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On Currents Associated with Topological Charges

by

J. Herrel

II. Institut für Theoretische Physik der Universität Hamburg

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We ask whether there are currents associated with these charges.

For D = n a conserved, local current is known in the literature.

Our investigation is for D = 2(n-1) in particular D = 4 and n = 3.

The results are:

a conserved current exists, but is not local; the corresponding charge is given by Hopf's invariant of the field; the charges are additive.

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I Introduction

Usually a conserved charge Q in classical or quantum field theory is associated with an internal symmetry of the Lagrangian. Noether's theorem then furnishes a current j^μ such that $Q=\int d\sigma^\mu j_\mu$ is a constant of motion.

Recently however the possibility of topological charges has found renewed interest.They are not related to a symmetry of any Lagrangian.

We study whether one can find currents associated with them.

We consider classical n-component unit vector-fields on a D dimensional space-time:

$$\vec\phi(\underline{x},t)\in S_\phi^{n-1};\quad \vec\phi=(\phi^1,\dots,\phi^n);\ \vec\phi\vec\phi=1\qquad \underline{x}\in\mathbb{R}^{D-1} \qquad (1)$$

Let us write $\hat\phi=(1,0,\dots,0)$ for the north-pole (NP) of S_ϕ^{n-1}. Assume that $\vec\phi(\underline{x},t)$ is acontinuous function of the arguments $(\underline{x},t)$ and that the Lagrangian is such that the total energy of a given field configuration $\vec\phi(\cdot,t)$ at a given time t can be finite only if

$$\vec\phi(\underline{x},t)\longrightarrow\hat\phi \qquad \text{as } x\longrightarrow\infty$$

We compactify the (D-1) dimensional space by adding the point at infinity and write $\mathbb{R}^{D-1}\cup\{\infty\}\simeq S^{D-1}$ by virtue of a stereographic projection.The field $\vec\phi$ at any time t may now be considered as a continuous map:

$$\phi(t):\ S^{D-1}\longrightarrow S_\phi^{n-1} \qquad (2)$$

which leaves the NP invariant.

Topological charges characterize the homotopy-classes of the map (2).

Since $\phi(t)$ is continuous in t (by assumption) time-developement does not change the homotopy class:topological charges are conserved.

We assumed that the field $\vec\phi$ represent a finite energy configuration.The existence of topological charges indicates that the space of finite energy fields is disconnected.These soliton-sectors are in one to one correspondence with the homotopy classes of the map (2) and the topological charges are homotopy invariants of the map (2).

Two cases are of interest;in both of them there exists a topolo gical charge Q which takes as values all intergers: $Q=0,\pm 1,\pm 2,\ldots$

<u>Case A</u>: $D=n$;$D>1$; $Q_A = \deg \phi(t)$ (degree of $\phi(t)$)

<u>Case B</u>: $D= 2(n-1)$; $D=4,6,8,10,\ldots$; $Q_B = \gamma(\phi)$(Hopf invariant γ of $\phi(t)$)

Note: for $D=6,10,\ldots$ $\gamma(\phi)\equiv 0$

In (case A) a local conserved current was found by Patani et al [1]

$$j^{\mu}(\underline{x},t) = \epsilon_{a_1,\ldots,a_n}\, \epsilon^{\mu\mu_2,\ldots,\mu_n}\, \phi^{a_1}\partial_{\mu_2}\phi^{a_2}\ldots\partial_{\mu_n}\phi^{a_n}$$

We will find that the situation is quite different in case B.
Of main physical interest is D=4 ;let us therefore specialize to this value.
One can again find a conserved current $j^{\mu}(\underline{x},t)$ such that $Q=\int d^3x\, j^0(\underline{x},t)$ determines the homotopy classes of $\phi(t)$.However j^{μ} is not alocal functional of $\vec{\phi}$.

