)$, hence $AX_\alpha = a_\alpha \cdot X_{-\alpha}$. Furthermore $A[X_\alpha, X_{-\alpha}] = -a_\alpha a_{-\alpha}[X_\alpha, X_{-\alpha}]$ and since $[X_\alpha, X_{-\alpha}] \in h_c$ it follows that $a_\alpha \cdot a_{-\alpha} = 1$ so $A^2 = 1$.
Now if $\alpha \in \Delta_k$ then $-\alpha \in \Delta_k$ and if $\alpha \in \Delta_p$ then $-\alpha \in \Delta_p$. Thus $Ak_c = k_c$, $Ap_c = p_c$ and because $Au = u$, A leaves k and p invariant. This implies that $Ag \subset g$ i.e. the restriction of A to g is an automorphism of g and I will use for it the same symbol. Let $\tilde{G}$ the universal covering of G. A then lifts uniquely to an automorphism $\tilde{A}$ of $\tilde{G}$ such that $\tilde{A}(\exp X) = \exp \tilde{A}X$ for all $X \in g$ [8]. Let $\tilde{T}$ the connected Lie subgroup of $\tilde{G}$ with Lie algebra h and $\Gamma \subset \tilde{\mathfrak{z}}$ (the center of $\tilde{G}$) a discrete (normal) subgroup of $\tilde{G}$ such that $G \cong \tilde{G}/\Gamma$. Then $\Gamma \subset \tilde{T}$ ([6] 7.2.5 and 3.8.3), hence $\tilde{A}(\Gamma) \subset \Gamma$. For any $x = \tilde{x} \cdot \Gamma \in G$ we can therefore unambiguously define $A(x) = \tilde{A}(\tilde{x})\Gamma$. This yields the automorphism looked for.
Next, assume that there is given an irreducible, analytic representation $U(\cdot)$ of G. The arguments of sec II then imply that either $\frac{1}{i} U(H) \geq 0$ or $\frac{1}{i} U(H) \leq 0$. Choose a representation with $\frac{1}{i} U(H) \geq 0$. Now $e^{n2\pi H}$ $(n \in \mathbb{Z})$ is a central element of G since $Ad(e^{n2\pi H}) = Id_g$ (using the normalization $(AdH)^2|_p = -Id_p$). $U(\cdot)$ being irreducible this implies

$$(4.1) \qquad U(\exp n2\pi H) = e^{in2\pi d}, \quad d \in \mathbb{R}, \quad n \in \mathbb{Z}$$

so $\frac{1}{i} U(H)$ has discrete spectrum $d + \ell$, $\ell \in \mathbb{Z}$. We may choose d to be the lowest eigenvalue of $\frac{1}{i} U(H)$. Thus the spectrum of $\frac{1}{i} U(H)$ contains at most the points d, d + 1, d + 2, ..., $d \geq 0$. Let $\delta \in \hat{K}$ such that $\mathcal{H}(\delta) \neq \emptyset$ and $\frac{1}{i} U(H) \cdot v = d \cdot v$ for any $v \in \mathcal{H}(\delta)$ (such δ exists). Fix a subspace $\mathcal{H}'(\delta)$ in which $U(\cdot)|_K$ acts irreducibly. In fact $\mathcal{H}'(\delta)$ carries an irreducible representation of K', the connected Lie subgroup of K with Lie algebra k'. k' is compact and $\mathcal{H}'(\delta)$ finite dimensional, hence the representation of k'_c on $\mathcal{H}'(\delta)$ has a highest weight Λ' with weight vector Ω. Set $\Lambda \doteq d \cdot \Lambda_H + \Lambda'$ (through $\Lambda'(H) \doteq 0$, Λ' is identified with a linear form on h_c). Then $U(Z)\Omega = \Lambda(Z)\Omega$ for all $Z \in h_c$. Furthermore if $\alpha \in \Delta_k^+$ then $U(X_\alpha)\Omega = 0$ by definition of Ω. If $\alpha \in \Delta_p^+$ then $\frac{1}{i} U(H) U(X_\alpha)\Omega = \frac{1}{i} U([H, X_\alpha])\Omega +$
$+ U(X_\alpha) \frac{1}{i} U(H)\Omega = (\frac{1}{i} \alpha(H) + d) U(X_\alpha)\Omega$.
Now $\frac{1}{i} \alpha(H) = -1$ by definition of Δ_p^+. Hence $U(X_\alpha)\Omega = 0$ because d is the lowest eigenvalue of $\frac{1}{i} U(H)$. Using the algebraic irreducibility of the representation of g_c on $\mathcal{H}_K$ the theorem follows. ∎

Naturally, there arises the question for what $d \geq 0$ and Λ' dominant integral an irreducible representation U(·) of G actually exists. A partial answer to this question is well known: if Λ' is arbitrary and d large enough (the bound will be specified later) then there exists an analytic representation of the universal covering $\tilde{G}$ of G with highest weight $\Lambda = d\Lambda_H + \Lambda'$ and this representation belongs to the interpolated holomorphic discrete series. For completeness, I will now describe these representations (e.g. [1] and references cited therein). Let G_c be a complex Lie group and G a real form of G_c with hermitian Lie algebra g, k, p, H, p_+, p_-, K, K_c, P_+, P_- as usual. Furthermore let $\tilde{G}$, $\tilde{K}$, $\tilde{K}_c$ be the connected universal coverings of G, K, K_c respectively. Denote by V_Λ the unitary irreducible finite dimensional representation of $\tilde{K}$ in the complex vectorspace E_Λ with highest weight $\Lambda = d\Lambda_H + \Lambda'$. Now $G \subset P_- \cdot K_c \cdot P_+$ i.e. for any $x \in G$ we have $x = x_- \cdot k(x) \cdot x_+$ in obvious notation. The mapping $x \to k(x)$ uniquely lifts to a mapping $\tilde{x} \to \tilde{k}(x)$ of $\tilde{G}$ into $\tilde{K}_c$ such that $\tilde{k}(1) = 1$. By analytic continuation V_Λ extends to a holomorphic representation of $\tilde{K}_c$. Define

$$(4.2) \qquad \langle v, w\rangle_{\Lambda,z} \doteq \langle V_\Lambda(\tilde{k}(g_z))^{-1} \cdot v,\ V_\Lambda(\tilde{k}(g_z))^{-1} \cdot w\rangle \quad \text{for all } v, w \in E_\Lambda$$

$\langle .,. \rangle$ being a $\tilde{K}$-invariant scalar product on E_Λ and $g_z \in \tilde{G}$ such that $z = g_z \cdot \tilde{K} \in \tilde{G}/\tilde{K}$. With the help of a $\tilde{G}$-invariant measure $d\mu(z)$ on $\tilde{G}/\tilde{K}$ (it is unique up to a constant) one is able to define a Hilbert space $H(\Lambda)$ consisting of holomorphic functions $\phi(z)$ on $\tilde{G}/\tilde{K} \cong G/K \cong \mathbb{D}$ (compare sec. II) with values in E_Λ such that

$$(4.3) \qquad \|\phi\|^2_\Lambda \doteq \int_{G/K} d\mu(z)\ \|\phi(z)\|^2_{\Lambda,z} < \infty$$

