DEUTSCHES ELEKTRONEN-SYNCHROTRON DESY

DESY 76/07
February 1976

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Excitation of Betatron-Synchrotron Resonances by a Dispersion in the Cavities

by

A. Piwinski and A. Wrulich

Abstract:

Satellite resonances due to a dispersion in the cavities are investigated. It is shown that both betatron and synchrotron oscillations are excited. For betatron wave numbers above an integer the excitation is stronger than below an integer. Also magnetic imperfections or correcting coils can lead to these resonances but only if they produce a dispersion in the cavities. Rise times for DORIS and PETRA are calculated.

1. Introduction

In this paper betatron-synchrotron resonances are investigated which are produced by the dispersion in the cavities and which do not depend on the chromaticity which is assumed to be compensated. The frequencies at the resonance are given by

$$Q_{x,z} = n \pm m\, Q_s \tag{1.1}$$

where n and m are integers.

Those resonances were observed in NINA[1] and SPEAR[2,3], and theoretical investigations were given in[1,3,4,5,6]. But two points have not been considered which determine the resonance mechanism.

First, only the change of the betatron amplitude was considered whereas the synchrotron amplitude was assumed to remain constant. We will show that both betatron and synchrotron amplitudes are changed at the same time. Two cases can then be distinguished: Betatron and synchrotron amplitudes increase or decrease together, depending on the phase between the two oscillations. This case occurs for betatron numbers above an integer. If the betatron number is below an integer, only one of the two amplitudes can increase whereas the other one decreases. In that case betatron and synchrotron oscillation exchange their oscillation energy periodically.

Since the synchrotron oscillation is considerably nonlinear the synchrotron frequency changes with increasing amplitude. The particles will come out of resonance and the amplitudes will decrease. When the initial amplitudes are reached the process starts again. Those oscillations were studied with a simulation on a computer.

The second point we want to discuss is the question how magnetic imperfections or correcting elements can excite these satellite resonances. We will show that distortions of the closed orbit excite the resonance only if they produce a dispersion in a cavity.

There are several possibilities to suppress these resonances. The normal dispersion (produced by the bending magnets) can be tuned to zero in all cavity sections. Another way is to compensate the effect by a suitable distribution of accelerating sections over the ring and by corresponding Q-values. The cavities in one straight section cannot compensate each other. Their contributions must be added which will be shown below. The dispersion caused by magnetic imperfections can be suppressed in the cavities with help of correcting coils. One needs at least two correcting coils in all section (with curvature) between two cavity sections.

2. Solution for Linear Oscillations

2.1 Matrix Representation for Betatron and Synchrotron Oscillations

In the linear case, i.e. for the first harmonic (m = 1 in Eq.(1.1)) and for small synchrotron amplitudes an exact solution for the coupled oscillations can be derived by using a matrix formalism for the betatron and synchrotron motion. We describe the betatron motion with the two variables

$$y_1 = x$$
$$y_2 = x'\beta - x\beta'/2$$

with

$$\beta = \text{amplitude function}$$
$$(\ )' = \frac{d}{ds} \quad (s = \text{longitudinal coordinate}) \ .$$

To simplify the calculation we assume that the amplitude function β, the dispersion D and the derivatives β' and D' have the same distribution in all accelerating sections. This assumption is not necessary for an exact solution, but it reduces the rate of calculation considerably.
The transformation from the end of one cavity section to the end of the next one is then given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{j+1} = \begin{pmatrix} \cos\mu_{\beta_j} & \sin\mu_{\beta_j} \\ -\sin\mu_{\beta_j} & \cos\mu_{\beta_j} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_j \qquad (2.1)$$

with μ_{β_j} = betatron phase advance .

For the description of the synchrotron oscillation it is necessary to take into account the discontinuous distribution of the accelerating field. Therefore we subdivide the ring into sections with curvature and into sections with acceleration.

In a curved section the energy deviation ΔE with respect to an equilibrium particle remains constant, whereas the synchrotron phase φ is changed by

$$\delta\varphi = -2\pi k \alpha_j \frac{\Delta E}{E} \tag{2.2}$$

with k = harmonic number

α_j = momentum compaction factor of j-th curved section $(\sum_j \alpha_j = \alpha)$

In a cavity section the synchrotron phase remains constant whereas the energy is changed by

$$\delta E = e U_j \cos\psi \, \varphi \tag{2.3}$$

with e = elementary charge

U_j = voltage of the accelerating section

ψ = synchronous phase

The transfer matrix for a curved section and the following accelerating section is

$$\begin{pmatrix} \Delta E \\ \varphi \end{pmatrix}_{j+1} = \begin{pmatrix} 1 & e U_j \cos\psi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2\pi k \alpha_j / E & 1 \end{pmatrix} \begin{pmatrix} \Delta E \\ \varphi \end{pmatrix}_j$$

$$= \begin{pmatrix} 1 - 2\pi k \alpha_j e U_j \cos\psi / E & e U_j \cos\psi \\ -2\pi k \alpha_j / E & 1 \end{pmatrix} \begin{pmatrix} \Delta E \\ \varphi \end{pmatrix}_j \tag{2.4}$$

with the abbreviations

$$y_3 = \frac{\Delta E}{E}$$

$$y_4 = \frac{\varphi}{2\pi k \alpha}$$

we get for the transfer matrix of the synchrotron oscillation the expression

$$\begin{pmatrix} y_3 \\ y_4 \end{pmatrix}_{j+1} = \begin{pmatrix} 2\cos\mu_{s_j} - 1 & 2(1-\cos\mu_{s_j})\alpha/\alpha_j \\ -\alpha_j/\alpha & 1 \end{pmatrix} \begin{pmatrix} y_3 \\ y_4 \end{pmatrix}_j \qquad (2.5)$$

with

$$\cos\mu_{s_j} = 1 - \pi k \alpha_j e U_j \cos\psi/E \qquad (2.6)$$

For small synchrotron frequencies we have

$$\mu_{s_j} \approx \sqrt{2\pi k \alpha_j e U_j \cos\psi/E}$$

2.2 Coupling of Betatron and Synchrotron Oscillations

If the particle energy E is changed by δE on the small path length δs, the betatron coordinates are changed by

$$\begin{aligned} \delta y_1 = \delta x &= -D\,\delta E/E \\ &= -D\dot{E}\,\delta s/E/v \end{aligned} \qquad (2.7)$$

and

$$\begin{aligned} \delta y_2 = \delta x'\beta - \delta x\beta'/2 &= -F\,\delta E/E \\ &= -F\dot{E}\,\delta s/E/v \end{aligned} \qquad (2.8)$$

with

$$F = D'\beta - D\beta'/2 \qquad (2.9)$$

v = particle velocity, $(\;)^{\bullet} = \frac{d}{dt}$.

The total change of the betatron coordinates in a cavity section is obtained by transforming with the matrix

$$M(s_o,s) = \sqrt{\frac{\beta(s_o)}{\beta(s)}} \begin{pmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{pmatrix} \qquad (2.10)$$

with

$$\Delta\phi = \phi(s_o) - \phi(s)$$

and by integration along the path within the cavity section. s_o denotes an arbitrary reference point within the section.

The integrals are

$$\delta y_1 = -\frac{1}{vE}\int\sqrt{\frac{\beta(s_o)}{\beta(s)}}\,(D\cos\Delta\phi + F\sin\Delta\phi)\,\dot{E}\,ds \qquad (2.11)$$

$$\delta y_2 = -\frac{1}{vE}\int\sqrt{\frac{\beta(s_o)}{\beta(s)}}\,(-D\sin\Delta\phi + F\cos\Delta\phi)\,\dot{E}\,ds \qquad (2.12)$$

To evaluate the integrals we use an integral representation for the dispersion D(s)[7)]

$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\int_s^{s+C}\frac{1}{\rho}\sqrt{\beta(s)}\,\cos(\phi(\sigma)-\phi(s)-\pi Q)\,d\sigma \qquad (2.13)$$

with ρ = radius of curvature

With Eqs.(2.9) and (2.13) one obtains

$$F(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\int_s^{s+C}\frac{1}{\rho}\sqrt{\beta(s)}\,\sin(\phi(\sigma)-\phi(s)-\pi Q)\,d\sigma \qquad (2.14)$$

Inserting Eqs. (2.13) and (2.14) in Eqs.(2.11) and (2.12) yields

$$\delta y_1 = -\frac{D(s_0)}{v E}\int \dot{E}\, ds = -D(s_0)\frac{\delta E}{E} \qquad (2.15)$$

and

$$\delta y_2 = -\frac{F(s_0)}{v E}\int \dot{E}\, ds = -F(s_0)\frac{\delta E}{E} \qquad (2.16)$$

Here we have employed the fact that $1/\rho$ is zero in a straight section.