We introduce the "field-strength"

$$f_{\mu\nu}(\underline{x},t) = \epsilon_{abc}\,\phi^a\partial_\mu\phi^b\partial_\nu\phi^c(\underline{x},t) \tag{3}$$

Its dual $\tilde{f}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma}$ is conserved(see Sec.IIIb).

$$\partial^{\mu}\tilde{f}_{\mu\nu} = 0 \tag{4}$$

Therefore there exists a potential A_μ such that

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5}$$

Of course A_μ is determined only up to a local gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$ since $f_{\mu\nu}$ is explicitly gauge invariant.
Using standard techniques of algebraic topology,one shows that there exist gauges such that (5) is valid even at infinity $\underline{x} = \infty$ in a sense that will be made precise.We will give an explicit expression for the potential A which achieves this.

Given such achoice of potential the current

$$j^{\mu} = \tilde{f}^{\mu\nu}A_\nu$$

is conserved,because $\tilde{f}^{\mu\nu}f_{\mu\nu} = 0$ (seeIIIb).
The chrge density $j^0(\underline{x},t)$ is given by

$$j^0 = \vec{A}\,\mathrm{curl}\,\vec{A} \qquad \text{and} \qquad Q = \int d^3x\,(\vec{A}\,\mathrm{curl}\,\vec{A}) = 4\pi\gamma(\phi(t))$$

However A_μ is not a local functional of $\vec{\phi}$ and therefore the current is not,either.
Moreover the current is not gauge invariant:

if $A_\mu \mapsto A_\mu + \partial_\mu \Lambda$

then the current changes by a total divergence:

$$j^{\mu} \mapsto j^{\mu} + \partial_\nu(\tilde{f}^{\mu\nu}\Lambda)$$

For these reasons the current cannot be considered as an observable,for its value cannot be determined by a measurement of $\vec{\phi}$ and its derivations at time t.
It appeares impossible to find an observable current associated with the Hopf charge.
Nevertheless one cannot rule out the appearance of topological charges of the Hopf type in nature.
To fix ideas,think of baryon number as a Hopf charge.This is not absurd:the e.m. charge distribution within an elementary particle can be measured (e.m. form factor),but the baryonic charge distribution cannot be measured experimentally for lack of a vector meson that would couple to it.One only measures baryon number by counting particles and adding up the baryon numbers associated with them.But this is also quite possible for topological charges which take as values <u>all</u> integers,especially for charges of the Hopf type.They have the appropriate additivity properties as we shall see in Sec.IV.
In Sec.II the concept of Hopf's invariant in introduced;
in Sec.III the current is constucted;
in Sec.V an alternative formula for the current is given;here the current is concentrated on closed curves but is again non local and not gauge invariant.

II The Hopf-invariant [2]

Let $\vec{\phi}(x,t)$ be given as described in the introduction. For D=4 the associated map $\phi(t)$ (eq.2) maps S^3 in coordinate space onto S^2_ϕ in field space:

$$\phi(t): S^3 \longrightarrow S^2_\phi \qquad \text{(surjective)}$$

The spheres S^3 and S^2_ϕ are manifolds. Let $\mathcal{F}^{(p)}(M)$ be the space of p-forms on the manifold M; if $p > \dim M$, each p-form vanishes identically on M.

Choose $\sigma \in \mathcal{F}^{(2)}(S^2_\phi)$ to be the normed surface form on S^2_ϕ, i.e. :

$$\int_{S^2} \sigma = 1$$

since $d\sigma \in \mathcal{F}^{(3)}(S^2)$ it follows that $d\sigma = 0$.

The induced 2-form $\phi^*\sigma \in \mathcal{F}^{(2)}(S^3)$ is closed: $d(\phi^*\sigma) = \phi^*(d\sigma) = 0$.

By application of de Rham's second theorem, one finds that on S^3 each closed form is exact. Hence there exists an 1-form $\alpha \in \mathcal{F}^{(1)}(S^3)$ such that $d\alpha = \phi^*\sigma$ on S^3 .