Harish-Chandra has given necessary and sufficient conditions for Λ in order that H (Λ) is nontrivial. If $\pi = \{\alpha_1, \dots, \alpha_n\}$ is the system of simple roots for the Weylchamber P we may assume that $\pi \cap \Delta^+_p = \{\alpha_1\}$. Let γ be the largest root when ordering Δ lexicographically with respect to π (for any root β, $\beta > 0$ iff $\beta = \sum_{i=1}^{n} n_i \alpha_i$, $n_1, \dots, n_k = 0$, $n_{k+1} > 0$ for some k). Obviously $\gamma \in \Delta^+_p$ and $H_\gamma = iH + H'_\gamma$, $H'_\gamma \in ih'$ ($B(H_\gamma, z) = \gamma(z)$ for all $z \in h_c$). Harish-Chandra's result is the following:

$H(\Lambda)$ is nontrivial iff $(\Lambda+\varrho_p)(H_\gamma) < 0$

where $\varrho_p = \frac{1}{2}\sum_{\alpha\in\Delta_p^+}\alpha$. According to Lemma 3.5 $\varrho_p(H_\gamma) > 0$ and $\Lambda(H_\gamma) = (d\cdot\Lambda_H + \Lambda')(H_\gamma) = -d + \Lambda'(H_\gamma)$. Because Λ' is dominant integral with respect to (h_c', Δ_k^+) and $B(H_{\alpha_i}, H_\gamma) \geq 0$ we have $\Lambda'(H_\gamma) \geq 0$. Setting $d_c \doteq \Lambda'(H_\gamma) + \varrho_p(H_\gamma)$ one finds $d_c > 0$ and a nontrivial $H(\Lambda)$ iff $d > d_c$. In this case $H(\Lambda)$ carries an irreducible representation of $\tilde{G}$ which belongs to the interpolated discrete series of $\tilde{G}$. The unitary operators $U(x)$, $x \in \tilde{G}$, are defined by:

$$(4.4) \qquad (U_\Lambda(x)\,\phi)(z) = S_\Lambda(x,z)\,\phi(x^{-1}\cdot z)$$

where

$$(4.5) \qquad S_\Lambda(x,z) \doteq V_\Lambda(\tilde{k}(g_z))\,V_\Lambda(\tilde{k}(x^{-1}g_z))^{-1} \qquad (z = g_z\cdot\tilde{K},\ g_z \in \tilde{G})$$

I will prove now that $S_\Lambda(x,z)$ is indeed a holomorphic (matrix-valued) function of z. Let $\tilde{\pi}_{\tilde{K}_c}$ the projection of $P_- \times \tilde{K}_c \times P_+$ on $\tilde{K}_c$. Identify G/K with the open subset $e^{\mathbb{D}}$ of P_-. Then G acts on $e^{\mathbb{D}} \times K_c \times P_+$ by left multiplication and this action may be lifted to an action of $\tilde{G}$ on $e^{\mathbb{D}} \times \tilde{K}_c \times P_+$. For any two elements x, y of $\tilde{G}$ we have:

$$\tilde{k}(x^{-1}\cdot y) = \tilde{\pi}_{\tilde{K}_c}[(x^{-1}\cdot y)(1)] = \tilde{\pi}_{\tilde{K}_c}(x^{-1}[y(1)])$$

$(1 \doteq 1 \times 1 \times 1 \in e^{\mathbb{D}} \times K_c \times P_+)$. Especially for $y = g_z$ we have $g_z(1) = e^{X_-}\times q$, $X_- = X_-(z) \in \mathbb{D}$ a holomorphic function of z and $q \in \tilde{K}_c \times P_+$. Clearly:

$$\tilde{\pi}_{\tilde{K}_c}[x^{-1}(g_z(1))] = \tilde{\pi}_{\tilde{K}_c}[x^{-1}(e^{X_-})]\cdot\tilde{\pi}_{\tilde{K}_c}[q]$$

and hence:

$$(4.6) \qquad \begin{aligned} S_\Lambda(x,z) &= V_\Lambda[\tilde{\pi}_{\tilde{K}_c}(q)]\cdot V_\Lambda[\tilde{\pi}_{\tilde{K}_c}(x^{-1}(e^{X_-}))\cdot\tilde{\pi}_{\tilde{K}_c}(q)]^{-1} = \\ &= V_\Lambda[\tilde{\pi}_{\tilde{K}_c}(x^{-1}(e^{X_-}))]^{-1} \end{aligned}$$

Since the action of $\tilde{G}$ on $e^{\mathbb{D}} \times \tilde{K}_c \times P_+$ is holomorphic and so is the projection $\pi_{\tilde{K}_c}$) $S_\Lambda(x, z)$ depends holomorphically on X_- and hence on z.

<u>Theorem 4.3:</u> If $d > d_c$ then the representation U_Λ of $\tilde{G}$ on $H(\Lambda)$ is the analytic representation with $\frac{1}{i} U(H) \geq 0$ and highest weight Λ.

<u>Proof:</u> (sketch) It is known that for $d > d_c$ $H(\Lambda)$ contains the constant functions $\phi_v(z) = v \in E_\Lambda$. The closed subspace of all these functions carries the irreducible representation of $\tilde{K}$ with highest weight Λ, in fact: $U_\Lambda(\ell)\,\phi_v = V_\Lambda(\ell)\cdot v \quad (\ell \in \tilde{K})$. Taking Ω to be the vector with highest weight in this subspace we have $U_\Lambda(Z)\Omega = \Lambda(Z)\Omega$ for all $Z \in h_c$. To prove that Ω is also the vector with highest weight with respect to $U_\Lambda(x)$, $x \in \tilde{G}$, we have to show that $U_\Lambda(X)\Omega = 0$ for $X \in p_-$. Now $X \notin g$ which forbids the direct calculation $U_\Lambda(X)\Omega = \frac{d}{dt} U_\Lambda(\exp tX)\Omega \big|_{t=0}$. However $U_\Lambda(X)$ is defined for all $X \in g_c$ by complex linearity and a glance at (4.6) shows that for fixed $z \in G/K$ i.e. fixed X_- one can analytically continue $S_\Lambda(x, z)$ into a neighborhood of 1 in G_c. Thus

$$(U_\Lambda(X)\,\phi_v)(z) = \frac{d}{dt}\left(S_\Lambda(e^{tX}, z)\cdot v\right)\Big|_{t=0} = 0$$

since $\pi_{\tilde{K}_c}(e^{-tX}(e^{X_-})) = 1$ for $X \in p_-$. ∎

We have thus seen, that a simple Lie group has analytic representations iff its Lie algebra is hermitian.
In that case the holomorphic (interpolated) discrete series **are** not empty and these representations are all analytic. It is known and is a subject of present mathematical research that the representations U_Λ may be analytically continued (after multiplying with a convenient meromorphic factor c(d)) in the variable d to values smaller than the critical value d_c [1]. As far as I know no complete answer to the question of determining the lowest value of d (for fixed Λ') has been given yet. For the group $\widetilde{SU(1,1)}$ the problem has been solved long ago and more recently for the group $\widetilde{SU(2,2)}$, too [3].