Eqs.(2.15) and (2.16) show that the change of the betatron coordinates in one straight section does not depend on the distribution of the accelerating cavities. The change can be calculated as if it were produced in one cavity, where the position of the cavity is arbitrary within the accelerating section. This can also be seen if one takes into consideration that in a straight section, the dispersion is a particle trajectory performing a betatron oscillation. The transfer matrix of the coupled oscillations for a accelerating section can be written in the form

$$M_{acc,j} = \begin{pmatrix} 1 & 0 & 0 & -2 D_0 S_j \alpha/\alpha_j \\ 0 & 1 & 0 & -2 F_0 S_j \alpha/\alpha_j \\ 0 & 0 & 1 & 2 S_j \alpha/\alpha_j \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.17)$$

$$S_j = 1 - \cos \mu_{s_j}$$

The path length between two cavity sections depends not only on the energy but also on the betatron oscillation of a particle. The contribution due to the betatron oscillation is given by

$$\delta L = \int_{s_j}^{s_{j+1}} \frac{X(s)}{\rho(s)}\, ds \qquad (2.18)$$

The integral can be expressed by the dispersion and its derivative at the beginning or at the end of the section with curvature. This can be performed with help of Eqs.(2.13) and (2.14) and yields:

$$\delta L = \frac{1}{\beta_c}\left[(D_0 \sin\mu_{\beta_j} - F_0 B_j) y_1 + (D_0 B_j + F_0 \sin\mu_{\beta_j}) y_2 \right] \qquad (2.19)$$

with

$$B_j = 1 - \cos\mu_{\beta_j} \tag{2.20}$$

and

$$\beta_o = \beta(s_o)\,, \qquad D_o = D(s_o)\,, \qquad F_o = F(s_o)$$

y_1 and y_2 are the betatron coordinates at the beginning of the curved section. Similar as for the exitation of the betatron oscillation in a cavity one can show that the lengthening δL does not depend on the position of the reference point s_o , i.e. on the beginning and on the end of the integral in Eq.(2.18), as long as these points are in the neighbouring straight sections. With

$$\delta y_{j4} = \frac{\delta\varphi}{2\pi k \alpha} = -\frac{\delta L}{\alpha C}$$

and C = circumference

the transfer matrix for a curved section can be written as

$$M_{curv,j} = \begin{pmatrix} \cos\mu_{\beta_j} & \sin\mu_{\beta_j} & 0 & 0 \\ -\sin\mu_{\beta_j} & \cos\mu_{\beta_j} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -g(D_o\sin\mu_{\beta_j} - F_o B_j) & -g(D_o B_j + F_o\sin\mu_{\beta_j}) & -\alpha_j/\alpha & 1 \end{pmatrix} \tag{2.21}$$

with

$$g = \frac{1}{\alpha C \beta_o} \tag{2.22}$$

The transfer matrix for an accelerating section and the following curved section is given by

$$M_j = M_{acc,j} \cdot M_{curv,j}$$

$$= \begin{pmatrix} \cos\mu_{\beta_j} + D_o b_{1j} & \sin\mu_{\beta_j} + D_o b_{2j} & 2D_o S_j & -2D_o S_j\,\alpha/\alpha_j \\ -\sin\mu_{\beta_j} + F_o b_{1j} & \cos\mu_{\beta_j} + F_o b_{2j} & 2F_o S_j & -2F_o S_j\,\alpha/\alpha_j \\ -b_{1j} & -b_{2j} & 1-2S_j & 2S_j\,\alpha/\alpha_j \\ -b_{1j}\alpha_j/\alpha/S_j/2 & -b_{2j}\alpha_j/\alpha/S_j/2 & -\alpha_j/\alpha & 1 \end{pmatrix} \tag{2.23}$$

with

$$b_{1j} = \frac{S_j}{2\alpha_j C\beta_o} (D_o \sin\mu_{\beta j} - F_o B_j)$$

$$b_{2j} = \frac{S_j}{2\alpha_j C\beta_o} (D_o B_j + F_o \sin\mu_{\beta j})$$

2.3 Eigenvalues and Rise Times

In order to investigate the stability of the coupled oscillations we have to calculate the eigenvalues of the revolution matrix. The revolution matrix is the product of, in general, several different matrices M_j (Eq.(2.23)). We will confine ourself to the case of N equal matrices M_j. In this case is

$$\mu_{\beta j} = \mu_\beta / N , \qquad \mu_{sj} = \mu_s / N$$

The eigenvalues of M_j are defined by

$$\left(\lambda + \frac{1}{\lambda} - 2\cos\mu_{\beta j}\right)\left(\lambda + \frac{1}{\lambda} - 2\cos\mu_{sj}\right) = 2\lambda \left(1 - \frac{1}{\lambda}\right)^2 \sin\mu_{\beta j} S_j g (D_o^2 + F_o^2) \frac{\alpha}{\alpha_j} \qquad (2.24)$$

With
$$\lambda = e^{\pm i\mu_{1,2}}$$

Eq.(2.24) can be written as

$$(\cos\mu_{1,2} - \cos\mu_{\beta j})(\cos\mu_{1,2} - \cos\mu_{sj}) = (\cos\mu_{1,2} - 1)\sin\mu_{\beta j} S_j g (D_o^2 + F_o^2)\frac{\alpha}{\alpha_j} \qquad (2.25)$$

Eq.(2.25) determines an exact solution for the frequencies of the coupled betatron and synchrotron oscillations. A resonance occurs for

$$\cos\mu_{\beta j} \approx \cos\mu_{sj} \qquad (2.26)$$

or
$$\mu_{\beta j} \approx 2\pi n \pm \mu_{sj}$$

To determine the solution in the vicinity of the resonance it will be convenient to define the deviation of μ_{β_j} from a multiple of 2π :

$$\mu_{\beta_j} = 2\pi n + \delta\mu_{\beta_j} \tag{2.27}$$

with

$$-\pi < \delta\mu_{\beta_j} \leq \pi$$

Further we define the change of μ_{β_j} and μ_{s_j} due to the coupling by

$$\delta\mu_1 = \mu_1 - \mu_{\beta_j} \tag{2.28}$$

$$\delta\mu_2 = \mu_2 - \mu_{s_j} \tag{2.29}$$

with the assumption that the coupling is small, i.e.

$$\frac{1}{2}\left|\delta\mu_{1,2} \cot\mu_{\beta,s_j}\right| \ll 1$$

one obtains for $\delta\mu_{1,2}$

$$\delta\mu_{1,2} = \pm\sqrt{\frac{1}{4}\left(|\delta\mu_{\beta_j}| - \mu_{s_j}\right)^2 - \frac{(1-\cos\mu_{\beta_j})^2 g\alpha(D_o^2+F_o^2)}{\alpha_j \sin\mu_{\beta_j}}} \tag{2.30}$$

Eq.(2.30) shows that the phase advance per section can become complex, if $\sin\mu_{\beta_j}$ is positive, that means for Q_β/N values above an integer. In that case the oscillations are antidamped and unstable. The shortest rise time per revolution on the resonance is given by

$$\begin{aligned}\frac{1}{T_r} &= N f_o \left|\delta\mu_{1,2}\right| \\ &= N f_o (1-\cos\mu_{\beta_j})\sqrt{\frac{N g (D_o^2+F_o^2)}{\sin\mu_{\beta_j}}}\end{aligned} \tag{2.31}$$

with f_o = revolution frequency, N = number of equal sections. The width of the resonance is given by

$$\Delta f_{\beta,s} = \pm\frac{2}{T_r} \tag{2.32}$$

For small synchrotron frequencies the trigonometric functions may be expanded and one obtains for the rise time

$$\frac{1}{T_r} = \frac{1}{2} f_o (2\pi Q_s)^{3/2}\sqrt{\frac{D_o^2+F_o^2}{\alpha C\beta_o}} \tag{2.33}$$

Eq.(2.33) shows the dependence of the strength of the resonance on the synchrotron frequency, i.e. on the accelerating voltage and on the harmonic number.

3. Nonlinear Synchrotron Oscillations

For large amplitudes the synchrotron oscillation becomes nonlinear. The nonlinearity results from the sinusoidal variation of the accelerating voltage and from wake fields induced by the beam. With increasing amplitude the synchrotron frequency will change and the particle comes out of resonances. Thus the nonlinearity provides a limitation for the increase of the amplitude of betatron and synchrotron oscillations. On the other hand the nonlinearity produces other resonances at the harmonics of the synchrotron frequency.

A complete analytical solution which takes into account the coupling between betatron and synchrotron oscillations and the nonlinearity of the synchrotron oscillation is very complicated. Numerical solutions are obtained by a simulation with a digital computer (Sec.5). In this section we will derive an analytical expression for the rise times on a resonance.

