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Abstract: Using the correspondence between classical O_3-invariant field theory in one space dimension and sine-Gordon field theory, the O_3-invariant field is shown to decay into solitons in analogy to the decay of the sine-Gordon field.

1. Introduction

The following paper is based on the correspondence between classical chiral O3-invariant field theory in two space-time dimensions and sine-Gordon field theory [1]. With help of this correspondence some features of the sine-Gordon field, according to it's soliton character [2], are translated onto the chiral field.

Every chiral field can be reduced by a simple transformation of space-time coordinates to a normalized chiral field with energy density 1/2 and vanishing momentum density. These normalized fields shall be investigated in this paper. They are in one-to-one correspondence, modulo rotations, to the sine-Gordon fields [1].

The sine-Gordon field is known to consist of a soliton and a dispersive part. For large times the latter fades away. The soliton part however decays into what here is called building blocks. They are running away from each other, each of them having constant velocity. Every building block is made up by those single solitons and breathers, which have its velocity [2].

For the so called n-soliton solutions of the sine-Gordon equation, which decay into their building blocks, the corresponding chiral field can be shown to decay into the corresponding building blocks. A sufficient condition will be given, under which this result holds for more general solutions. Finally some chiral fields are given explicitely.

The paper is organized as follows: in part 2 we repeat the connection between chiral and sine-Gordon field theory and in part 3 the n-soliton solutions of sine-Gordon equation. In part 4 the asymptotic behaviour of chiral field theory is treated.

2. Connection between chiral and sine-Gordon field theory

2.1. The chiral invariant field [1]

The chiral invariant field $\vec{q}$ obeys the equation

$$\vec{q}_{tt} - \vec{q}_{xx} + (\vec{q}_t^{\,2} - \vec{q}_x^{\,2})\vec{q} = 0 , \qquad \vec{q}^{\,2} = 1 \tag{2.1}$$

Here x,t are space-time coordinates, $\vec{q}$ is a real 3 dimensional vector and $\vec{q}_t$ means $\partial q/\partial t$. One can look upon (2.1) as equation of motion of a string, constrained to move along the sphere $\vec{q}^{\,2} = 1$. For $\vec{q}_{tt} - \vec{q}_{xx} = 0$ holds for the free string. Because of the constraint $\vec{q}^{\,2} = 1$ the projection onto the tangential plane of the sphere has to be taken: $\vec{q}_{tt} - \vec{q}_{xx} - \vec{q}(\vec{q}\vec{q}_{tt} - \vec{q}\vec{q}_{xx}) = 0$. That is just (2.1), with $\vec{q}\vec{q}_{tt} = \partial(\vec{q}\vec{q}_t)/\partial t - \vec{q}_t^{\,2} = -\vec{q}_t^{\,2}$ and $\vec{q}\vec{q}_{xx} = -\vec{q}_x^{\,2}$. Introducing characteristic coordinates

$$\xi = (t+x)/2 , \quad \eta = (t-x)/2 \tag{2.2}$$

(2.1) passes into

$$\vec{q}_{\xi\eta} + (\vec{q}_\xi \vec{q}_\eta)\vec{q} = 0 , \quad \vec{q}^{\,2} = 1 \tag{2.3}$$

As (2.3) is forminvariant under transformations

$$(\xi, \eta) \to (\xi'(\xi), \eta'(\eta)) \tag{2.4}$$

leaving the light cone fixed, and as we have energy and momentum conservation

$$(q_\eta^2)_\xi = 0 = (q_\xi^2)_\eta \tag{2.5}$$

the field $\vec{q}$ can be reduced by a transformation (2.4) to a normalized field with energy density 1/2 and momentum density vanishing:

$$\vec{q}_\xi^{\,2} = 1 = q_\eta^2 , \quad \vec{q}^{\,2} = 1 \tag{2.6}$$

or in space-time coordinates

$$\vec{q}_t\vec{q}_x = 0\,,\quad \vec{q}_t^{\,2}+\vec{q}_x^{\,2} = 1\,,\quad \vec{q}^{\,2} = 1 \tag{2.7}$$

Notice that (2.7) implies (2.1).

2.2. Correspondence between chiral and sine-Gordon field

The last equation allows to attach an angle α to the normalized chiral field $\vec{q}$ by

$$(\vec{q}_t \times \vec{q}_x)\vec{q} = (1/2)\sin\alpha\,,\quad \vec{q}_t^{\,2} = \cos^2(\alpha/2)\,,\quad \vec{q}_x^{\,2} = \sin^2(\alpha/2) \tag{2.8}$$

$\vec{q}_t \times \vec{q}_x$ denotes the cross product between $\vec{q}_t$ and $\vec{q}_x$ and α obeys the sine-Gordon equation [1]:

$$\alpha_{tt} - \alpha_{xx} = -\sin\alpha \tag{2.9}$$

The angle α is determined by $\vec{q}$ uniquely modulo 2π.

Conversely to every solution α of (2.9) exactly one, modulo rotations, normalized chiral field $\vec{q}$ exists, so that (2.8) holds. This leads to the ordinary differential equations (2.11) and (2.12) for the chiral field. They are achieved by choosing a path $(x(\tau),t(\tau))$ in space and time with parameter τ. Thus a path $q(x(\tau),t(\tau))$ on the sphere $\vec{q}^{\,2} = 1$ is fixed. Using the arc length s along this path, defined through

$$ds/d\tau = |d\vec{q}/d\tau| \tag{2.10}$$

we have the equation

$$\vec{q}_{ss} = -\vec{q} + [(\vec{q}\times\vec{q}_s)\vec{q}_{ss}](\vec{q}\times\vec{q}_s) \tag{2.11}$$

$s(\tau)$ and $(\vec{q}\times\vec{q}_s)\vec{q}_{ss}$ are determined by the field α(+):

$$ds/d\tau = \sqrt{t'^2\cos^2(\alpha/2)+x'^2\sin^2(\alpha/2)}\,,$$

$$(\vec{q}\times\vec{q}_s)\vec{q}_{ss} = (t'\alpha_x + x'\alpha_t)/2s' - [(x't''-x''t')\sin\alpha - x't'\alpha']/(2s'^3) \tag{2.12}$$

(+) See appendix 1

and " ′ " here means " $d/d\tau$ ".