V Coverings of polarly decomposable Lie semigroups

Let G be a real, connected Lie group with hermitian Lie algebra g, $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ a Cartandecomposition and K the connected subgroup of G with Lie algebra k. Assume that G is a real form of a complex, connected Lie group G_c and define the semigroups S^o and S as in sec.III. Define the center $\mathfrak{z}_S$ of S by

$$\mathfrak{z}_S = \{ s \in S \mid s \cdot x = x \cdot s \quad \text{for all} \quad x \in S \}$$

Lemma 5.1: The center of G_c equals $\mathfrak{z}_S$ and $\mathfrak{z}_S$ is also the center of G (in particular: $\mathfrak{z}_S \subset G$)

Proof: Fix a maximal abelian subalgebra $\mathfrak{h}$ of k and set $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. Then h is a maximal abelian subalgebra of $\mathfrak{u}$.
Let U resp. T be the connected subgroups of G_c with Lie algebra $\mathfrak{u}$ resp. $\mathfrak{h}$. T is a maximal torus of the maximal compact subgroup U of G_c. Thus the center $\mathfrak{z}_{G_c}$ of G_c is contained in T ([6] 7.2.5 and 3.8.3). Since T is a subgroup of G we have shown that $\mathfrak{z}_{G_c} \subset \mathfrak{z}_G$. Conversely, if $z \in \mathfrak{z}_G$ then xz = zx for all $x \in G$ and by analyticity in x it follows that $z \in \mathfrak{z}_{G_c}$, hence $\mathfrak{z}_{G_c} = \mathfrak{z}_G$. ∎

This lemma shows, that the universal covering $\tilde{G}_c$ of G_c contains the largest semigroup of type S that is constructable along the scheme explained in sec. III. In fact, let $\check{G}$ the connected subgroup of $\tilde{G}_c$ with Lie algebra g and $\check{S}$ the corresponding semigroup ($\check{G}$ is not simply-connected). Clearly then:

Lemma 5.2: There exists a discrete central subgroup Γ of $\check{G}$ such that $G_c = \tilde{G}_c/\Gamma$, $G = \check{G}/\Gamma$, $S = \check{S}/\Gamma$.

The question now arises of how to construct a holomorphic semigroup in case G is not a real form of some complex Lie group G_c, e.g. for $\tilde{G}$, the universal covering of G. Of course, the appropriate framework for dealing with such groups would be a general theory of Lie semigroups without refering to a complex carrier Lie group. However in doing so one encounters some obscurities concerning the local and global differentialgeometric structure of the semigroups and it seems to be difficult to obtain practically valuable results (especially in view of the "continuation theorem", sec. VI). I will thus make the assumption that S is polarly decomposable (G_c,G,S like above). This restricts the discussion to semigroups associated with the Lie algebras

$$\mathfrak{su}(n,m) \quad (n,m \geq 1) \quad ; \quad \mathfrak{so}^*(2n) \quad (n \geq 3) \quad ; \quad \mathfrak{sp}(n,\mathbb{R}) \quad (n \geq 1)$$

and by isomorphism

$$\mathfrak{so}(1,2) \; ; \mathfrak{so}(3,2) \; ; \mathfrak{so}(4,2) \; ; \mathfrak{so}(6,2)$$

(maybe all the holomorphic semigroups of sec. III are polarly decomposable; however...)

Let V be the open, G-invariant cone generated by H ($k = k' \oplus \mathbb{R} \cdot H$, k' compact). Then $S^o \cong G \times V$ where $\cong$ denotes the surjective diffeomorphism $(u,X) \in G \times V \to ue^{iX} \in S^o$. Obviously the universal covering $\tilde{S}^o$ of S^o is diffeomorphic to $\tilde{G} \times V$. The set $\tilde{\bar{S}} \doteq \tilde{G} \cup \tilde{S}^o$ is then made into a topological space by defining $\mathcal{O} \subset \tilde{\bar{S}}$ to be open iff $\pi(\mathcal{O})$ is open, where π: $\tilde{\bar{S}} \to \bar{S}$ is the natural projection ($\bar{S}$ is given the subspace topology, i.e. $\mathcal{O} \subset \bar{S}$ is open iff $\mathcal{O} = \bar{S} \cap \mathcal{O}'$, $\mathcal{O}'$ an open subset of G_c). When looking at $\tilde{G}$ as a topological subspace of $\tilde{\bar{S}}$ the ordinary topology is recovered, since:

<u>Lemma 5.3:</u> G is a closed subgroup of G_c.

<u>Proof:</u> Let $\theta: \mathfrak{g}_c \to \mathfrak{g}_c$ be the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{g}$. Then θ lifts to an automorphism of $\tilde{G}_c$ such that $\theta(\exp X) = \exp \theta X$. By lemma 5.2 $G_c \cong \tilde{G}_c / \Gamma$, Γ a central discrete subgroup of $\tilde{G}$. Hence $\theta \Gamma = \Gamma$ which means that θ can also be defined on G_c. Now one easily shows that G is the connected component of $\{x \in G_c \mid \theta(x) = x\}$ containing 1, which is closed.∎

Obviously $\tilde{S}^o$, when equipped with the natural topology arising from its differentiable structure, is also a topological subspace of $\tilde{S}$.
The local structure of $\tilde{S}$ is furthermore clarified by the notion of charts. There exist two types of charts, both are made up from an open set $\mathcal{O} \subset \tilde{S}$ and a mapping $\varphi : \mathcal{O} \longrightarrow \mathbb{R}^{2n}$ (n = dimg). In the first case $\varphi(\mathcal{O}) = \mathbb{R}^{2n}$ and φ is a homöomorphism, i.e. $(\mathcal{O}, \varphi)$ is an ordinary chart of $\tilde{S}^o$. In the other case $\varphi(\mathcal{O}) = \mathbb{R}^n \times (V \cup \{o\})$ and by giving this set the subspace topology from $\mathbb{R}^{2n}$ φ is also a homöomorphism. Obviously $\varphi^{-1}(\mathbb{R}^n \times \{o\}) = \mathcal{O} \cap \tilde{G}$ and $\varphi^{-1}(\mathbb{R}^n \times V) \subset \subset \tilde{S}^o$. In fact, $(\mathcal{O} \cap \tilde{G}, \varphi|_{\mathcal{O} \cap \tilde{G}})$ is a chart for $\tilde{G}$.

Theorem 5.4: $\tilde{S}$ carries a unique semigroup structure such that the multiplication viewed as a mapping from $\tilde{S} \times \tilde{S} \to \tilde{S}$ is continuous resp. holomorphic when restricted to $\tilde{S}^o \times \tilde{S}^o$ and such that

$$\left.\begin{aligned} \tilde{\pi}(s_1 \cdot s_2) &= \tilde{\pi}(s_1) \cdot \tilde{\pi}(s_2) \\ 1 \cdot s = s \cdot 1 &= s \end{aligned}\right\} \text{ for all } s_1, s_2, s \in \tilde{S}$$

Here, 1 denotes the unit for the (canonical) multiplication in $\tilde{G}$ which coincides with that inherited from $\tilde{S}$.