For the nonlinear case we assume that the synchrotron wave number Q_s is small as compared to 1 and that the change of the amplitudes due to the coupling is small for one revolution. We further assume that the ring consists of N equal sections with acceleration and curvature.

We write the betatron oscillation at successive passages of a cavity in the form

$$X = \sqrt{\varepsilon\beta_c}\ \sin(2\pi p Q_\beta + \gamma_\beta) \tag{3.1}$$

$$X'\beta_c - X\beta_c'/2 = \sqrt{\varepsilon\beta_c}\ \cos(2\pi p Q_\beta + \gamma_\beta) \tag{3.2}$$

with ε = emittance, γ_β = constant phase angle
p = number of revolutions

The emittance can be written as

$$\varepsilon\beta_c = X^2 + (X'\beta_c - X\beta_c'/2)^2 \tag{3.3}$$

The nonlinear synchrotron oscillation can be written in the form

$$\frac{\Delta E}{E} = \zeta = \sum_{m=1}^{\infty} \zeta_m \sin(m(2\pi p Q_s + \gamma_s)) \qquad (3.4)$$

$$\varphi = \sum_{m=0}^{\infty} \varphi_m \cos(m(2\pi p Q_s + \gamma_s)) \qquad (3.5)$$

where γ_s is a constant phase angle. The Hamiltonian of the synchrotron oscillation is

$$H = \zeta^2 + 2\left(\frac{Q_{s\ell}}{k\alpha}\right)^2 \int_0^{\varphi} f(\varphi)\, d\varphi \qquad (3.6)$$

$f_o Q_{s\ell}$ is the synchrotron frequency in linear approximation which is not the same as the frequency $f_o\, Q_s$ for the nonlinear oscillation. $\int f(\phi)d\phi$ is the potential of the nonlinear oscillation with

$$f(\varphi) = \varphi + f_2\varphi^2 + f_3\varphi^3 + \dots$$

The change of the betatron amplitude per revolution is then given by

$$\delta\sqrt{\varepsilon\beta_o} = -\frac{2\pi}{N} Q_s \sum_{j=1}^{N} \left[D_o \sin(2\pi Q_\beta(p+\tfrac{j}{N})+\gamma_\beta) + F_o \cos(2\pi Q_\beta(p+\tfrac{j}{N})+\gamma_\beta)\right] \cdot$$

$$\cdot \sum_{m=1}^{\infty} m\, \zeta_m \cos(2\pi m Q_s(p+\tfrac{j}{N}) + m\gamma_s) \qquad (3.7)$$

We take the average of Eq.(3.7) over many revolutions in the case of a resonance, i.e. if Eq.(1.1) is satisfied. When n/N is an integer we get

$$\delta\sqrt{\varepsilon\beta_o} = -m\pi Q_s \left[D_o \sin(\gamma_\beta \mp m\gamma_s) + F_o \cos(\gamma_\beta \mp m\gamma_s)\right] \zeta_m \qquad (3.8)$$

When n/N is not an integer the average change is zero and no resonance can occur. This is, of course, valid only if the optical parameters in the accelerating sections are equal and if the distances between the sections are equal.

The exitation of the synchrotron oscillation is produced by a path lengthening due to the betatron oscillation which follows from Eq.(2.19) to be

$$\delta L = \frac{1}{\beta_o}\left[D_o x + F_o (x'\beta_o - x\beta_o'/2)\right] \frac{\delta\mu_o}{N} \qquad (3.9)$$

since we have assumed that Q_s, and in the resonance case also $\delta\mu_\beta$ is small. The change of the Hamiltonian is given by

$$\delta H = 2\left(\frac{Q_{se}}{k\alpha}\right)^2 f(\varphi)\,\delta\varphi$$

$$= -\frac{2}{(2\pi f_0 k\alpha)^2}\,\ddot{\varphi}\,\delta\varphi \qquad (3.10)$$

if one introduces the usual differential equation for the nonlinear synchrotron oscillation

$$\ddot{\varphi} + 4\pi^2 f_0^2 Q_{se}^2\, f(\varphi) = 0. \qquad (3.11)$$

With

$$\delta\varphi = -\frac{2\pi k}{C}\,\delta L$$

and after averaging over many revolutions one obtains for the change per revolution

$$\delta H = -\frac{2\pi m^2 Q_s^2\,\delta\mu_\beta}{k\alpha^2 C\beta_0}\left[D_0 \sin(\chi_\beta \mp m\chi_s) + F_c\cos(\chi_\beta \mp m\chi_s)\right]\varphi_m\sqrt{\varepsilon\beta_0} \qquad (3.12)$$

With

$$m\,Q_s\,\varphi_m = k\alpha\,\zeta_m$$

Eq.(3.12) can also be written in the form

$$\delta H = -\frac{2\pi m Q_s\,\delta\mu_\beta}{\alpha C\beta_0}\left[D_0\sin(\chi_\beta \mp m\chi_s) + F_c\cos(\chi_\beta \mp m\chi_s)\right]\zeta_m\sqrt{\varepsilon\beta_0} \qquad (3.13)$$

From Eqs.(3.8) and (3.13) follows

$$\delta H - 2\,\frac{\delta\mu_\beta}{\alpha C\beta_0}\sqrt{\varepsilon\beta_0}\;\delta\sqrt{\varepsilon\beta_0} = 0$$

or

$$H - \frac{\delta\mu_\beta}{\alpha C\beta_0}\,\varepsilon\beta_0 = const. \qquad (3.14)$$

For small nonlinearities, if the relation

$$v_m = \left|\frac{\zeta_m}{\zeta_1}\right| \ll 1 \quad (m = 2, 3, \ldots) \tag{3.15}$$

holds, Eq.(3.14) may also be written as

$$\left\langle \frac{\Delta^2 E}{E^2} \right\rangle - \frac{\delta\mu_0}{\alpha C \beta_0} \langle X^2 \rangle = const. \tag{3.16}$$

Eqs.(3.14) and (3.16) show the influence of the sign of $\delta\mu_\beta$ on the resonance. For Q_β below an integer ($\delta\mu_\beta < 0$) the amplitudes of the oscillations are limited. Betatron and synchrotron oscillation can only exchange their energies. For Q_β above an integer ($\delta\mu_\beta > 0$) both amplitudes can increase so far as they do not exceed other limitations. Therefore satellite resonances for Q_β above an integer should be stronger than for Q_β below an integer.

The maximum increase of the amplitudes per unit time follows from Eqs.(3.8) and (3.13) as

$$\dot{\sqrt{\varepsilon\beta_c}} = A\,\zeta_1\,v_m \tag{3.17}$$

$$\dot{\zeta}_1 = A\,\frac{|\delta\mu_1|}{\alpha C \beta_c}\sqrt{\varepsilon\beta_c}\;v_m \tag{3.18}$$

with

$$A = m\pi f_c\,Q_s\sqrt{D_c^2 + F_c^2}$$

4. Satellite Resonances and Magnetic Imperfections

In this section we want to show that the excitation of satellite resonances in a machine with dipole errors can occur only in an accelerating cavity. The resonance will be excited if the dipole errors produce a dispersion in the cavity. In order to compensate the effect one has to compensate the dispersion in the cavity with correcting elements.

The emittance ε is a constant of motion in a linear uncoupled machine with a time-independent magnetic field. ε can only be changed by a change of energy due to radiation losses or due to accelerating fields. The energy loss due to quantum radiation cannot lead to a resonance since the loss does not vary with the synchrotron frequency (except due to damping which is very small). The only energy change with the synchrotron frequency occurs in a cavity. Thus dipole errors can excite a satellite resonance only if they produce a dispersion in a cavity.