Hence the integral of $\alpha \wedge \phi^*\sigma$ over S^3 is defined:

$$\alpha \wedge \phi^*\sigma \in \mathcal{F}^{(3)}(S^3) \;;\; \gamma(\phi) \equiv \int_{S^3} \alpha \wedge \phi^*\sigma \qquad (6)$$

$\gamma(\phi)$ has the property of being an integer and of being independent of the special choice of

(i) $\sigma \in \mathcal{F}^{(2)}(S^2_\phi)$ such that $\int_{S^2} \sigma = 1$

(ii) $\alpha \in \mathcal{F}^{(1)}(S^3)$ such that $d\alpha = \phi^*\sigma$

The integral $\gamma(\phi)$ is called Hopf invariant γ of $\phi(t)$ and had been discovered by H.Hopf [3] .

III The Hopf current

a) Construction of the potential

Let $f_{\mu\nu}$ be any skew-symmetric tensor defined on $S^3 \times \mathbb{R}$.

We map $f_{\mu\nu}$ into the 2-form $f \in \mathcal{F}^{(2)}(S^3 \times \mathbb{R})$ given by

$$f = f_{\mu\nu}\, dx^\mu dx^\nu$$

Let the dual $\tilde{f}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma}$ be conserved. Then f is a closed form:

$$df = 0 \qquad (*)$$

The results of (secII) say that f is exact on S^3; since $\mathbb{R}$ is contractible to a point , f is exact on $\mathbb{R}$ by Poincare's lemma. Hence there exist an 1-form $A \in \mathcal{F}^{(1)}(S^3 \times \mathbb{R})$ such that

$$dA = f \quad \text{on} \quad S^3 \times \mathbb{R}. \qquad (8a)$$

We write $A = A_\mu dx^\mu$; locally (eq.8a) can be written as the common formula:

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{on} \quad \mathbb{R}^3 \times \mathbb{R} \qquad (8b)$$

For the construction of the potential A some notations are needed:

manifold $S^3 := \{(\xi^1,\ldots,\xi^4) \in \mathbb{R}^4 : \sum_1^4 (\xi^i)^2 = 1\}$

NP =(0,0,0,1) SP=(0,0,0,-1)

open covering	local coordinate system
$\mathcal{U} := S^3 \setminus \{NP\}$	$x^i = \dfrac{\xi^i}{1-\xi^4}$
$\bar{\mathcal{U}} := S^3 \setminus \{SP\}$	$y^i = \dfrac{\xi^i}{1+\xi^4}$ $\quad (i=1,2,3)$

$\mathcal{U}, \bar{\mathcal{U}}$ are contractible; the local coordinate system (x^i); (y^i) is the stereographic projection on $\mathbb{R}^3$. Hence on $\mathcal{U} \times \mathbb{R}$; $\bar{\mathcal{U}} \times \mathbb{R}$ a solution of (eq.8b) is given by the Poincare lemma:

$$\text{on } \mathcal{U}\times\mathbb{R}\;:\quad dB=f \iff f_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu \qquad (\underline{x},t)\in\mathcal{U}\times\mathbb{R}.$$

$$\text{on } \bar{\mathcal{U}}\times\mathbb{R}\;:\quad d\bar{B}=f \iff f_{\mu\nu}=\partial_\mu \bar{B}_\nu-\partial_\nu \bar{B}_\mu \qquad (\underline{y},t)\in\bar{\mathcal{U}}\times\mathbb{R}.$$

On the intersection $E:=(\mathcal{U}\cap\bar{\mathcal{U}})\times\mathbb{R}$ we find a closed form:

$$d(\bar{B}-B)=0$$

Since on $\mathcal{U}\cap\bar{\mathcal{U}}\simeq S^3\setminus\{NP;SP\}\simeq\mathbb{R}^3\setminus\{0\}$ each 1-cycle z is a boundary b of a two dimensional domain, we apply de Rham's second theorem again and conclude that the closed 1-form $B^E:=\bar{B}-B$ is exact on E, so that $\bar{B}-B=d\lambda$.