Proof: (scetch) With the help of the **monodromy** principle [9] one lifts the mapping $m: \tilde{S}^o \times \tilde{S}^o \to S^o$, $m(s_1, s_2) = \tilde{\pi}(s_1) \cdot \tilde{\pi}(s_2)$ to a mapping $\tilde{m}: \tilde{S}^o \times \tilde{S}^o \to \tilde{S}^o$ such that

$$\tilde{m}((1,X), (1,X)) = (1, 2X) \quad , \quad X \in V.$$

By a consideration of charts one easily shows that m can be continuously extended to a mapping from $\tilde{S} \times \tilde{S}$ into $\tilde{S}$ and this is just the multiplication looked for. ∎

The existence of the simply connected semigroup $\tilde{S}$ gives rise to a semigroup S_G for any Lie group G with Lie algebra g, namely by setting $S_G \cong \tilde{S}/\Gamma$, Γ a the central discrete subgroup of $\tilde{G}$ with $G \cong \tilde{G}/\Gamma$. Clearly then $S_G^o \cong G \times V$ has the same local differentialgeometric features as $\tilde{S}$.

Let G_c, G, S be as usual. Then the following antiautomorphism of G_c leaves S invariant:

$$x \longrightarrow \bar{x} \doteq \theta(x^{-1})$$

(θ is the conjugation of G_c with respect to G, see proof of Lemma 5.3). Clearly:

$$\overline{x_1 \cdot x_2} = \bar{x}_2 \cdot \bar{x}_1 \qquad \text{and} \qquad \bar{x} = x^{-1} \quad \text{for } x \in G$$

This antiautomorphism lifts uniquely to an antiautomorphism of $\tilde{S}$ such that $\bar{1} = 1$, and thus may also be defined on any semigroup $S_{G'}$, because $\bar{\Gamma} = \Gamma$ ($G' \cong \tilde{G}/\Gamma$).

VI. Holomorphic, contractive representations of polarly decomposable semigroups and the continuation theorem

Let G be a connected Lie group with hermitian Lie algebra g and assume that the semigroups associated with g are polarly decomposable. Thus there exists a holomorphic semigroup S_G having G at the boundary (sec.V).

Definition 6.1: A holomorphic, contractive representation of S_G is a mapping $T: S_G \to L(\mathcal{H},\mathcal{H})$, $\mathcal{H}$ a Hilbertspace, $L(\mathcal{H},\mathcal{H})$ the space of all bounded, linear operators in $\mathcal{H}$, having the following properties:

(1) $T(s_1 \cdot s_2) = T(s_1) \cdot T(s_2)$ for all $s_1, s_2 \in S_G$

(2) $||T(s)|| \leq 1$ for all $s \in S_G$

(3) $T(s)^+ = T(\bar{s})$ for all $s \in S_G$

(4) for any $\phi \in \mathcal{H}$, $T(s)\phi$ is a continuous vectorvalued function on S_G and when restricted to the interior S_G^o of S_G this function is even holomorphic.

(Here, A^+ denotes the Hilbert space adjoint of the linear operator A and $s \to \bar{s}$ is the antiautomorphism defined in the preceding section)

Of course, if one restricts a holomorphic, contractive representation of S_G to G, one obtains a unitary, continuous representation of G. This representation is an analytic representation of G, in fact $\frac{1}{i}U(X) \geq 0$ for all $X \in V$. Indeed the elements $(\exp(\mathrm{Re}\, z\, X), \mathrm{Im}\, z\, X) \in G \times (V \cup \{0\}) \cong S_G$ $(z \in \mathbb{C},\ \mathrm{Im}\, z \geq 0)$ build up a holomorphic one parameter semigroup which is represented contractively and holomorphically by T. Hence its generator is represented positively (modulo $\sqrt{-1}$).

Conversely we have:

Theorem 6.2: Suppose $U(x), x \in G$, is an analytic representation of G in a Hilbertspace $\mathcal{H}$ such that $\frac{1}{i}U(H) \geq 0$. Then $U(\cdot)$ can be analytically continued to a holomorphic, contractive representation of S_G.

(As usual, H is a central element of a maximal compactly embedded subalgebra k of g and V the cone generated by H). This theorem is in a way the converse of a theorem stated and proved in reference [8] on the possibility of analytic continuation of contractive representations of a "euclidean" semigroup.

Proof: We may assume that G is simply connected, i.e. $G = \tilde{G}$, $S_G = \tilde{S}$ (if $S_G = \tilde{S}/\Gamma$, $G = \tilde{G}/\Gamma$ then Γ is represented trivially). I will henceforth use the letter G for a Lie group with Lie algebra g being a real form of a complex Lie group G_c. Let S be the holomorphic semigroup contained in G_c that is generated by G and V. Then there are open neighborhoods $N \subset N' \subset \mathfrak{g}$ of 0 such that:

(i) $\exp' : N' \times N' \to G_c$, $\exp'(X,Y) = e^X \cdot e^{iY}$ is a diffeomorphism and $\exp'(X,Y) \in S^O$ iff $Y \in V$.

(ii) $\exp'(N,N) \cdot \exp'(N,N) \subset \exp'(N',N')$

(iii) $\lambda \cdot N' \subset N'$ for all $\lambda \in [0,1]$

(iv) there exists a dense set $D \subset \mathcal{H}$ of equianalytic vectors for $U(x)$, $x \in \tilde{G}$, such that $U(x)\phi$ has an analytic continuation to all of $\exp'(N',N')$ for all $\phi \in D$ [5], [8], [11].

For $x \in S \cap \exp'(N',N')$ define bounded operators $T(x)$ in the following way: set $x = \exp'(X,Y)$, $X \in N'$, $Y \in V \cap N'$ or $Y = 0$. In the latter case put $T(x) = U(x)$. If $Y \neq 0$ set $T(x) = U(\exp X) \cdot e^{-\frac{1}{i} U(Y)}$. Clearly $T(x)$ is well defined and $\|T(x)\| \leq 1$. For any $\phi \in D$, $U(x)\phi$ $(x \in \exp N')$ can be analytically continued to all of $\exp'(N', N')$ and the value of this continuation at $(X, Y) \in N' \times N'$ will be denoted by $\phi(X, Y)$.

Lemma 6.3: If $X \in N'$, $Y \in N' \cap V$ or $Y = 0$, we have $\phi(X,Y) = T(\exp'(X,Y)) \cdot \phi$. Moreover for any $\Psi \in \mathcal{H}$ $T(x)\Psi$ is a continuous function of $x \in S \cap \exp'(N',N')$ and a holomorphic function of $x \in S^{\circ} \cap \exp'(N',N')$.