2.3. About the topology of the correspondence between chiral and sine-Gordon fields

In (2.18) an upper bound for the distance of two normalized fields $\vec{q}_1$ and $\vec{q}_2$ is given, expressed as function of the distance of the two corresponding sine-Gordon fields α_1 and α_2. For this purpose let us introduce the two dreibein fields $\vec{e}_{ia}$ (i=1,2,3; a=1,2) by

$$\vec{e}_{1a} = \vec{q}_a\,,\quad \vec{e}_{2a} = \vec{q}_{at}/\cos(\alpha_a/2)\,,\quad \vec{e}_{3a} = \vec{q}_{ax}/\sin(\alpha_a/2)\,,\quad (a=1,2) \tag{2.13}$$

$\vec{e}_{i1}$ and $\vec{e}_{i2}$ are two orthonormal systems, as can be seen from (2.7) and (2.8). Now the distance between $\vec{q}_1$ and $\vec{q}_2$ can be described by a field $\exp(\vec{\omega}(x,t)\vec{I})$ of rotations:

$$\exp[\vec{\omega}(x,t)\vec{I}]\,\vec{e}_{i1}(x,t) = \vec{e}_{i2}(x,t)\,,\quad (i=1,2,3) \tag{2.14}$$

Here $\vec{I} = (I_1,I_2,I_3)$ are the rotation generators, with $[I_\mu, I_\nu] = \varepsilon_{\mu\nu\rho}I_\rho$. If $(x(\tau),t(\tau))$ is a path in space and time and " ′ " = " $d/d\tau$ ", then one finds(+):

$$[(d/d\tau)\exp(\vec{\omega}\vec{I})]\exp(-\vec{\omega}\vec{I})\,\vec{e}_{i2} = [(\vec{\gamma}_2-\vec{\gamma}_1)\vec{I}]_i^{\,k}\,\vec{e}_{k2}\,,\quad (i=1,2,3) \tag{2.15}$$

with

$$\vec{\gamma}_a = [(t'\alpha_{ax}+x'\alpha_{at})/2\,,\ -x'\sin(\alpha_a/2)\,,\ t'\cos(\alpha_a/2)]\,,\quad (a=1,2) \tag{2.16}$$

If still (2.17) is used, where $\omega = |\vec{\omega}|$ and $\vec{n} = \vec{\omega}/\omega$,

$$[(d/d\tau)\exp(\vec{\omega}\vec{I})]\exp(-\vec{\omega}\vec{I}) = \vec{I}\,[\omega'\vec{n} + \sin\omega\,\vec{n}' + (1-\cos\omega)\,\vec{n}\times\vec{n}'] \tag{2.17}$$

then a comparison with (2.15) yields the result:

$$|d\vec{\omega}/d\tau| \le [\,|\alpha_{1x}-\alpha_{2x}||t'| + |\alpha_{1t}-\alpha_{2t}||x'| + |\alpha_1-\alpha_2|(|x'|+|t'|)\,]\,3\pi/4 \tag{2.18}$$

For $|\vec{\omega}'| \le |\omega'\vec{n}| + \omega|\vec{n}'|$ and $\omega \le (\sin\omega + 1 - \cos\omega)\pi/2$ in $\omega\in(0,\pi)$, hence $|\vec{\omega}'| \le (|\omega'|\,|\vec{n}| + \sin\omega|\vec{n}'| + (1-\cos\omega)|\vec{n}\times\vec{n}'|)\frac{\pi}{2} \le |\vec{\gamma}_2-\vec{\gamma}_1|\,3\pi/2$.

(+) See appendix 1

3. The n-soliton solutions of the sine-Gordon equation

In part 3.1 the explicit formula for these solutions is repeated and in 3.2 their asymptotic decay is described.

3.1. Explicit formula for the n-soliton solutions [2]

The n-soliton solutions are determined by two sequences $(m_1,\dots m_n)$ and $(\lambda_1,\dots\lambda_n)$ of complex parameters. The m_i and λ_i here either are purely real or form conjugate complex pairs. Further on the real parts of all λ_i are positive . Accordingly the m_i and λ_i may be arrayed:

$$\begin{aligned}&\lambda_1,\bar\lambda_1,\cdots\lambda_{2k-1},\bar\lambda_{2k-1},\lambda_{2k+1}=\bar\lambda_{2k+1},\cdots\lambda_n=\bar\lambda_n\ ;\ \mathrm{Re}(\lambda_i)>0\ ;\\ &m_1,\bar m_1,\cdots m_{2k-1},\bar m_{2k-1},m_{2k+1}=\bar m_{2k+1},\cdots m_n=\bar m_n\end{aligned}\qquad(3.1)$$

Every real λ_i corresponds to one single soliton occuring in the n-soliton solution under consideration, every conjugate complex pair to one single breather.

$n\times n$-matrices Λ, M, R, S, L, Q, P are introduced by

$$\Lambda=\mathrm{diag}[\lambda_1,\cdots\lambda_n]\ ,\quad M=\mathrm{diag}[m_1,\cdots m_n]\ ,$$

$$R=2(\Lambda+1/16\Lambda)\quad,\quad S=2(\Lambda-1/16\Lambda)\quad,$$

$$L:\ L_{ik}=1/(\lambda_i+\lambda_k)\ ,\quad (i,k=1,\cdots n)\ ,$$

$$Q(x,t)=i\,M\,[\exp(Rx+St)]\,L\ ,\qquad(3.2)$$

$$P=\begin{array}{c}\left[\begin{array}{cc|c}\begin{pmatrix}0&1\\1&0\end{pmatrix}\ddots&0&0\\0&\begin{pmatrix}0&1\\1&0\end{pmatrix}&\\\hline 0&&\begin{matrix}1&0\\0&\ddots\ 1\end{matrix}\end{array}\right]\end{array}\begin{array}{l}1\\\vdots\\2k\\2k+1\\\vdots\\n\end{array}$$

The n-soliton solution $\alpha(x,t)$ and its derivatives α' with " $'$ " = " $\partial/\partial t$ " or " $\partial/\partial x$ " then are

$$\begin{aligned}\alpha(x,t)&=2i\,\log\det[(1-Q)/(1+Q)]=-4\arg\det(1-Q)\\ \alpha'&=2i\,[\det{}'(1-Q)/\det(1-Q)-\overline{\det}{}'(1-Q)/\overline{\det}(1-Q)]\end{aligned}\qquad(3.3)$$

where the notations " $^-$ " and " $^+$ " mean complex and Hermitian conjugation. The connection to the notation in [2], formula on page 1056 is as follows: Q = V, $\lambda_k=-i\zeta_k$ and our m_k is $-i$ times the m_k of [2].