But there is the argument that the calculation of the satellite resonance with help of the dispersion in a cavity produced by the dipole error gives the same result as the calculation with the dipole error as a driving force for the betatron oscillation. This was shown in [6] for a single kick and by neglecting the local chromaticity, i.e. the curvature of the distorted orbit in the quadrupoles. The local chromaticity, however, essentially determines the dispersion due to the distortion and its variation along the circumference. The excitation of the resonance depends now on the position of the cavity and the two methods of calculation give different results. But even neglecting the local chromaticity, the two methods give different results if one considers more than one kick. This can easily be seen in the case of two kicks. The dispersion which is produced by two kicks with the deflections Θ_1 and Θ_2 is obtained from Eq.(2.13) with

$$\frac{1}{\rho} = \Theta_1 \delta(s - s_1) + \Theta_2 \delta(s - s_2) \qquad (4.1)$$

where $s_{1,2}$ denotes the position of the dipole error. If the cavity is in the section $\{s_2 , s_1\}$ one obtains

$$D_0 = \frac{\sqrt{\beta_0}}{2 \sin \pi Q_\beta} \left[\Theta_1 \sqrt{\beta_1} \cos(\phi_1 - \phi_0 - \pi Q_\beta) + \Theta_2 \sqrt{\beta_2} \cos(\phi_2 - \phi_0 - \pi Q_\beta) \right] \qquad (4.2)$$

where the index 0 denotes the cavity. Similar one gets from Eq.(2.14)

$$F_0 = \frac{\sqrt{\beta_0}}{2 \sin \pi Q_\beta} \left[\Theta_1 \sqrt{\beta_1} \sin(\phi_1 - \phi_0 - \pi Q_\beta) + \Theta_2 \sqrt{\beta_2} \sin(\phi_2 - \phi_0 - \pi Q_\beta) \right] \qquad (4.3)$$

With

$$X_0 = \sqrt{\varepsilon \beta_0} \sin(\phi_0 + \chi_\beta) \qquad (4.4)$$

the change of the emittance follows from Eqs.(2.15) and (2.16) to be

$$\delta\varepsilon = - \frac{\sqrt{\varepsilon}}{\sin \pi Q_\beta} \left[\Theta_1 \sqrt{\beta_1} \sin(\phi_1 - \pi Q_\beta + \chi_\beta) + \Theta_2 \sqrt{\beta_2} \sin(\phi_2 - \pi Q_\beta + \chi_\beta) \right] \frac{\delta E}{E} \qquad (4.5)$$

The change of the emittance does not depend on the position of the cavity so long as it is in the section $\{s_2, s_1\}$. But if the cavity is in the section $\{s_1, s_2\}$ one obtains from Eqs.(2.13) and (2.14)

$$D_c = \frac{\sqrt{\beta_c}}{2\sin\pi Q_\rho}\left[\Theta_1\sqrt{\beta_1}\cos(\phi_1-\phi_c+\pi Q_\rho)+\Theta_2\sqrt{\beta_2}\cos(\phi_2-\phi_c-\pi Q_\rho)\right] \tag{4.6}$$

$$F_c = \frac{\sqrt{\beta_c}}{2\sin\pi Q_\rho}\left[\Theta_1\sqrt{\beta_1}\sin(\phi_1-\phi_c+\pi Q_\rho)+\Theta_2\sqrt{\beta_2}\sin(\phi_2-\phi_c-\pi Q_\rho)\right] \tag{4.7}$$

The change of the emittence is now given by

$$\delta\varepsilon = \frac{-\sqrt{\varepsilon}}{\sin\pi Q_\rho}\left[\Theta_1\sqrt{\beta_1}\sin(\phi_1+\pi Q_\rho+\chi_\rho)+\Theta_2\sqrt{\beta_2}\sin(\phi_2-\pi Q_\rho+\chi_\rho)\right]\frac{\delta E}{E} \tag{4.8}$$

In general Eq.(4.5) is different from Eq.(4.8). If one chooses

$$\Theta_1\sqrt{\beta_1} = \Theta_2\sqrt{\beta_2}$$

and

$$\phi_2 = \phi_1 + \pi$$

the dispersion and $\delta\varepsilon$ vanish in $\{s_2, s_1\}$, but not in $\{s_1, s_2\}$. The excitation of the resonance depends on the position of the cavity. Therefore the calculation of the resonance with the dipole error as the driving force gives a different result, since it does not take into account the position of the cavity.

5. Computer Simulation and Calculation of Rise Times

A simulation of the betatron and synchrotron oscillations on a digital computer has been done. The coupling of the oscillations due to a dipersion in a cavity and the nonlinearity of the synchrotron oscillation have been taken into accout. The following equations are used in the computer program.

In a cavity the coordinates of the oscillations are changed by

$$E_{n+1} = E_n + eU[\sin(\psi+\phi_n) - \sin\psi] \tag{5.1}$$

$$\phi_{n+1} = \phi_n \qquad (5.2)$$

$$x_{n+1} = x_n - D_o\,(E_{n+1} - E_n)/E_e \qquad (5.3)$$

$$x'_{n+1} = x'_n \qquad (5.4)$$

with $\quad E_e$ = equilibrium energy

D' and β' are assumed to be zero in the cavity to simplify the calculation. Between two cavity sections the change is given by

$$E_{n+2} = E_{n+1} \qquad (5.5)$$

$$\phi_{n+2} = \phi_{n+1} - 2\pi k\left[\alpha\,\frac{E_{n+1}-E_e}{E_e} + \frac{D_o}{C}\left(\sin\mu_\beta\,\frac{x_{n+1}}{\beta_o} + (1-\cos\mu_\beta)\,x'_{n+1}\right)\right] \qquad (5.6)$$

$$x_{n+2} = \cos\mu_\beta\,x_{n+1} + \sin\mu_\beta\,\beta_o\,x'_{n+1} \qquad (5.7)$$

$$x'_{n+2} = -\sin\mu_\beta\,x_{n+1}/\beta_o + \cos\mu_\beta\,x'_{n+1} \qquad (5.8)$$

The following parameters from DORIS were used and kept constant in all figures:

$D_o = 1.8$ m, $\beta_o = 15$ m, $C = 288$ m, $\alpha = 0.022$, $k = 480$

Eqs.(5.1) to (5.8) were employed for 1200 revolutions. To guarantee the resonance condition, the frequency of the nonlinear synchrotron oscillation was calculated without coupling for a sufficiently large number of revolutions. The initial phase relations were varied to look for the maximum amplitude. Figs.1 to 4 show the typical behaviour of the amplitudes during the 1200 revolutions. In Figs.1 and 2 the betatron and synchrotron amplitudes are plotted for a Q_β-value below an integer. Here, the betatron amplitude increases while the synchrotron amplitude decreases. Figs.3 and 4 show the amplitudes for a Q_β-value above an integer. In this case we have a growth of both oscillations which is then limited by the nonlinearity of the synchrotron oscillation.

The maximum betatron amplitudes that appear during the 1200 revolutions were calculated as a function of the betatron wave number whereby the synchrotron frequency was kept constant. These functions are plotted in Figs.5 to 8 for two different Q_s-values and for two different energy deviations. It can be seen from the figures that Q_β-values above an integer lead to larger amplitudes than Q_β-values below an integer.

The dependence of the maximum amplitude on the initial amplitude is plotted in Figs.9 and 10 for resonances on three different harmonics of the synchrotron frequency.

For PETRA the rise time of the betatron oscillation was calculated for two different cases with dispersion at the cavities. In the first case the excitation of the resonance is not compensated by the cavity distribution which occurs, for example, for two cavity sections in opposite positions of the ring and Q_x-values above an even integer. In the second case the excitation is compensated by two cavity sections in opposite positions and Q_x above an odd integer.

The following PETRA-parameters were used:

E = 23 GeV, C = 2304 m, α = 0.00365, k = 3840, β_o = 20 m, σ_x = 2.7 mm
$\sigma_E = 1.4 \cdot 10^{-3}$, Q_s = 0.125, ψ = 38°

The dispersion was pessimistically assumed to be D_o = 2m. The rise time T_β is defined by

$$\dot{\sigma}_x = \frac{\sigma_x}{T_\beta}$$

The calculation was done for a particle with 1 and for a particle with 6 standard deviations, respectively, in energy and betatron amplitude distribution.

In the first case we get:

order of resonances	$T_\beta(1\sigma_x, 1\sigma_E)$	$T_\beta(6\sigma_x, 6\sigma_E)$
1	19 μsec	24 μsec
2	360 μsec	48 μsec
3	34 msec	312 μsec
4	-	1.02 msec
5	-	1.68 msec

The results from Eq.(3.17) and from the computer simulation are in a good agreement.

The computer simulation yields the following amplitudes:

order of resonance	$X_\beta(1\sigma_x,\ 1\sigma_E)$	$X_\beta(6\sigma_x,\ 6\sigma_E)$
1	unstable	unstable
2	15 mm	unstable
3	4.2 mm	46 mm
4	4.0 mm	35 mm
5	4.3 mm	32 mm

For the second case, for Q_β-values above an odd integer, no increase of the amplitudes was found. However, full compensation is only obtained for a perfect symmetry in betatron phase advance and rf voltage. We estimated that, for example, an asymmetry of 10% in the rf voltage leads to rise times twenty times larger than those given in the above table. To avoid these difficulties the PETRA optics was changed to zero dispersion in the cavities.

Due to the small vertical beam emittance a small vertical dispersion caused by distortions can also be dangerous. This dispersion can be compensated at the cavities by means of vertical correcting coils.

Further calculations for several operating conditions of PETRA are prepared.

ACKNOWLEDGEMENT

The authors would like to thank Prof. G.-A. Voss for many useful discussions.