Proof: If $Y = 0$, clearly $\phi(X,Y) = T(\exp'(X,Y))\phi$. If $Y \in V \cap N'$, then $U(\exp X) \exp[z \cdot U(Y)]\, \phi$, $\operatorname{Im} z > 0$, is a holomorphic function of z that is continuous for $\operatorname{Im} z \searrow 0$. Moreover there is a connected open neighborhood $\mathcal{O} \subset \mathbb{C}$

of zero, such that $\exp X \cdot \exp zY \in \exp'(N', N')$ for all $z \in \mathcal{O}$ and $i \in \mathcal{O}$. Now $z \in \mathcal{O} \longrightarrow \exp X \cdot \exp zY \in G_c$ is holomorphic, hence by uniqueness of analytic continuation: $\phi(X,Y) = U(\exp X) \exp iU(Y)\phi = T(\exp'(X,Y))\cdot\phi$.
This also proves that for $\phi \in D$, $T(x)\phi$ is continuous resp. holomorphic for $x \in S \cap \exp'(N',N')$ resp. $x \in S^\circ \cap \exp'(N',N')$, a statement which extends by uniform continuity of $T(\cdot)$ to all of $\mathcal{H}$. ∎ (Lemma)

We may choose N so small that if $x \in \exp'(N,N)$ then $\bar{x} = \Theta(x^{-1}) \in \exp'(N',N')$. For $x \in S \cap \exp'(N,N)$ we have

$$(6.1) \quad T(x)^+ = e^{iU(Y)}\, U(\exp(-X)) = U(\exp(-X)) \cdot e^{iU[\mathrm{Ad}\exp X \cdot Y]} = T(\bar{x})$$

(where $x = \exp'(X,Y)$). Hence for $x \in S^\circ \cap \exp'(N,N)$, $\psi \in \mathcal{H}$, $T(x)^+\psi$ is an antianalytic function of x that is continuous for $Y \to 0\ (Y \in V)$. For any $\phi, \psi \in \mathcal{H}$ the scalar product

$$(T(x)^+\psi, T(y)\phi) \quad ; \quad x, y \in S \cap \exp'(N,N)$$

is jointly continuous (resp. holomorphic) in x, y and so is $(\psi, T(x\cdot y)\phi)$

Moreover the two expressions coincide for $x, y \in G \cap \exp'(N,N)$. Again by uniqueness of analytic continuation one concludes that

$$(6.2) \quad T(x)\cdot T(y) = T(x\cdot y) \quad \text{for all} \quad x, y \in S \cap \exp'(N,N)$$

Putting everything together we have found a local holomorphic, contractive representation of S and hence of $\tilde{S}$. Although $\tilde{S}$ is simply connected the extension of a local representation of $\tilde{S}$ to all of $\tilde{S}$ is complicated through the fact that one can only continue T along special paths in $\tilde{S}$ namely the causal paths. A path[+)] $\gamma(t)$, $t \in [0,1]$, is called <u>causal</u> iff for $0 \le t_1 < t_2 \le 1$ $\gamma(t_1) < \gamma(t_2)$ i.e. there is some $s \in \tilde{S}^\circ$ such that $\gamma(t_2) = s \cdot \gamma(t_1)$.

[+)] By a path γ in $\tilde{S}$ I shall always mean a continuous mapping $\gamma : [0,1] \longrightarrow \tilde{S}$.

<u>Lemma 6.4:</u> For any path u(t) in $\tilde{G}$ and any $X \in V$ the curve $\gamma(t) = (u(t), tX) \in \tilde{G} \times (V \cup \{0\}) \cong \tilde{S}$ is a causal path in $\tilde{S}$.

<u>Proof:</u> Let $0 \leq t_1 < t_2 \leq 1$, $\Delta = t_2 - t_1 > 0$. Then

$$\begin{aligned}(u(t_2), t_2 X) &= (u(t_2), 0)\cdot(1, \Delta\cdot X)\cdot(1, t_1\cdot X) = \\ &= (u(t_2), 0)\cdot(1, \Delta\cdot X)\cdot(u(t_1)^{-1}, 0)\cdot(u(t_1), 0)\cdot(1, t_1\cdot X) = \\ &= s\cdot(u(t_1), t_1\cdot X) \qquad ; \qquad s = (u(t_2), \Delta\cdot X)\cdot(u(t_1)^{-1}, 0) \in \tilde{S}^\circ.\end{aligned}$$

Hence $\gamma(t)$ is causal. ∎ (Lemma)

Set $\mathcal{O} \doteq (\exp N, (N \cap V) \cup \{0\}) \subset \tilde{G} \times (V \cup \{0\}) \cong \tilde{S}$ and $\mathcal{O}' \doteq (\exp N', (N' \cap V) \cup \{0\})$.

An <u>allowed subdivision</u> of a causal path $\gamma(t), t \in [0,1]$, is a sequence $s_0 = \gamma(0)$, $s_1 = \gamma(t_1), \ldots, s_n = \gamma(1)$ such that $\gamma(t) \in \mathcal{O}\cdot s_j$ for $t \in [t_j, t_{j+1}]$, $(j = 0, \ldots, n-1)$. From continuity and causality of γ it follows easily that γ has allowed subdivisions.

Let $x \in \tilde{S}^\circ$ and $\gamma(t), t \in [0,1]$, a causal path in $\tilde{S}$ with $\gamma(0) = 1$, $\gamma(1) = x$. By Lemma 6.4 such a path exists. For any allowed subdivision $s_0, s_1 \ldots s_n$ define

$$T_\gamma(x) \doteq T(r_n)\cdot T(r_{n-1})\cdot \ldots \cdot T(r_1) \tag{6.3}$$

where the r_j's are defined by: $s_j = r_j \cdot s_{j-1}$, $j = 1, \ldots, n$

Observe, that $r_j \in \mathcal{O}$ and so $T(r_j)$ is well defined.

<u>Lemma 6.5:</u> $T_\gamma(x)$ depends only on γ and x but not on the particular choice of the allowed subdivision.

<u>Proof:</u> Given two allowed subdivisions $s_0, \ldots, s_n$ and $s'_0, \ldots, s'_{n'}$ of γ one can build a new allowed subdivision of γ by superposition of the given ones. Using

(6.2) it is then readily verified that the products (6.3) corresponding to the three subdivisions of γ are equal. ∎ (Lemma)

To show that $T_\gamma(x)$ does in fact not depend on γ too, I need one more Lemma:

Lemma 6.6: If γ and γ' are two causal curves in $\tilde{S}$ such that $\gamma(0)=\gamma'(0)=1$ and $\gamma(1)=\gamma'(1)=x$ then there exists a continuous mapping $\Psi: [0,1]\times[0,1] \to \tilde{S}$ satisfying:

(i) $\Psi(0,\lambda)=1$, $\Psi(1,\lambda)=x$ for all $\lambda \in [0,1]$

(ii) for fixed $\lambda \in [0,1]$, $\Psi(t,\lambda)$, $t \in [0,1]$ is a causal curve in $\tilde{S}$.