Because of (3.1) following symmetry relations hold:

$$\bar\Lambda=\Lambda^+=P\Lambda P,\ \bar M=M^+=PMP,\ \bar R=R^+=PRP,\ \bar S=S^+=PSP,\ \bar L=L^+=PLP\qquad(3.4)$$

and we have the lemma (+):

All eigenvalues of Q are purely imaginary. (L3.1)

This guarantees the regularity of (3.3). Finally we mention the determinant formula [3]:

$$\det[1/(a_i+b_k)]=\prod_{i<k}(a_i-a_k)(b_i-b_k)/\prod_{i,k}(a_i+b_k)\qquad(3.5)$$

which is useful for the evaluation of (3.3).

3.2. Asymptotic decay of the n-soliton solution

The essence of this section are the theorems 3.1 and 3.2. They concern the decay of the field into its building blocks and the decrease of the building blocks.

In the following the λ_i of (3.1) are assumed to be pairwise distinct and the m_i to be $\neq 0$. At first some technical details and definitions:

" $'$ " means " $\partial/\partial t$ " or " $\partial/\partial x$ ". I, J, K,... denotes partitions of the index set $N=\{1,\dots n\}$. $|I|$ is the

(+) See appendix 2

symbol for the cardinality of I. For L from (3.2) L_I is the sub-determinant of L with row and column indices in I. Analogously M_I is defined. To $R = \operatorname{diag}[R_1,\dots R_n]$ in (3.2) belongs the abbreviation $r_I = \sum_{i\in I} R_i$ and to S belongs s_I. The velocities v_i and angles φ_i are given by

$$\begin{aligned} v_k &= (1-16|\lambda_k^2|)/(1+16|\lambda_k^2|)\,, \\ \lambda_k &= |\lambda_k|\exp(i\varphi_k)\,,\quad (\varphi_k\in(-\pi/2,\pi/2))\,, \end{aligned} \qquad (k=1,\dots n) \qquad (3.6)$$

and thus

$$R_k x + S_k t = \left[\cos\varphi_k(x-v_k t) + i\sin\varphi_k(-v_k x+t)\right]\big/\sqrt{1-v_k^2} \qquad (3.7)$$

Let u be one of the velocities in (3.6). It divides the index set N into three classes K,L,M:

$$\begin{aligned} &N = K\cup L\cup M,\quad K\cap L = K\cap M = L\cap M = \emptyset\,, \\ &i\in K \Leftrightarrow v_i < u,\quad i\in L \Leftrightarrow v_i = u,\quad i\in M \Leftrightarrow v_i > u,\quad (i\in N) \end{aligned} \qquad (3.8)$$

$u_<$, $u_>$ denote the two velocities from (3.6) next to u, so that $u_< < u$ and $u_> > u$. The $|L|\times|L|$-matrix $\tilde Q$ is defined by

$$\begin{aligned} &\mu_j = \prod_{i\in K}(\lambda_i-\lambda_j)^2/(\lambda_i+\lambda_j)^2,\quad \tilde M = \operatorname{diag}[\mu_j M_j]\,,\ (j\in L)\,, \\ &\tilde R = \operatorname{diag}[R_j]\,,\quad \tilde S = \operatorname{diag}[S_j]\,,\quad (j\in L)\,, \\ &\tilde L:\ \tilde L_{jk} = 1/(\lambda_j+\lambda_k)\,,\quad (j,k\in L)\,, \\ &\tilde Q = i\tilde M\left[\exp(\tilde R x+\tilde S t)\right]\tilde L\,. \end{aligned} \qquad (3.9)$$

The building block of the field α, moving with velocity u, is now

$$\tilde\alpha = -4\arg\det(1-\tilde Q) = -4\arg(-i)^{|K|}M_K L_K\left[\exp(r_K x+s_K t)\right]\det(1-\tilde Q) \qquad (3.10)$$

The second =-sign holds (modulo 2π), as M_K, L_K, r_K, s_K are real.

<u>Asymptotic decay of the field into its building blocks:</u>

Using known rules for the development of the characteristic polynomial of a matrix we get

$$\det(1-Q) = \det\left[1-iM\exp(Rx+St)L\right] = \sum_{\emptyset\subseteq I\subseteq N}(-i)^{|I|}M_I L_I\exp(r_I x+s_I t).$$

If this sum is split up into the part $K\subseteq I\subseteq K\cup L$ and the rest and if (3.5) is used, one arrives at

$$\det(1-Q) = \overbrace{(-i)^K M_K L_K\left[\exp(r_K x+s_K t)\right]\det(1-\tilde Q)}^{z} + \overbrace{\sum_{\substack{\emptyset\subseteq I\subseteq N \\ \text{not } K\subseteq I\subseteq K\cup L}}(-i)^{|I|}M_I L_I\exp(r_I x+s_I t)}^{\varepsilon} \qquad (3.11)$$

We still need the relation

$$|\arg(z+\varepsilon)-\arg(z)| \le (\pi/2)|\varepsilon/z|\,,\qquad (|\varepsilon/z|\le 1) \qquad (3.12)$$

for the imaginary part " arg " of the logarithm, to be able to derive from (3.3), (3.10), (3.11) and (3.12) the two inequalities

$$|\alpha-\tilde\alpha| \le (\pi/2)|\varepsilon/z| \le \begin{cases} \text{const.}\ \varepsilon\ \exp(-r_K x - s_K t) \\ \text{const.}\ \varepsilon\ \exp\left[-(r_K+r_L)x-(s_K+s_L)t\right] \end{cases} \qquad (3.13)$$

Here we worked with the relation $\det\tilde Q^{-1}$ = const. $\exp(-r_L x - s_L t)$ and used as consequence of lemma 3.1 that $|\det(\tilde Q^{-1}-1)|$ and $|\det(1-\tilde Q)|$ are ≥ 1. The domain of validity of (3.13) is reached, if its right side becomes small enough. Inserting (3.7) into the right side of (3.13) one arrives at theorem 3.1 (+):

Pre-conditions:

u is one of the velocities in (3.6).

$u_<$ ($u_>$) is the velocity from (3.6) next to and less (greater) than u. If u is already minimal (maximal), set $u_< = -\infty$ ($u_> = +\infty$).

α is defined by (3.3) and $\tilde\alpha$ by (3.10). (T3.1)

Consequence:

There exist constants $\delta > 0$, $c > 0$, $\varkappa > 0$, so that in the cone $\{(x,t)|\ t > 0,\ u_< t < x < u_> t\}$ holds

$$|\alpha - \tilde\alpha| \le c\left\{\exp[-\varkappa(x-u_< t)] + \exp[-\varkappa(u_> t - x)]\right\},$$

if the right side of this inequality is less than δ.