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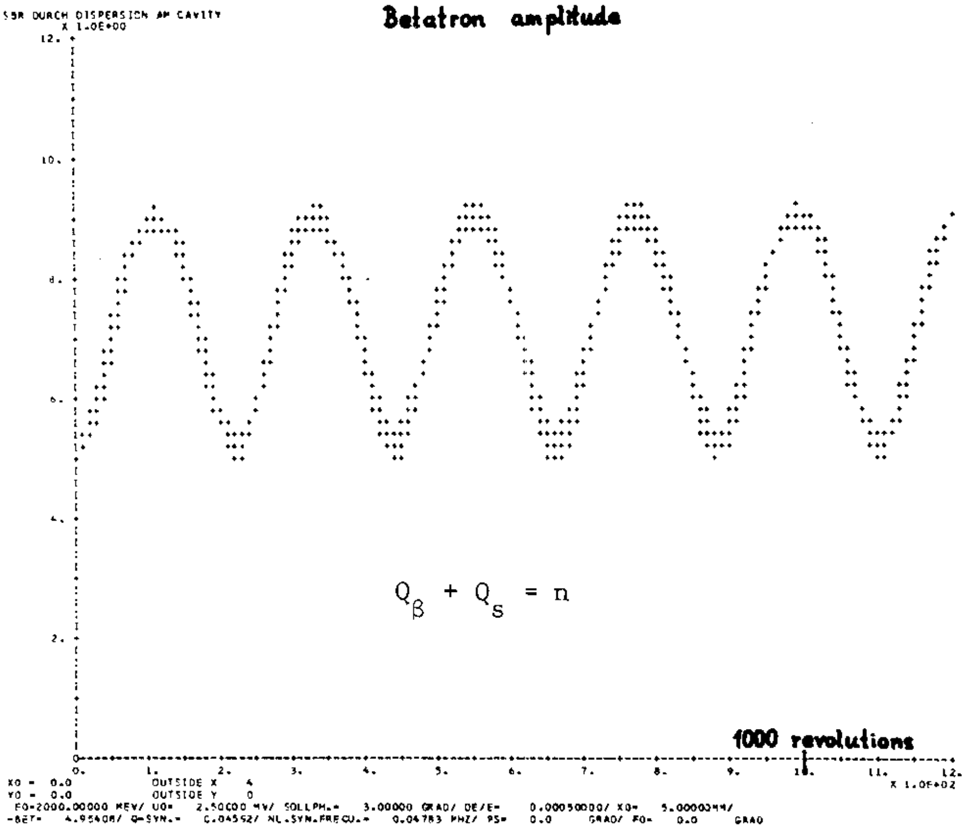
Betatron amplitude
SBR DURCH DISPERSION AM CAVITY
$Q_\beta + Q_s = n$
1000 revolutions

Fig 1

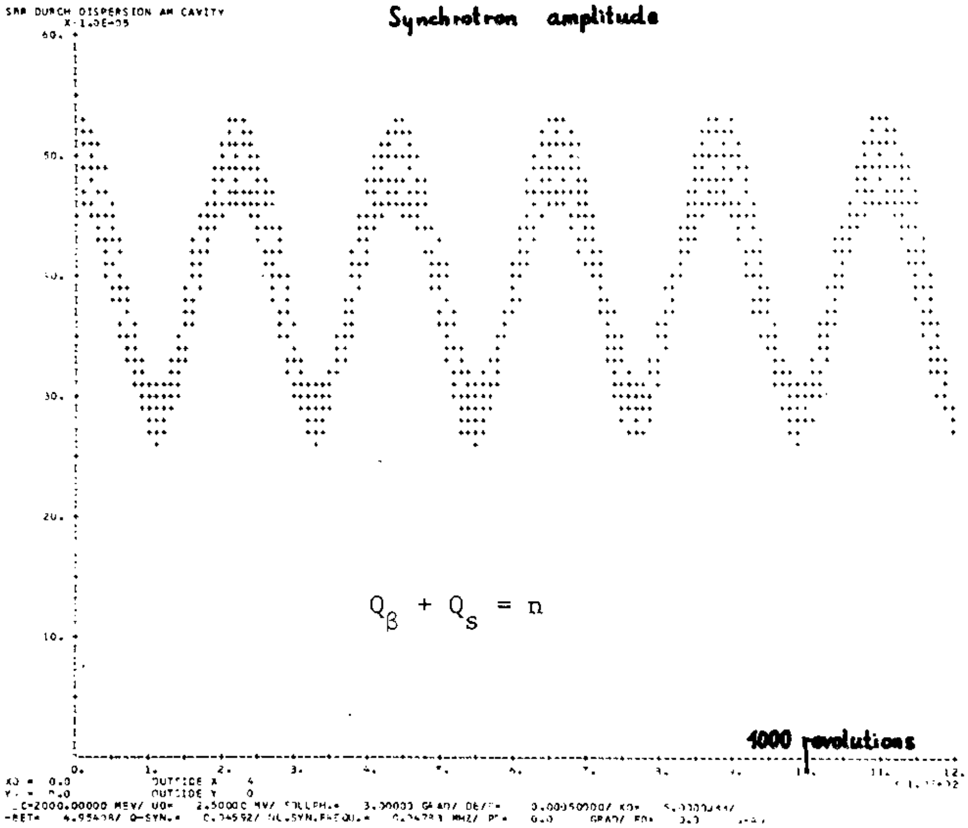
Synchrotron amplitude
SBR DURCH DISPERSION AM CAVITY
$Q_\beta + Q_s = n$
1000 revolutions

Fig 2

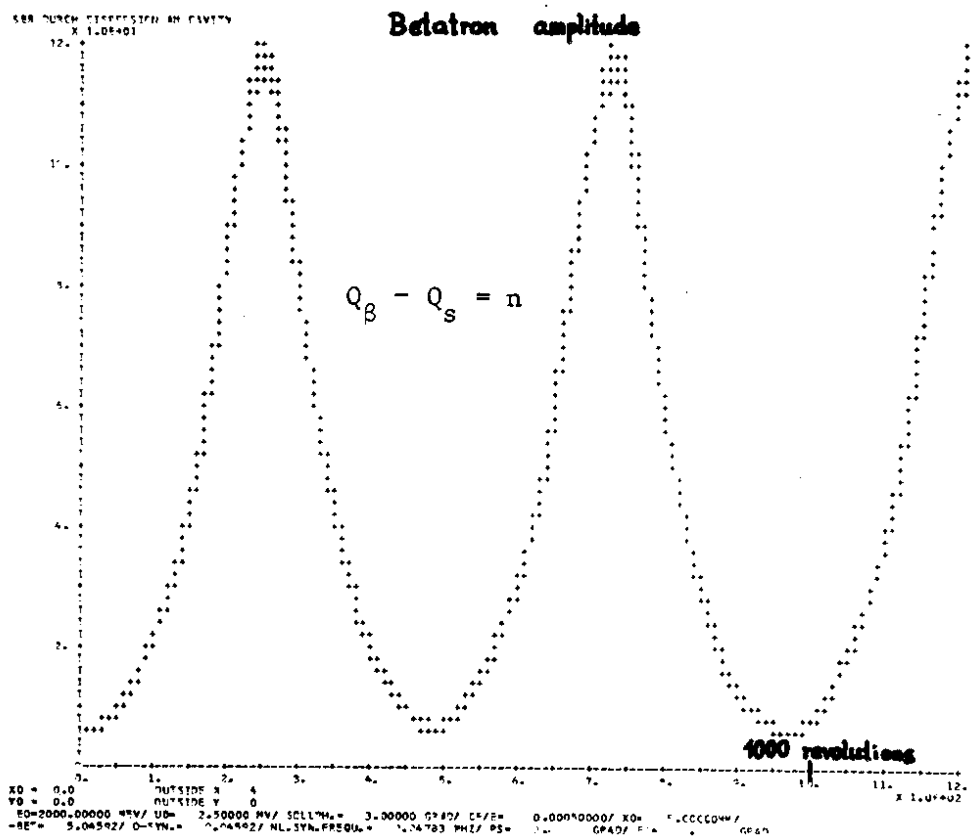
Betatron amplitude
$Q_\beta - Q_s = n$
4000 revolutions