$\Psi(t,0)=\gamma(t)$; $\Psi(t,1)=\gamma'(t)$

(Ψ will be called a causal homotopy)

Proof: Since $\tilde{S} \cong \tilde{G}\times(V\cup\{0\})$, $\gamma(t)$ may be written as $(u(t), X(t))$, where $u(t)$ is a curve in $\tilde{G}$ and $X(t)$ a curve in $V\cup\{0\}$. Note that $\gamma(t)\in \tilde{S}^{0}$ for $t > 0$ and so $X(t)\in V$ for $t>0$ and $X(0)=0$. For $(t,\lambda)\in[0,1]\times[0,1]$ set

$$\phi(t,\lambda) = \begin{cases} \gamma(t) & \lambda = 0 \\ (u(t), \frac{t}{\lambda}X(\lambda)) & \lambda \neq 0 \quad , \quad t\in[0,\lambda] \\ \gamma(t) & \lambda \neq 0 \quad , \quad t \in [\lambda,1] \end{cases}$$

Then $\phi(t,\lambda)$ is a causal homotopy such that

$$\phi(t,0) = \gamma(t)\ , \quad \phi(t,1) = (u(t), t\cdot X(1))$$

The same can be done with γ' yielding a causal homotopy $\phi'(t,\lambda)$ such that

$$\phi'(t,0) = \gamma'(t)\ , \quad \phi'(t,1) = (u'(t), tX(1))$$

Because $\tilde{G}$ is simply connected and $u(0)=u'(0)=1$, $u(1)=u'(1)$ one can deform $\phi'(t,1)$ into $\phi(t,1)$ by another causal homotopy and by composition of the three homotopies one obtains $\Psi(t,\lambda)$ ∎ (Lemma)

Let γ, γ' be two paths in $\tilde{S}$ having common endpoints 1 and $x \in \tilde{S}^{o}$ and let $\Psi(t,\lambda)$, $t,\lambda \in [0,1]$ a causal homotopy such that $\Psi(t,0) = \gamma(t)$ and $\Psi(t,1) = \gamma'(t)$. If we can prove that for any $\lambda \in [0,1]$ one can find a positive number ε such that

$$(6.4) \quad T_{\Psi(\cdot,\lambda)}(x) = T_{\Psi(\cdot,\lambda')}(x) \quad \text{for all } \lambda' \in [0,1] \text{ with } |\lambda' - \lambda| < \varepsilon$$

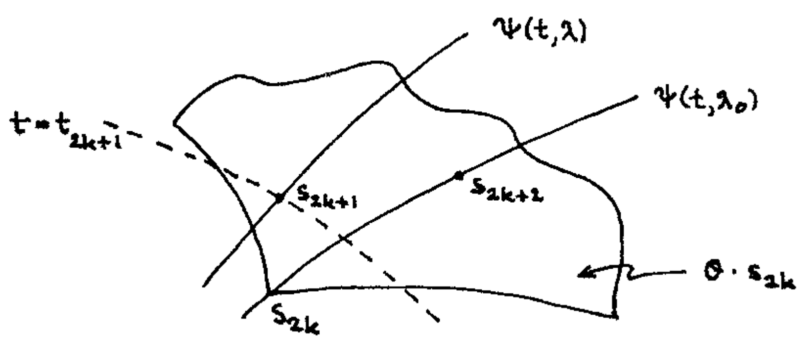
then clearly $T_{\gamma}(x) = T_{\gamma'}(x) \doteq T(x)$.
The topology of G_c can be described by a right invariant distance and this can be done as well for $\tilde{S}$. Since the set $[0,1] \times [0,1]$ is compact, $\Psi(t,\lambda)$ is uniformly continuous with respect to the right invariant distance. Hence there are positive numbers ω, δ such that

if $|t-t'| < \omega$, $|\lambda - \lambda'| < \delta$ and $\Psi(t,\lambda) < \Psi(t',\lambda')$
then $\Psi(t',\lambda') \in \mathcal{O} \cdot \Psi(t,\lambda)$.

Now fix $\lambda = \lambda_o \in [0,1]$ and choose $0 = t_o, t_1, \dots, t_{2n} = 1$ such that $0 < t_{j+1} - t_j < \omega/2$ and set $s_{2j} = \Psi(t_{2j}, \lambda_o)$. Obviously $s_o, s_2, \dots, s_{2n}$ is an allowed subdivision of $\Psi(t,\lambda_o)$. By construction $\Psi(t_{2k+1}, \lambda_o) \in \mathcal{O}^o \cdot s_{2k}$, $\mathcal{O}^o$ denoting the interior of $\mathcal{O}$ $(k = 0,1,\dots, n-1)$. Because of continuity of $\Psi(t,\lambda)$ there exists some $\delta_k > 0$ such that $\Psi(t_{2k+1},\lambda) \in \mathcal{O}^o \cdot s_{2k}$ for $|\lambda - \lambda_o| < \delta_k$. Set

$$\varepsilon = \min\{\delta, \delta_1, \delta_2, \dots, \delta_{n-1}\}$$

and let $\lambda \in [0,1]$, $|\lambda - \lambda_o| < \varepsilon$. Define $s_{2k+1} \doteq \Psi(t_{2k+1}, \lambda)$.
Then $s_o, s_1, s_3, \dots, s_{2n-1}, s_{2n}$ is an allowed subdivision of the curve $\Psi(t,\lambda)$.



Define r_k through $s_{k+2} = r_k \cdot s_k$ $(k = 0, \dots, 2n-2)$ and ϱ_{2k} by $s_{2k+1} = \varrho_{2k} \cdot s_{2k}$ $(k = 0, \dots, n-1)$. The r_k's and ϱ_{2k}'s are all elements of $\mathcal{O}$ and they satisfy:

$$r_{2k+1} \cdot \varrho_{2k} = \varrho_{2k+2} \cdot r_{2k} \quad \in \mathcal{O}' \qquad (k = 0, 1, \dots, n-2)$$

$$r_{2n-1} \cdot \varrho_{2n-2} = r_{2n-2} \quad \in \mathcal{O}'$$

hence by (6.2):

$$T(r_{2k+1}) T(\varrho_{2k}) = T(\varrho_{2k+2}) T(r_{2k})$$

$$T(r_{2n-1}) \cdot T(\varrho_{2n-2}) = T(r_{2n-2})$$

Thus by induction one proves that

$$T(r_{2k-1}) T(r_{2k-3}) \cdot \ldots \cdot T(r_1) \cdot T(s_1) =$$

$$= T(\varrho_{2k}) \cdot T(r_{2k-2}) \cdot T(r_{2k-4}) \cdot \ldots \cdot T(r_2) \cdot T(r_0)$$

for k = 0, 1, ..., n - 1 and then

$$T(r_{2n-1}) \cdot T(r_{2n-3}) \cdot \ldots \cdot T(r_1) T(s_1) = T(r_{2n-2}) \cdot T(r_{2n-4}) \cdot \ldots \cdot T(r_2) \cdot T(r_0)$$

But the left hand side of this equation equals $T_{\psi(\cdot,\lambda)}(x)$ and the right hand side $T_{\psi(\cdot,\lambda_0)}(x)$. This proves (6.4) and thus establishes the independence of $T_\gamma(x)$ on γ.
Set $T(x) \doteq T_\gamma(x)$. Obviously $T(x) \cdot T(y) = T(x \cdot y)$ for all $x, y \in \tilde{S}^\circ$ and since T(x) is continuous near 1 we can define T(x) for all $x \in \tilde{S}$ by continuity. Furthermore T(x) inherits all the properties concerning holomorphy, contractivity and reality ($T(x)^+ = T(\bar{x})$) from the

local representation ∎ (Theorem).