(+) See appendix 3

Additional remark:

The consequence remains valid, if instead of α and $\tilde{\alpha}$ finite derivatives of them are taken. (T3.1)
The constants δ and c then are dependent on the order of the derivatives, $\varkappa$ not.

Decrease of the building blocks:

It is formulated in form of theorem 3.2:

Pre-conditions:

u is one of the velocities in (3.6).

$\tilde{\alpha}$ is defined by (3.10).

Consequence:

Constants $c > 0$, $\varkappa > 0$ exist, so that

$$|\tilde{\alpha}(\mathrm{mod}\,2\pi)| \leq c\exp(-\varkappa|x-ut|). \qquad (T3.2)$$

Additional remark:

The consequence remains valid for a finite derivative of $\tilde{\alpha}$ instead of $\tilde{\alpha}$, c depending on the order of derivative, $\varkappa$ not, and now " mod2π " has to be omitted.

For a proof use the relations:

$$\tilde{\alpha} = -4\arg\det(1-\tilde{Q}) = -4\arg\det\tilde{Q}^{-1}(1-\tilde{Q}),$$

$$\tilde{\alpha}' = 2i\left[\det{}'(1-\tilde{Q})/\det(1-\tilde{Q}) - \text{conjugate complex}\right] = 2i\left[\det{}'(\tilde{Q}^{-1}-1)/\det(\tilde{Q}^{-1}-1) - \text{c.c.}\right],$$

$$\det(1-\tilde{Q}) = \sum_{\emptyset \subseteq I \subseteq L}(-i)^{|I|}\hat{M}_I\hat{L}_I\exp\left\{\sum_{j\in I}\left[\cos\varphi_j(x-ut)+i\sin\varphi_j(-ux+t)\right]/\sqrt{1-u^2}\right\}, \qquad (3.14)$$

$$\det\tilde{Q}^{-1} = (i)^{-|L|}\det{}^{-1}\hat{M}\cdot\det{}^{-1}\hat{L}\cdot\exp\left\{-\sum_{j\in L}\left[\cos\varphi_j(x-ut)+i\sin\varphi_j(-ux+t)\right]/\sqrt{1-u^2}\right\},$$

$$|\det(1-\tilde{Q})| \geq 1, \quad |\det(\tilde{Q}^{-1}-1)| \geq 1, \quad \cos\varphi_j > 0, \quad (j=1,\dots n)$$

4. Asymptotic behaviour of the chiral field

Part 4.1 is about the building blocks of the chiral field, 4.2 about the asymptotic decay of the field into its building blocks, and in 4.3 some explicit solutions of the field equation (2.7) are given.

4.1. Building blocks of the chiral field

$\vec{q}_-(t) = \lim_{x\to-\infty}\vec{q}(x,t)$ and $\vec{q}_+(t)$ denote the left and right end of the string $\vec{q}(x,t)$. Now theorem 4.1 can be formulated:

Pre-conditions:

u is one of the velocities in (3.6).

$\tilde{\alpha}$ is given by (3.10).

$\tilde{q}$ corresponds to $\tilde{\alpha}$ via (2.8).

Consequence: (T4.1)

The left and right end $\tilde{q}_-(t)$ and $\tilde{q}_+(t)$ of the string $\tilde{q}(x,t)$ are moving both on one great circle of the sphere $\vec{q}^2 = 1$ with velocity 1 in the same direction.

Proof: It is easy to see that $\tilde{q}_+$ and $\tilde{q}_-$ each move on its <u>own</u> circle with velocity 1. For because of theorem 3.2 $\tilde{q}$, $\tilde{q}_t$, $\tilde{q}_{tt}$ converge for $x\to\pm\infty$ uniform in finite t-intervals, and $\lim_{x\to\pm\infty}\tilde{q}_x = \lim_{x\to\pm\infty}\tilde{q}_{xx} = 0$. This leads together with (2.1) and (2.7) to the statement $\ddot{\tilde{q}}_\pm + \tilde{q}_\pm = 0$, $\dot{\tilde{q}}_\pm^2 = 1$. To show that the left and right end both run on <u>one</u> great circle in the <u>same</u> direction, let us first assume they do not, which can be formulated as

<u>assumption:</u> $(\tilde{q}_+(t) - \tilde{q}_-(t))^2 = a + b\sin(2(t-t_0)),\ b \neq 0.$

To derive a contradiction, $\tilde{\alpha}(x,t,h) = \tilde{\alpha}(x+uh,\ t+h)$ and
$d(x,t,h) = |\tilde{\alpha}(x,t,h)-\tilde{\alpha}(x,t)| + |\tilde{\alpha}_x(x,t,h)-\tilde{\alpha}_x(t,h)| + |\tilde{\alpha}_t(x,t,h)-\tilde{\alpha}_t(x,t)|$ are introduced. Along the path $(x(\tau),t(\tau)) = (\tau,0)$ theorem 3.2 yields $|\tilde{\alpha}(x,t,h)(\mathrm{mod}\,2\pi)|$, $|\tilde{\alpha}_x(x,t,h)|$, $|\tilde{\alpha}_t(x,t,h)| \le c\exp(-\varkappa\tau)$, and thus holds the lemma:

To every $\varepsilon>0$ exists $\tau(\varepsilon)>0$, so that

$$\int_{-\infty}^{-\tau} d\tau'\ d(x,t,h) + \int_{\tau}^{\varepsilon} d\tau'\ d(x,t,h) < \varepsilon\,. \tag{L4.1}$$

Use of (3.14) leads to:

For $\tau>0$ and $\varepsilon>0$ exists $\delta(\tau,\varepsilon)$ so that

$$\int_{-\tau}^{\tau} d\tau'\ d(x,t,h) < \varepsilon\,, \tag{L4.2}$$

if $|\exp(ih(1-u^2)^{1/2}\sin\varphi_1)-1| < \delta(\tau,\varepsilon)$, $(1\in L)$.

and as consequence of (L4.1) together with (L4.2) we have:

For $\varepsilon>0$ exists $\delta(\varepsilon)>0$, so that

$$\int_{-\infty}^{\infty} d\tau'\ d(x,t,h) < \varepsilon\,, \tag{L4.3}$$

if $|\exp(ih(1-u^2)^{1/2}\sin\varphi_1)-1| < \delta(\varepsilon)$, $(1\in L)$.