Fig 3

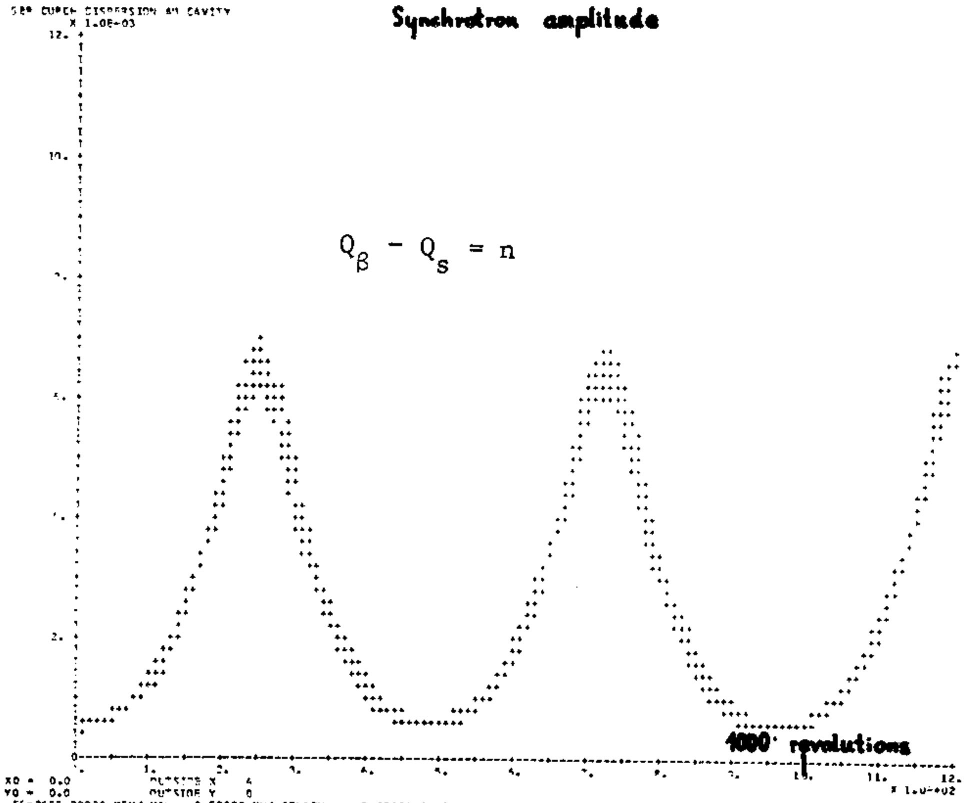
Synchrotron amplitude
$Q_\beta - Q_s = n$
4000 revolutions

Fig 4

$Q_\beta < n$

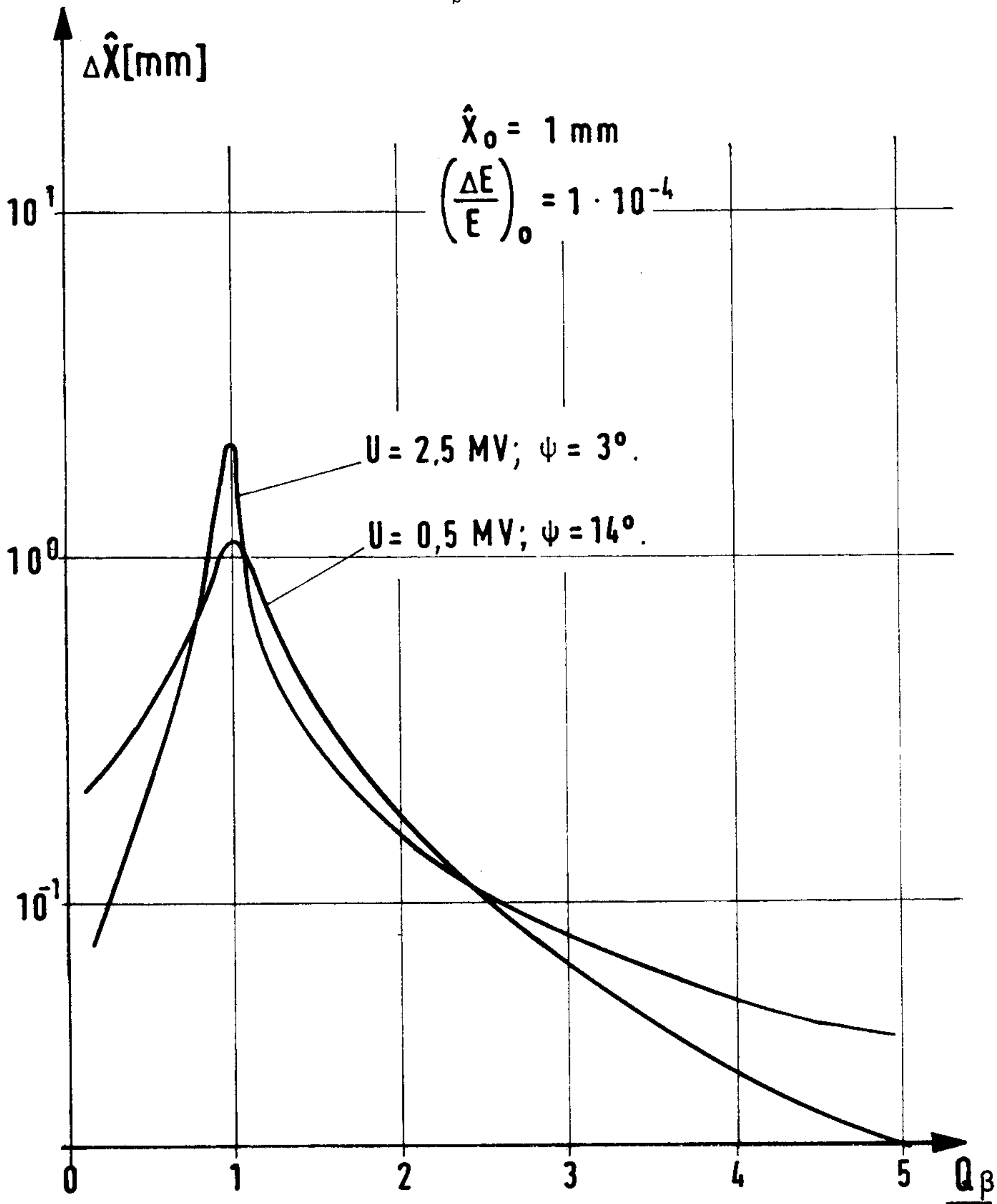

Fig 5: VARIATION of the MAXIMUM BETATRON AMPLITUDE with BETATRON FREQUENCY.