Let me briefly describe how the representations of $\tilde{S}$ are realized in the irreducible analytic representations U_Λ of $\tilde{G}$ carried by the Hilbertspace $H(\Lambda)$ (sec. IV). This is most easily seen by using the reproducing -(Bergman-) kernel $K_\Lambda(z,z')$ which is defined by

$$(K_\Lambda(z,\cdot),\phi)_\Lambda = \phi(z) \quad \text{for all} \quad \phi \in H(\Lambda) \ , \ z \in \tilde{G}/\tilde{K}$$

Now, applying $U_\Lambda(x)$, $x \in \tilde{G}$, yields (see(4.4))

$$(K_\Lambda(z,\cdot), U_\Lambda(x)\phi)_\Lambda = (U_\Lambda\phi)(z) = S_\Lambda(x,z)\cdot\phi(x^{-1}\cdot z)$$

By the above theorem, the left hand side of this equation has an analytic continuation namely $(K_\Lambda(z,\cdot), T_\Lambda(x)\phi)_\Lambda$ to all $x \in \tilde{S}$. Choosing $\phi(z) = v$ = constant it follows that $S_\Lambda(x,z)$ has an analytic continuation to all $x \in \tilde{S}$ and for this function I will use the same symbol.
Next, recall theorem 3.3, which said, that $S^* = S^{-1}$ acts on G/K from the left in a holomorphic way. Hence for $x \in \tilde{S}$, $\phi \in H(\Lambda)$, $\phi(\pi(x)^{-1}\cdot z)$ is well defined and holomorphic (continuous) in $x \in \tilde{S}^\circ$ $(\tilde{S})$.
(π denotes the canonical projection of $\tilde{S}$ onto S). Collecting all these arguments yields for all $x \in \tilde{S}$:

$$(T_\Lambda(x)\phi)(z) = S_\Lambda(x,z)\cdot\phi(\pi(x)^{-1}\cdot z)$$

VII Conclusions and unsolved problems

It has been shown that the general definition of an analytic representation given in sec. 1 is a powerful one. Only a restricted class of simple Lie groups actually have analytic representations. More precisely, analytic representations occur iff the Lie group G has a holomorphic (interpolated) discrete series of representations. Representations belonging to this series are analytic. These are however not all irreducible analytic representations of G ([1], [3]). Irreducible analytic representations are representations with highest weight Λ. If they do not belong to the discrete (interpolated) series they can be reached from the latter by analytic continuation in Λ. The problem of giving explicit realizations of all irreducible analytic representations is not yet solved completely (see however [1]).
Simple Lie groups G having analytic representations also have the special property that they are boundary groups of certain holomorphic Lie semigroups S_G. These semigroups act in a natural way on holomorphic homogeneous spaces G/K which play a key role in the construction of the holomorphic (interpolated) discrete series of representations of G.
The holomorphic semigroups S_G cannot always be embedded in a holomorphic Lie group G_c of the same dimension. Therefore the problem arises to give a general definition of a (nonabelian) Lie semigroup, a question which is open to future research.
The existence of analytic representations and holomorphic semigroups are not unrelated. In fact it has been shown in the preceeding section that the holomorphic, contractive representations of S_G all arise from analytic representations of the boundary group G. However this result could not be established for all Lie groups G with hermitian Lie algebra g, namely the Lie algebras $\mathfrak{so}(n,2)$ $(n > 6)$[+)] and $\mathfrak{e}_{6(-14)}$, $\mathfrak{e}_{7(-25)}$ were excluded. The missing step is a proof for the polar decomposability of the corresponding semigroups. The polar decomposition is to some extent a noncompact version of the global Cartan decomposition and it is extremely likely that it holds generally.
Once the continuation theorem holds, it follows that the vector valued functions $U(x)\phi$, $x \in G$, ($U(\cdot)$ an analytic representation of G) are boundary values of holomorphic functions defined on the huge domain S_G. Of course, it would be interesting to have some explicit, holomorphic parametrisations of S_G.

+) The case n = 5 needs a special argument to prove that its semigroups are polarly decomposable.

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Appendix A: The polar decomposition of some semigroups.

Let G_c be a connected simple Lie subgroup of $Gl(n,\mathbb{C})$ that is defined by a set of polynomial, holomorphic functions F_j , $j=1,\dots,m$, on $Gl(n,\mathbb{C})$ through: $\Lambda \in G_c$ iff $\Lambda \in Gl(n,\mathbb{C})$ and $F_j(\Lambda)=0$, $j=1,\dots,m$.
Clearly then, G_c is a complex closed subgroup of $Gl(n,\mathbb{C})$. Assume that there is a hermitian, nondegenerate sesquilinear form $(.,.)$ on $\mathbb{C}^n$ such that the group

(A1) $$G = \{\Lambda \in G_c \mid (\Lambda z_1, \Lambda z_2) = (z_1, z_2) \text{ for all } z_1, z_2 \in \mathbb{C}^n\}$$

is a connected real form of G_c. Denote by $\mathfrak{g}_c$ ($\mathfrak{g}$) the Lie algebra of G_c (G)+); obviously $\mathfrak{g}_c = \mathfrak{g} \oplus i\mathfrak{g}$. Define

$$V = \{\Sigma \in \mathfrak{g} \mid (z, i\Sigma z) < 0 \text{ for all } z \in \mathbb{C}^n,\ z \neq 0\}$$

(A2)

$$S^\circ = \{x \in G_c \mid x = u \cdot e^{iX_1} \cdot \ldots \cdot e^{iX_k};\ u \in G;\ X_1,\dots,X_k \in V;\ k \geq 1\}$$

Theorem: If $V \neq \emptyset$ then V is a nontrivial, open, G-invariant cone in $\mathfrak{g}$ and S° consists precisely of those elements $\Lambda \in G_c$ with $(\Lambda z, \Lambda z) < (z,z)$ for all $z \in \mathbb{C}^n$, $z \neq 0$. Moreover S° admits of a polar decomposition (see sec. III).

Proof: The first statement is trivial. Next let $X \in V$. Then
$\frac{d}{dt}(\exp itX \cdot z, \exp itX \cdot z) = 2(\exp itX \cdot z, iX \cdot \exp itX \cdot z) < 0 \quad (z \neq 0)$.
Hence $(\exp itX \cdot z, \exp itX \cdot z) < (z,z)$ $(t > 0,\ z \neq 0)$ so that
$S^\circ \subset \hat{S}^\circ \doteq \{\Lambda \in G_c \mid (\Lambda z, \Lambda z) < (z,z) \text{ for all } z \in \mathbb{C}^n,\ z \neq 0\}$.
For any endomorphism A of $\mathbb{C}^n$ define $\tilde{A}$ through $(\tilde{A} z_1, z_2) = (z_1, A z_2)$ for all $z_1, z_2 \in \mathbb{C}^n$. . We have $\tilde{X} = -X$ for all $X \in \mathfrak{g}$ and if $\Lambda \in G_c$ then $\tilde{\Lambda} \in G_c$.
Let $\Lambda \in \hat{S}^\circ$ and set $A \doteq \tilde{\Lambda} \cdot \Lambda$; $A \in G_c$ and $\tilde{A} = A$.