With (2.18) and $\tilde{q}(x,t)$ as corresponding chiral field to $\tilde{\alpha}(x,t)$ lemma 4.4 can be stated:

For $\varepsilon>0$ exists $\delta(\varepsilon)>0$, so that

$$|(\tilde{q}_+(0)-\tilde{q}_-(0))^2 - (\tilde{q}_+(h)-\tilde{q}_-(h))^2| < \varepsilon\,, \tag{L4.4}$$

if $\int_{-\infty}^{\infty} d\tau'\ d(x,t,h) < \delta(\varepsilon)$.

and because of (L4.3) and (L4.4) we at last arrive at:

For every $\varepsilon>0$ exists $\delta(\varepsilon)>0$, so that

$$|(\tilde{q}_+(t)-\tilde{q}_-(t))^2 - (\tilde{q}_+(0)-\tilde{q}_-(0))^2| < \varepsilon\,, \tag{L4.5}$$

if $|\exp(it(1-u^2)^{1/2}\sin\varphi_1)-1| < \delta(\varepsilon)$, $(1\in L)$.

The reals 1, $(1-u^2)^{1/2}\sin\varphi_1$, $(1\in L,\ \sin\varphi_1>0)$ are now assumed to be rational independent. Then Kronecker's Approximationssatz [4] assures to every sequence $\delta_\nu \xrightarrow[\nu\to\infty]{} 0$ and to every complex number c ($|c| = 1$) the existence of a sequence t_ν, so that $\lim_{\nu\to\infty} |\exp(i2t_\nu)-c| = 0$, $|\exp(it_\nu(1-u^2)^{1/2}\sin\varphi_1)-1| < \delta_\nu$, $(1\in L)$. Together with (L4.5) this contradicts the assumption, as c was arbitrary but $|c| = 1$. The case of rational dependent parameters is treated by approximation with independent parameters.

<u>4.2. Asymptotic behaviour</u>

Let us state theorem 4.2:

Pre-conditions:

The sine-Gordon field α, it's building block $\tilde{\alpha}$ with the velocity u and the neighbouring $u_<$ and $u_>$ are defined as in part 3.

$\vec{q}$, $\tilde{q}$ are the chiral fields corresponding to α, $\tilde{\alpha}$.

$\vec{\omega}(x,t)$ describes the distance between $\vec{q}$ and $\tilde{q}$ (T4.2)

as explained in (2.14).

Consequence: $\tilde{q}$ can be fixed by rotation, so that in the space-time region, where the estimate of theorem 3.1 holds, the distance $|\vec{\omega}(x,t)|$ is bounded by

$$|\vec{\omega}(x,t)| \le c\Big\{ \exp[-\varkappa(x-u_<t)]/\varkappa(x/t-u_<) + \exp[-\varkappa(u_>t-x)]/\varkappa(u_>-x/t)\Big\},\ (c>0,\ \varkappa>0).$$

The proof is based on (2.18) and theorem 3.1. Integration of $\vec{\omega}'$ along a straight line into the forward cone, which is limited by $x = u_< t$ and $x = u_> t$, yields the convergence of $\vec{\omega}$ for $t\to\infty$. This limit does not depend on the direction of the integration path, as can be seen by integration of $|\vec{\omega}'|$ along paths t = constant and then with t approaching infinity. As $\tilde{q}$ is determined by $\tilde{\alpha}$ except for rotations, $\vec{\omega}$ can be achieved to tend to 0. Integration of $|\vec{\omega}'|$ along the above mentioned path now gives the estimate of theorem 4.2.

Fixing all occuring building blocks $\tilde{q}$ of $\vec{q}$ in this way by rotations leads to a train of building blocks, towards which the field converges as t goes to infinity. The left and right ends of these building blocks are all moving on the equator of the sphere $\vec{q}^2 = 1$ with velocity 1, and all of them in the same direction. The left end of the first building block, belonging to the largest velocity u, coincides with the right end of the second building block, belonging to the next velocity, etc..

A remark about an extension of these results to more general than n-soliton solutions: Let β be a sine-Gordon field and $\beta = \alpha + \varepsilon$, where α is a n-soliton solution, and ε, ε_x, ε_t decrease in space-time cones uniform at least as fast as t^{-1-d} $(d>0)$. Then β decays in these cones into it's building blocks, because for the proof of theorem 4.2 the convergence of some integrals with integrand $|\vec{\omega}'|$ was sufficient. This convergence however is assured by the decrease $\sim t^{-1-d}$.

4.3. Explicit solutions

To find solutions of the field equation (2.7), the determinant $(\vec{q}\times\vec{q}_s)\vec{q}_{ss}$ shall be investigated. Taking $(x,t) = (x,\ t = \text{constant})$ as path in space and time with parameter x, one can conclude from (2.8):

$$\vec{q}_s = \vec{q}_x/\sin(\alpha/2),\ \vec{q}\times\vec{q}_s = -\vec{q}_t/\cos(\alpha/2),\ ds/dx = \sin(\alpha/2) \tag{4.1}$$

and therefore $(\vec{q}\times\vec{q}_s)\vec{q}_{ss} = -\vec{q}_t\vec{q}_{ss}/\cos(\alpha/2)$. But $\vec{q}_t\vec{q}_{xx} = \vec{q}_t(d/dx)(ds/dx)\vec{q}_s = (\vec{q}_t\vec{q}_{ss})(ds/dx)^2 = (\vec{q}_t\vec{q}_{ss})\sin^2(\alpha/2)$ and $\vec{q}_t\vec{q}_{xx} = (\partial/\partial x)(\vec{q}_t\vec{q}_x)-\vec{q}_{tx}\vec{q}_x = (-1/2)(\partial/\partial t)\vec{q}_x^{\,2} = -\sin(\alpha/2)\cos(\alpha/2)(\alpha_t/2)$. Thus we arrive at

$$(\vec{q}\times\vec{q}_s)\vec{q}_{ss} = (\alpha_t/2)/\sin(\alpha/2) = (\partial/\partial t)\log\tan(\alpha/4) \tag{4.2}$$

Geometrically $(\vec{q}\times\vec{q}_s)\vec{q}_{ss}$ means the change of direction of the path $\vec{q}(x,\ t = \text{constant})$, projected onto the tangential plane of the sphere $\vec{q}^2 = 1$ and reduced to the arc length s. Hence $\vec{q}(x,\ t = \text{constant})$ must be a circle, if this determinant is independent of x. (4.2) tells us, that it suffices when α has the form

$$\alpha(x,t) = 4\arctan[f(x)g(t)] \tag{4.3}$$

The 1-soliton solution, the 2-soliton solution with center of mass at rest and the breather at rest satisfy (4.3), and the corresponding chiral fields can be calculated.