$Q_\beta < n$

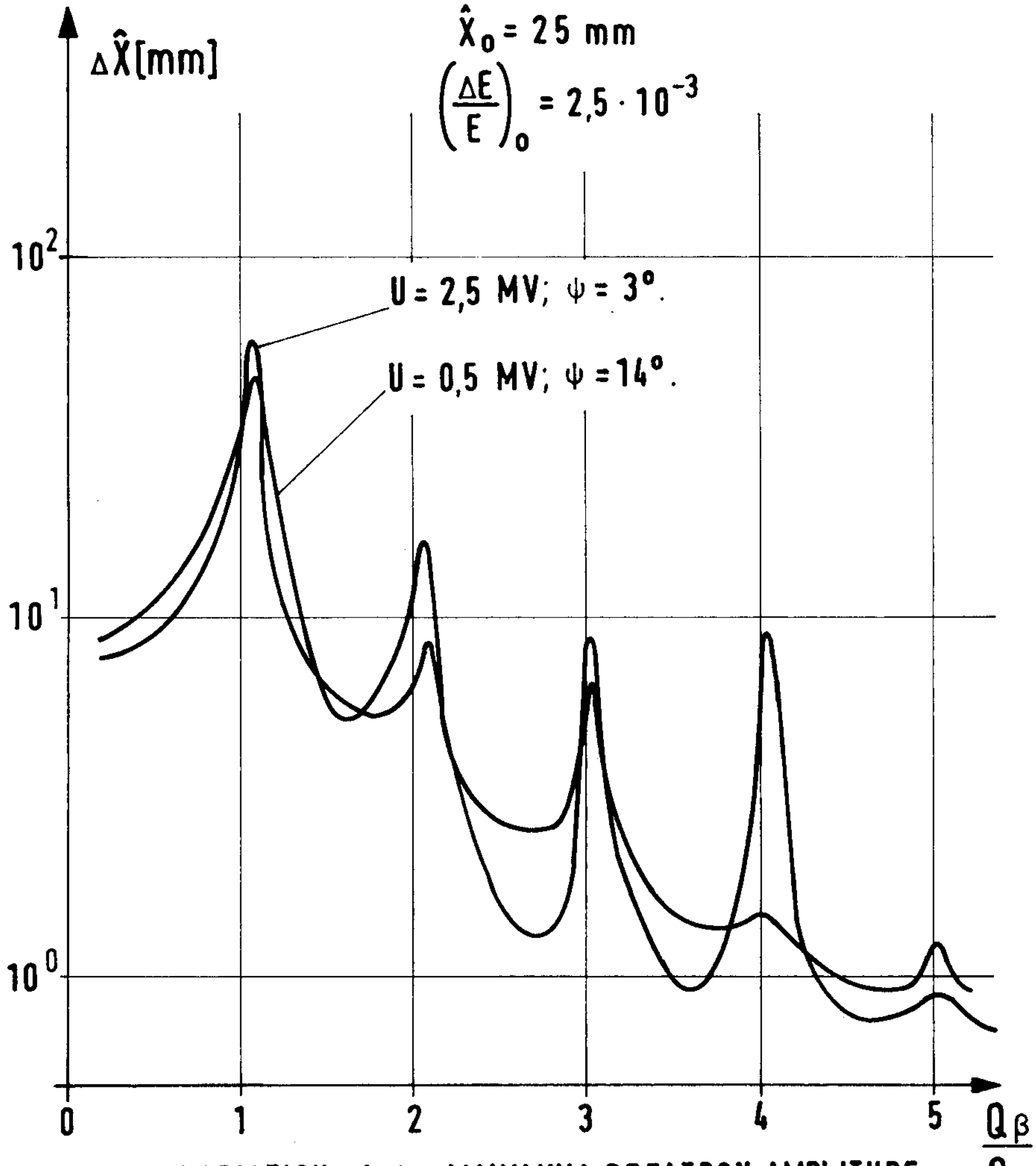

Fig 6: VARIATION of the MAXIMUM BETATRON AMPLITUDE with BETATRON FREQUENCY.

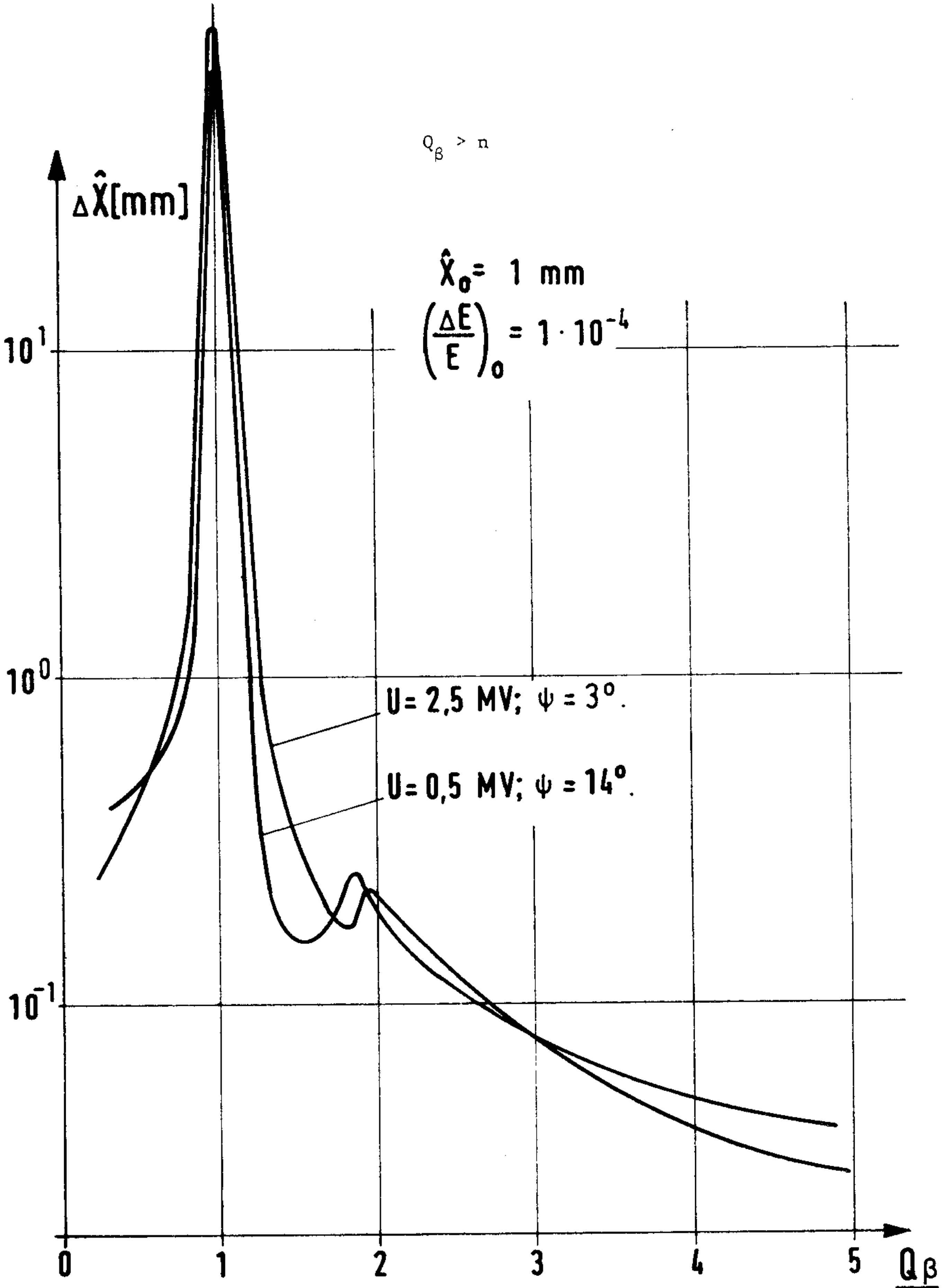

Fig. 7: VARIATION of the MAXIMUM BETATRON AMPLITUDE with BETATRON FREQUENCY

$Q_\beta > n$

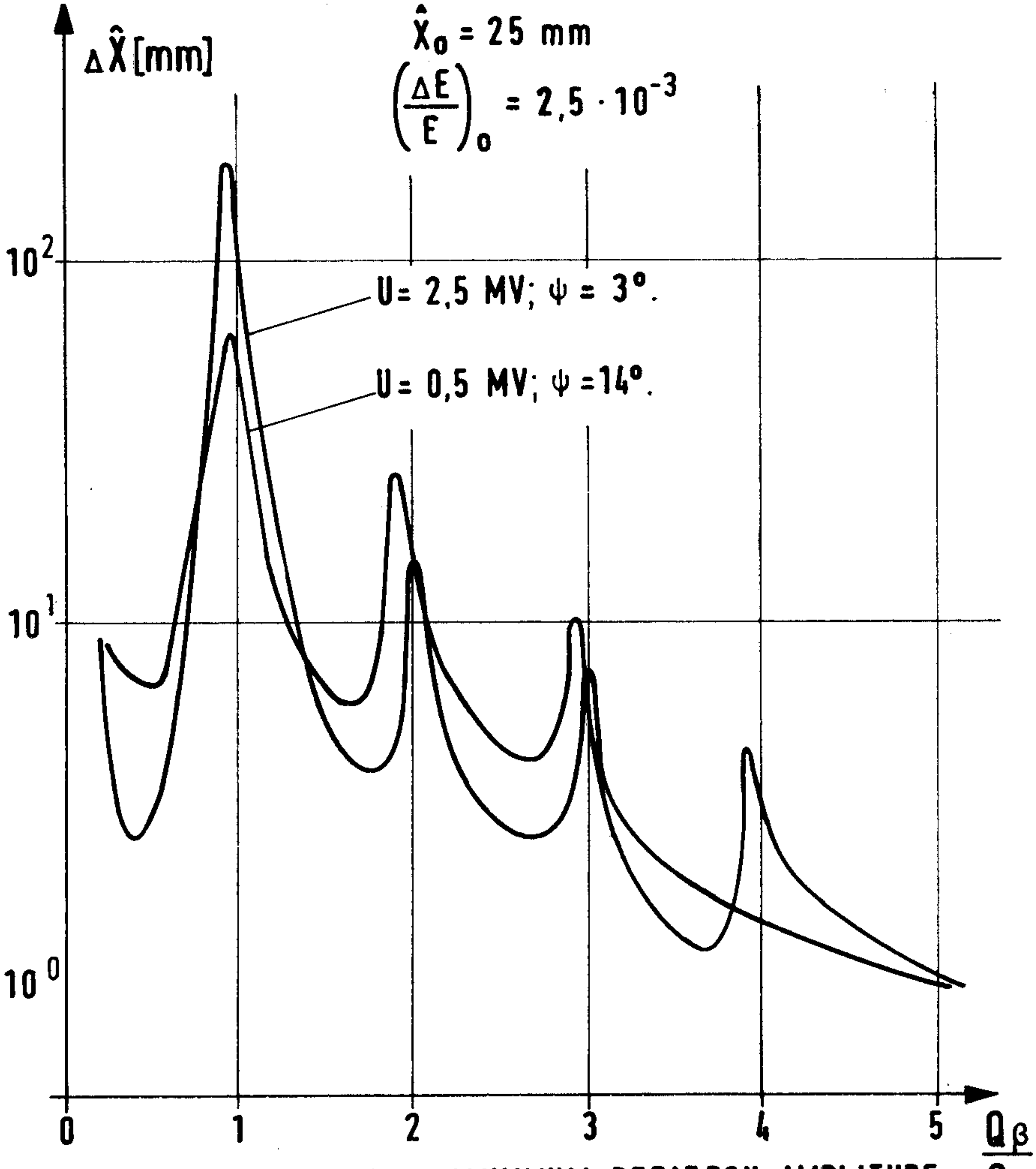

Fig 8: VARIATION of the MAXIMUM BETATRON AMPLITUDE with BETATRON FREQUENCY.