+) $\mathfrak{g}_c$ and $\mathfrak{g}$ are identified with their canonical $n \times n$-matrix representation.

Because $(z, Az) < (z,z)$ $(z \neq 0)$ one can diagonalize A: there exists a basis $z_1, \ldots, z_n$ of $\mathbb{C}^n$ such that

$$A z_j = \lambda_j z_j \quad ; \quad \lambda_j \in \mathbb{R} \, , \; \lambda_j (z_j, z_j) < (z_j, z_j)$$

$$(z_i, z_j) = 0 \qquad (i \neq j)$$

$$(z_i, z_i) = -1 \qquad (i = 1, \ldots, p)$$

$$(z_i, z_i) = +1 \qquad (i = p+1, \ldots, n)$$

The quadratic form $(z, Az) = (\Lambda z, \Lambda z)$ has the same signature as (z,z) has. For $j = 1, \ldots, p$ necessarily $\lambda_j > 1$ and hence $0 < \lambda_j < 1$ for $j = p+1, \ldots, n$. Thus one may write $A = e^X$ for some matrix X, $\tilde{X} = X$, $(z, X z) < 0$ for all $z \neq 0$. Let me now show that $X \in \mathfrak{g}_c$. Clearly, for all $t \in \mathbb{R}$, $\exp tX \in Gl(n, \mathbb{C})$ and for $k \in \mathbb{Z}$, $\exp kX \in G_c$.
Hence $Fj(\exp kX) = 0$ and thus $Fj(\exp tX) = 0, t \in \mathbb{R}$ (see e.g. [6] 3.2.7) proving that $\exp tX \in G_c$ and so $X \in \mathfrak{g}_c$. Because $\tilde{X} = X$ and $(z, Xz) < 0$ for all $z \neq 0$, $-iX$ is in fact an element of V. Now set $u \doteq \Lambda \cdot \exp(-\frac{1}{2}X)$ and we have by definition of X: $\tilde{u} \cdot u = 1$, $u \in G_c$ hence $u \in G$.
This establishes the inclusion $\hat{S}^o \subset S^o$ and therefore $\hat{S}^o = S^o$.
It has also been shown above that any $\Lambda \in S^o$ may be decomposed into $u \cdot \exp iX$, $u \in G$, $X \in V$ and this decomposition is unique. The fact that the mapping $(u, X) \longrightarrow u \cdot \exp iX$ is also regular can be proven by doubling standard arguments (e.g. proof of 3.2.10 in [6]).
Finally let $u \cdot \exp iX \in S^o$, $u \in G$, $X \in \mathfrak{g}$. Then also $\exp iX \in S^o$ and therefore one can diagonalize X yielding as above $X \in V$. Hence the local condition (ii) for S^o to be polarly decomposable is valid, too. ∎

Appendix B: Proof of Lemma 4.1

The notations are as in sec. IV. The proof presented here is not at all original but consists of collecting a bunch of results stated in Warner's book [11]. Without loss of generality one may assume that G is simply connected.

First notice that U(·) is topologically completely irreducible ([11] 4.3.1.7) and hence quasisimple (4.4.1.5). From Schur's Lemma (4.2.2.3) it follows then that $U(z) = \varkappa(z)\, U(1)$, $\varkappa(z) \in \mathbb{C}$, $z \in \mathfrak{Z}$, the center of G. Special elements of $\mathfrak{Z}$ are $\exp(n 2\pi H)$ (compare proof of Th. 4.2) thus proving that $\frac{1}{i}U(H)$ has discrete spectrum $\{d + \ell \mid \ell \in \mathbb{Z}\}$, $d \in \mathbb{R}$ fixed. Set $\Gamma = \{\exp(n 2\pi H) \mid n \in \mathbb{Z}\}$. Recalling $K = K' \times T$, K' the compact subgroup of K with Lie algebra $k' = [k,k]$ and T the one parametersubgroup of K generated by H, one observes that K/Γ is compact. Now define:

$$V(h) = U(h)\, e^{-idt} \quad \text{for} \quad h = h' \cdot \exp tH \in K \,, \; h' \in K'$$

In fact, V(·) is a unitary representation of K/Γ, i.e. $V(z) = V(1)$ for all $z \in \Gamma$. Let $\widehat{K/\Gamma}$ the set of all equivalence classes of finite dimensional irreducible representations of K/Γ. For $\delta \in \widehat{K/\Gamma}$ define $\chi_\delta(\cdot)$ resp. $d(\delta)$ to be the character resp. dimension of any representation of type δ and set

$$P(\delta) = d(\delta) \int_{K/\Gamma} \chi_\delta(h)^* \, V(h)\, dh$$

dh being the normalized Haar-measure on K/Γ.

Lemma B1: 1) the $P(\delta)$'s are projection operators in $\mathcal{H}$.

2) $\mathcal{H} = \bigoplus_{\delta \in \widehat{K/\Gamma}} \mathcal{H}(\delta)$; $\mathcal{H}(\delta) \doteq P(\delta)\mathcal{H}$ (unitary, direct sum)

3) Every $\mathcal{H}(\delta)$ is an algebraic sum of irreducible representations of K/Γ of type δ.

Proof: see [11], p.261 and [12].

The set of all analytic vectors $\mathcal{H}_\omega$ for U(·) is dense in $\mathcal{H}$. Defining $\mathcal{H}_\omega(\delta) \doteq \mathcal{H}_\omega \cap \mathcal{H}(\delta)$ we have:

Lemma B2: 1) $\mathcal{H}_\omega(\delta) = P(\delta)\mathcal{H}_\omega$

2) the linear space $\mathcal{H}_K = \sum_{\delta \in \hat{K}/\Gamma} \mathcal{H}_\omega(\delta)$ (algebraic sum) lies dense in $\mathcal{H}$.

3) $\mathcal{H}_K$ is stable under the action of $U(X)$, $X \in g_{\mathbf{c}}$.

Proof: 1) If $a \in \mathcal{H}_\omega$ then by definition of $P(\delta)$: $(x \in G)$

$$U(x)\,P(\delta)\,a = d(\delta) \int_{K'} dh' \int_0^{2\pi} \frac{dt}{2\pi}\, \chi_\delta^*(h' \cdot e^{tH})\, e^{-idt}\, U(x \cdot h' \cdot e^{tH})\, a$$

Since h' and e^{tH} vary through a compact region in G, $U(x \cdot h' \cdot e^{tH}) \cdot a$ can be analytically continued in a complex neighborhood of x uniformly in h', t and in such a way that this continuation is jointly continuous in x, h', t. Applying a standard result ([9] 13.8.6) now shows that $P(\delta)a \in \mathcal{H}_\omega$. 2) follows from 1) and Lemma B1 2).
Finally 3) is proven like 4.4.5.18 in Warner's book [11] ∎

Lemma 4.1 is now established by copying the proof of 4.5.2.9 [11] using the quasisimplicity of $U(\cdot)$.

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