Suitable coordinates χ, φ, ϑ are introduced on the sphere $\vec{q}^2 = 1$ by:

$$\begin{aligned}\vec{q}(\chi,\varphi,\vartheta) &= \exp(\chi I_1)\exp(\varphi I_3)\exp(\vartheta I_1)\vec{e}_3 = \\ &= \vec{e}_1\sin\vartheta\sin\varphi + \vec{e}_2(-\sin\chi\cos\vartheta - \cos\chi\sin\vartheta\cos\varphi) + \vec{e}_3(\cos\chi\cos\vartheta - \sin\chi\sin\vartheta\cos\varphi)\end{aligned} \tag{4.4}$$

$(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ denotes an orthonormal system and I_k the three generators of rotation. In Fig.4.1 (4.4) is sketched.

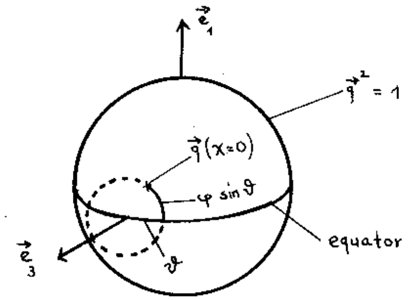

Fig.4.1

In the case $\vartheta = \vartheta(t)$, $\chi = \chi(t)$, $\varphi = \varphi(x,t)$, $\vec{m}(x,t) = \vec{e}_1\cos\varphi + \vec{e}_2\sin\varphi$ formula (4.5) holds:

$$\begin{aligned}
&\exp(-\chi I_1)\vec{q}_t = \varphi_t \sin\vartheta\,\vec{m} - \dot\chi\cos\vartheta\,\vec{e}_2 + \dot\vartheta\cos\vartheta\,\vec{m}\times\vec{e}_3 - (\dot\vartheta\sin\vartheta + \dot\chi\sin\vartheta\cos\varphi)\vec{e}_3,\\
&\exp(-\chi I_1)\vec{q}_x = \varphi_x \sin\vartheta\,\vec{m},\\
&\vec{q}_t\vec{q}_x = \varphi_x \sin\vartheta\,(\varphi_t\sin\vartheta - \dot\chi\sin\varphi\cos\vartheta),\\
&\vec{q}_t^{\,2} = \varphi_t^2\sin^2\vartheta + \dot\chi^2\cos^2\vartheta + \dot\vartheta^2 + \dot\chi^2\sin^2\vartheta\cos^2\varphi + 2\dot\vartheta\dot\chi\cos\varphi - 2\dot\chi\varphi_t\sin\vartheta\cos\vartheta\sin\varphi,\\
&\vec{q}_x^{\,2} = \varphi_x^2\sin^2\vartheta,\\
&(\vec{q}_t\times\vec{q}_x)\vec{q} = \varphi_x\sin\vartheta\,(\dot\vartheta + \dot\chi\cos\varphi)
\end{aligned} \tag{4.5}$$

and lemma (4.6):

$$\text{If } \varphi_t\sin\vartheta - \dot\chi\sin\varphi\cos\vartheta = 0, \quad \text{then } \vec{q}_t^{\,2} = (\dot\vartheta + \dot\chi\cos\varphi)^2 \tag{L4.6}$$

With (4.5) and (L4.6) we can prove that the following solutions obey (2.7) and (2.8):

<u>1-soliton solution (+):</u>

$$\alpha = 4\arctan\exp\left[(x-ut)/\sqrt{1-u^2}\right], \quad \sin\vartheta = \sqrt{1-u^2},\ \cos\vartheta = u,\ \varphi = \pi - \alpha/2,\ \chi = t \tag{4.6}$$

Fig.4.2 is a sketch of $\vec{q}(x,\, t = \text{constant})$.

(+) For u = 0 this solution was already constructed by M. Schlindwein. This was brought to the authors knowledge by K. Pohlmeyer.

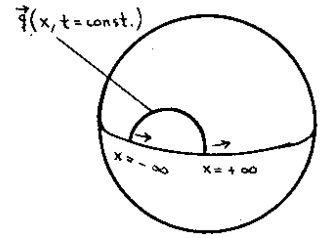

Fig.4.2

<u>2-soliton solution with center of mass at rest:</u>

$$\begin{aligned}
&\alpha = 4\arctan\left[u\cdot\sinh(x/\sqrt{1-u^2})/\cosh(ut/\sqrt{1-u^2})\right],\ \cot\vartheta = \left[u/\sqrt{1-u^2}\right]\tanh(ut/\sqrt{1-u^2}),\\
&\varphi = \pi - 2\arctan\left[\cosh(x/\sqrt{1-u^2})\Big/\sqrt{-1+\cosh^2(ut/\sqrt{1-u^2})/u^2}\,\right],\ \chi = t - \vartheta
\end{aligned} \tag{4.7}$$

In fig.4.3 $\vec{q}$ is sketched for different constant times.

A uniform rotation around the $\vec{e}_1$ axis is separated, thus the ends of the string seem to be at rest.

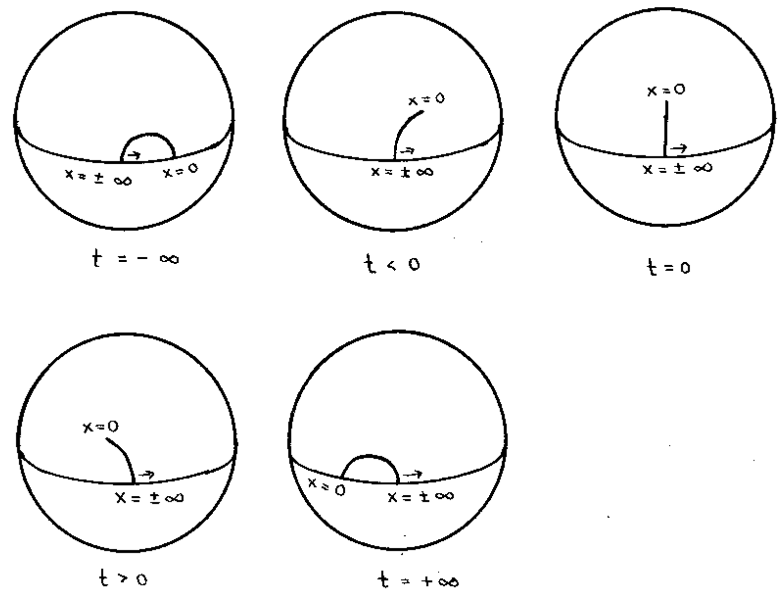

Fig.4.3

Breather at rest:

$$\alpha = 4\arctan\left[(\varkappa/\eta)\cos(t\eta/\lambda)/\cosh(x\varkappa/\lambda)\right], \quad \varkappa^2+\eta^2=\lambda^2,$$
$$\sin\vartheta = (\lambda/\eta)\cos(t\eta/\lambda)/\sqrt{1+(\varkappa/\eta)^2\cos^2(t\eta/\lambda)}, \quad \cos\vartheta = \sin(t\eta/\lambda)/\sqrt{\chi}, \qquad (4.8)$$
$$\chi = t+\vartheta, \quad \varphi = -2\arctan\left[\sinh(x\varkappa/\lambda)/\sqrt{1+(\varkappa/\eta)^2\cos^2(t\eta/\lambda)}\right]$$

In fig.4.4 we sketch the breather at constant times, and again have separated a uniform rotation.

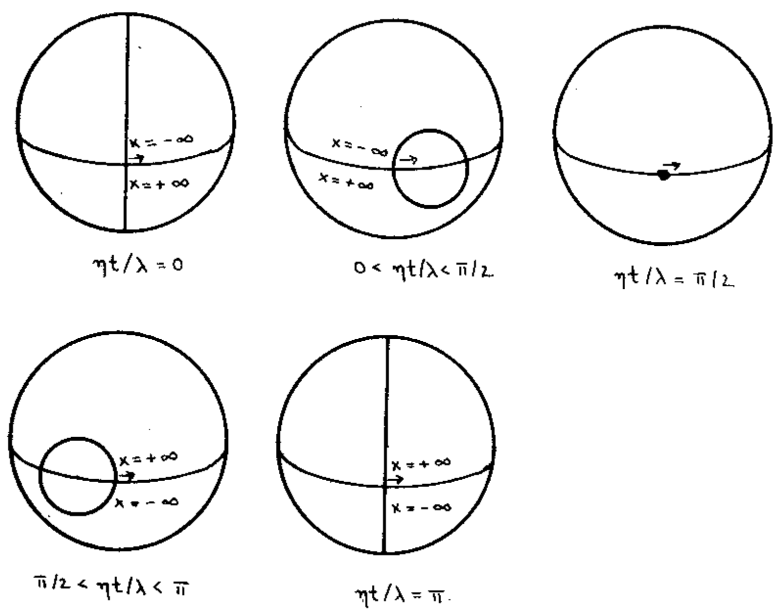

Fig.4.4

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Appendix 1

At first let us look for the derivatives $(d/d\tau)\vec{e}_i$ of the dreibein field in (2.13), expressed by the $\vec{e}_i$. For $\vec{e}_1' = \vec{q}'$ one at once finds

$$\vec{e}_1{}' = t'\cos(\alpha/2)\vec{e}_2 + x'\sin(\alpha/2)\vec{e}_3 \qquad (A1.1)$$

To get $\vec{e}_2'$ and $\vec{e}_3'$, $\vec{q}_{tt}$, $\vec{q}_{tx}$ and $\vec{q}_{xx}$ have to be calculated. From (2.7) and (2.8) follows

$$\vec{q}_{tt}\vec{q} = -\cos^2(\alpha/2), \quad \vec{q}_{tt}\vec{q}_t = -(\alpha_t/2)\sin(\alpha/2)\cos(\alpha/2), \quad \vec{q}_{tt}\vec{q}_x = (\alpha_x/2)\sin(\alpha/2)\cos(\alpha/2),$$
$$\vec{q}_{tx}\vec{q} = 0, \quad \vec{q}_{tx}\vec{q}_t = -(\alpha_x/2)\sin(\alpha/2)\cos(\alpha/2), \quad \vec{q}_{tx}\vec{q}_x = (\alpha_t/2)\sin(\alpha/2)\cos(\alpha/2), \qquad (A1.2)$$

and together with the equation of motion (2.1):

$$\vec{q}_{tt} = -\cos^2(\alpha/2)\vec{e}_1 - (\alpha_t/2)\sin(\alpha/2)\vec{e}_2 + (\alpha_x/2)\cos(\alpha/2)\vec{e}_3,$$
$$\vec{q}_{tx} = -(\alpha_x/2)\sin(\alpha/2)\vec{e}_2 + (\alpha_t/2)\cos(\alpha/2)\vec{e}_3, \qquad (A1.3)$$
$$\vec{q}_{xx} = -\sin^2(\alpha/2)\vec{e}_1 - (\alpha_t/2)\sin(\alpha/2)\vec{e}_2 + (\alpha_x/2)\cos(\alpha/2)\vec{e}_3$$

Thus $\vec{e}_2'$ and $\vec{e}_3'$ are equal to:

$$\vec{e}_2{}' = -t'\cos(\alpha/2)\vec{e}_1 + (1/2)(t'\alpha_x + x'\alpha_t)\vec{e}_3,$$
$$\vec{e}_3{}' = -x'\sin(\alpha/2)\vec{e}_1 - (1/2)(t'\alpha_x + x'\alpha_t)\vec{e}_2 \qquad (A1.4)$$

(A1.1) and (A1.4) can be written in a more compact form:

$$\vec{e}_\mu{}' = \gamma_\mu{}^\nu \vec{e}_\nu = (\vec{\gamma}\vec{I})_\mu{}^\nu \vec{e}_\nu, \quad (\mu = 1,2,3),$$

$$\vec{\gamma}\vec{I} = \begin{bmatrix} 0, & -t'\cos(\alpha/2), & -x'\sin(\alpha/2) \\ t'\cos(\alpha/2), & 0, & -(t'\alpha_x + x'\alpha_t)/2 \\ x'\sin(\alpha/2), & (t'\alpha_x + x'\alpha_t)/2, & 0 \end{bmatrix}, \quad \vec{\gamma} = \begin{bmatrix} (t'\alpha_x + x'\alpha_t)/2 \\ -x'\sin(\alpha/2) \\ t'\cos(\alpha/2) \end{bmatrix} \qquad (A1.5)$$

$\vec{I}$ denotes the generators of the rotation group.