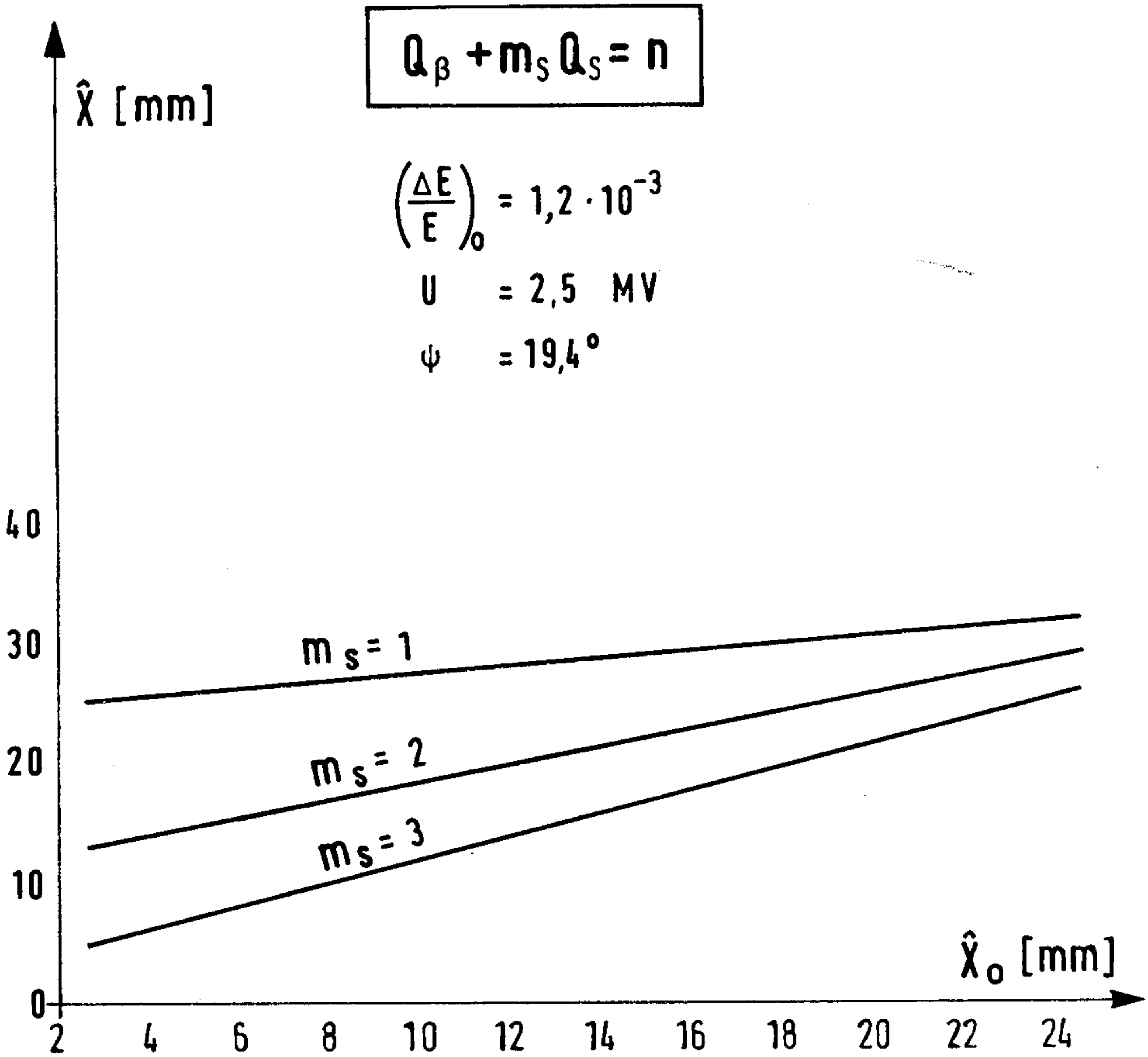

Fig 9: VARIATION of the MAXIMUM BETATRON AMPLITUDE with the INITIAL AMPLITUDE

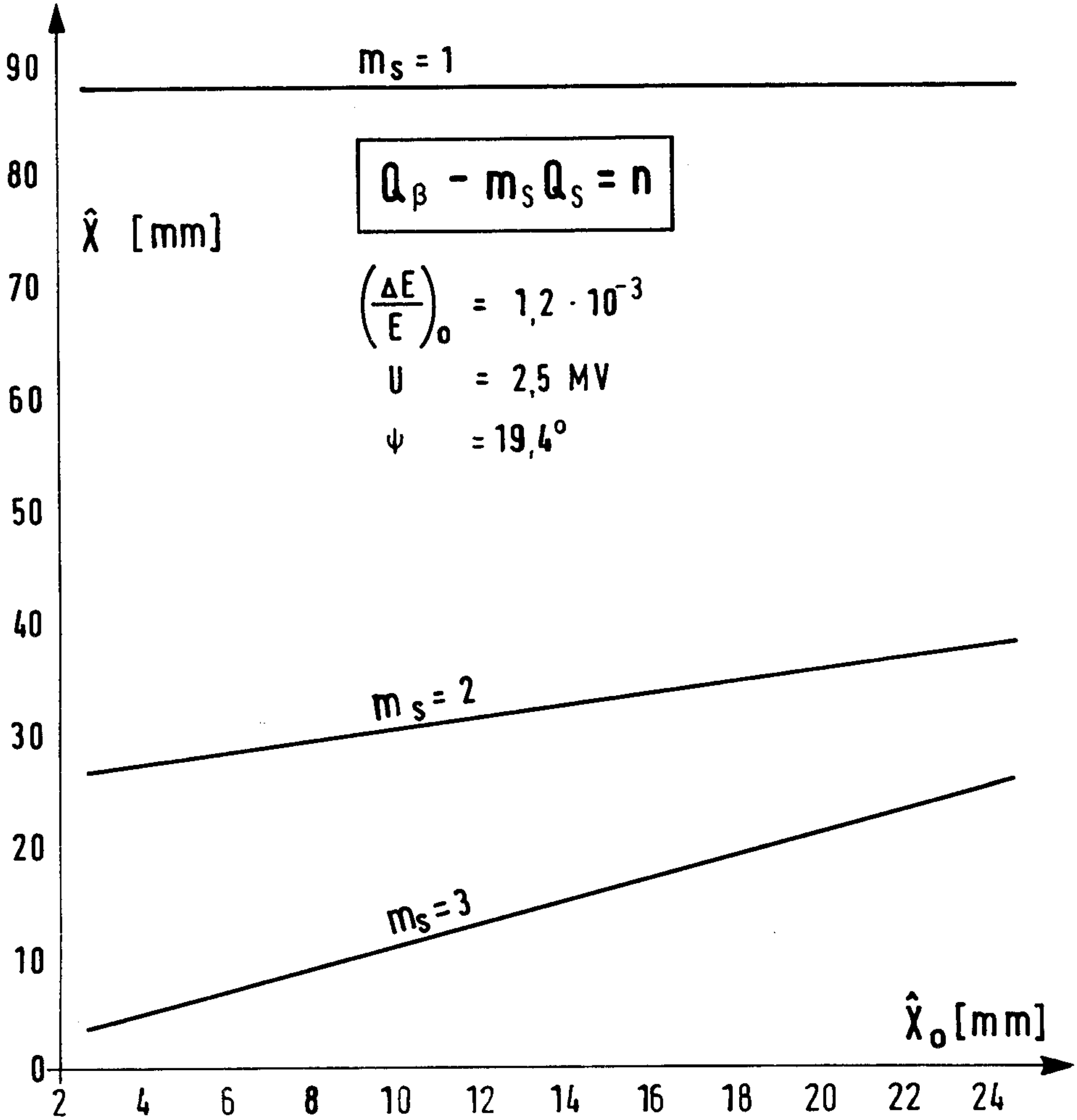

Fig. 10: VARIATION of the MAXIMUM BETATRON AMPLITUDE with the INITIAL AMPLITUDE