Now we turn to the calculation of $(\vec{q}\times\vec{q}_s)\vec{q}_{ss}$. From $s' = (t'^2\cos^2(\alpha/2)+x'^2\sin^2(\alpha/2))^{1/2}$ and $\vec{q}_s = \lambda^\mu \vec{e}_\mu$, where $\lambda^1 = 0$, $\lambda^2 = t'\cos(\alpha/2)/s'$ and $\lambda^3 = x'\sin(\alpha/2)/s'$ follows:

$$(\vec{q}\times\vec{q}_s)\vec{q}_{ss} = \det(\vec{e}_1, \lambda^\mu\vec{e}_\mu, \lambda^{\nu\prime}\vec{e}_\nu + \lambda^\nu \vec{e}_\nu')/s' =$$
$$= (\lambda^2\lambda^{3\prime}-\lambda^3\lambda^{2\prime})/s' + \lambda^\mu\lambda^\nu \det(\vec{e}_1,\vec{e}_\mu,\vec{e}_\nu')/s'.$$

The terms with $\mu \neq \nu$ vanish in the last summand, $\det(\vec{e}_1,\vec{e}_2,\vec{e}_2') = \det(\vec{e}_1,\vec{e}_3,\vec{e}_3') = (t'\alpha_x+x'\alpha_t)/2$ because of (A1.5), and hence $\det(\vec{q},\vec{q}_s,\vec{q}_{ss}) = (\lambda^2\lambda^{3\prime}-\lambda^3\lambda^{2\prime})/s' + (t'\alpha_x+x'\alpha_t)/2s'$. That is just the wanted formula (2.12), as can be found by evaluation of $\lambda^2\lambda^{3\prime}-\lambda^3\lambda^{2\prime}$.

Finally the proof of (2.15): The abbreviation $R = \exp(\vec{\omega}\vec{I})$ is used. Application of $(d/d\tau)$ onto $R\,\vec{e}_{i1} = \vec{e}_{i2}$, $(i = 1,2,3)$ yields with help of (A1.5):

$$R'\vec{e}_{i1}+R\,\vec{e}_{i1}' = \vec{e}_{i2}' = R'R^{-1}\vec{e}_{i2}+\gamma^k_{i1}\,R\,\vec{e}_{k1} = R'R^{-1}\vec{e}_{i2} + \gamma^k_{i1}\,\vec{e}_{k2},$$ therefore $$R'R^{-1}\,\vec{e}_{i2} = \vec{e}_{i2}'-\gamma^k_{i1}\,\vec{e}_{k2} = [(\vec{\gamma}_2-\vec{\gamma}_1)\vec{I}]^k_i\,\vec{e}_{k2}.$$

Appendix 2

Matrices of the type ML, with M, L from (3.2), shall be shown to have only real eigenvalues. For this reason the matrix P_- is introduced. It coincides with P from (3.2) except for those diagonal elements belonging to negative m_i. There P_- has a (-1). For the square root of M holds:

$$\sqrt{M}^+ = P\sqrt{M}\,P_- = P_-\sqrt{M}\,P \qquad \text{(A2.1)}$$

Because of $ML = M^{1/2}\,M^{1/2}\,L\,M^{1/2}\,M^{-1/2}$ it suffices to show that $M^{1/2}\,L\,M^{1/2}$ has real eigenvalues. For $H = P_-M^{1/2}L\,M^{1/2}$

we find with (A2.1) and (3.4)

$$H = H^+ = P_-\sqrt{M}\,L\sqrt{M} \qquad \text{(A2.2)}$$

$M^{1/2}\,L\,M^{1/2}$ is equal to $P_-\,H$. Let λ and x be an eigenvalue and eigenvector to this matrix, then $P_-\,H\,x = \lambda x$, $H\,x = \lambda P_-\,x$, $\langle x|Hx\rangle = \lambda\langle x|P_-x\rangle$, and if H is positive definite, λ has to be real because of $H = H^+$ and $P_- = P_-^+$. But H is $= P_-\,M^{1/2}\,L\,M^{1/2} = (M^{1/2})^+\,P\,L\,M^{1/2}$ and it suffices to prove P L positive definite. From linear algebra one knows that all the minors of P L, concentrated in the left upper corner of P L, have to be greater than 0. This is shown using (3.5) and (3.1).

Appendix 3

Here the way from (3.13) to the estimate in theorem 3.1 is shown. Substitution of (3.7) into the summands of the right hand side of (3.13) yields for these summands:

$$\text{const.}\exp\Big\{-\sum_{k\in K}\big[(x-v_kt)\cos\varphi_k+i(-v_kx+t)\sin\varphi_k\big]/(1-v_k^2)^{1/2} + \sum_{j\in I}\big[(x-v_jt)\quad\ldots\quad\big]/(1-v_j^2)^{1/2}\Big\}$$

and

$$\text{const.}\exp\Big\{-\sum_{k\in K\cup L}\big[(x-v_kt)\ \ldots\ \big]/(1-v_k^2)^{1/2} + \sum_{j\in I}\big[(x-v_jt)\ \ldots\ \big]/(1-v_j^2)^{1/2}\Big\},$$

where not $K \subseteq I \subseteq K\cup L$.

<u>For $t > 0$, $u_<t < x \leq ut$</u> take the first formula and distinguish the two cases:

1) $K=I$: Then because of (3.8) the real part of the exponent is less than $\varkappa(x-u_{>}t)$ with $\varkappa = \min_{j=1,\dots n} \cos\varphi_j/(1-v_j^2)^{1/2}$.
2) $K\neq I$: Real part of exponent $< -\varkappa(x-u_{<}t)$.

<u>For $t > 0$, $u t \leq x < u_{>}t$</u> the second formula yields the same results.

For a sketch of proof of the additional remark notice that for ε, z from (3.11) $|\varepsilon^{(N)}/z|$ has an upper bound of the form occuring in (3.13), and $|z^{(N)}/z|$ is uniform bounded in x, t. $z^{(N)}$ denotes a derivative of order N.

Literature cited:

[1] K. Pohlmeyer: Commun. math. Phys. 46, 207 (1976).
[2] L. A. Takhtadzhyan, L. D. Faddeev: Theoretical and Mathematical Physics 21, 1046 (1974).
[3] G. Pólya, G. Szegö: Problems and Theorems in Analysis II, 92 (Springer 1976).
[4] W. Maak: Fastperiodische Funktionen, 145 (Springer 1967).