Let c be any integration path in E then a suitable scalar function $\chi(\underline{x},t)$ is given by

$$\chi(\underline{x},t)=\int_{\hat{x}}^{x} dx'^\mu\, B^E_\mu(x') \qquad x=(\underline{x},t)\in E. \quad (11)$$

The integral is path independent since $(c-c')$ is a boundary and B^E is closed. $\chi(\underline{x},t)$ is determined up to an arbitrary constant.

In order to compose the complete solution in a smooth way, a C^∞ function φ on $S^3\times\mathbb{R}$ is used, which has the following properties:

let

V be a sufficiently small neighbourhood of NP; then $\varphi_{|V\times\mathbb{R}}=1$

$\bar{V}$ be a sufficiently small neighbourhood of SP; then $\varphi_{|\bar{V}\times\mathbb{R}}=0$.

The complete solution of (eq.8a) is defined by:

$$A(\underline{x},t):=B(\underline{x},t)+d(\varphi\chi) \qquad (12)$$

where the scalar function $\chi(\underline{x},t)$ is given by (eq 11)

That $A(\underline{x},t)$ is indeed defined on $S^3\times\mathbb{R}$ is seen by the following consistence check:

on $(\mathcal{U}\cap\bar{\mathcal{U}})\times\mathbb{R}$: $A=B+d(\varphi\chi)$ is defined, and $dA=dB=f$

on $(V\cap\bar{V}\cap V)\times\mathbb{R}$: $A=B+d\chi=\bar{B}$ is defined, especially in NP, and $dA=d\bar{B}=f$

on $(V\cap\bar{V}\cap\bar{V})\times\mathbb{R}$: $A=B$ is defined, especially in SP, and $dA=dB=f$

We write $A=A_\nu(x)dx^\nu$ formally as an integral in covariant form:

$$A_\nu(x)dx^\nu=\int_{\hat{x}}^{x} dx'^\mu f_{\mu\nu}(x')\,dx^\nu+d(\varphi\chi) \qquad x\in\mathcal{U}\times\mathbb{R} \quad (13)$$

Different paths of integration correspond to different choice of gauges for the potential, i.e.

$$A_\nu(c_1-c_2)=\partial_\nu\Upsilon \qquad c_i \text{ integration path; } \Upsilon \text{ scalar function}$$

But since the tensor $f_{\mu\nu}$ is gauge invariant, each choice of an integration path leads to a solution of (eq.8).

b) <u>Construction of the current</u>

In order to define a current associated with a charge of the Hopf type we introduce the skew-symmetric "field strength" tensor:

$$f_{\mu\nu}=\vec{\phi}\,(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi}) \qquad (14a)$$
$$=\epsilon_{abc}\,\phi^a\,\partial_\mu\phi^b\,\partial_\nu\phi^c$$

and its dual: $\tilde{f}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}f^{\rho\sigma}$ (14b)

Since $\vec{\phi}\cdot\vec{\phi}=1$ the functionalmatrix $(\partial_\mu\vec{\phi})$ must have a rank

$$rk(\partial_\mu\vec{\phi})<3 \qquad \text{i.e. :}$$
$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{abc}\,\partial_\mu\phi^a\,\partial_\nu\phi^b\,\partial_\rho\phi^c=0 \quad (15)$$

which implies the conservation law for the dual:

$$\partial_\mu\tilde{f}^{\mu\nu}=0 \qquad (16a)$$

The next step is to prove the lorentz-invariant orthogonality relation: $\tilde{f}^{\mu\nu}f_{\mu\nu}=0$ (16b)

Since $(\tilde{f}^{\mu\nu}f_{\mu\nu})^2\propto \det f_{\mu\nu}$ we have to show that $\det f_{\mu\nu}=0$. To verify this we only need the two geometric relations

$$\vec{\phi}\cdot\vec{\phi}=1 \qquad (17a)$$
$$\vec{\phi}\cdot\partial_\mu\vec{\phi}=0 \qquad (17b)$$

and the multiplication law for scalar triple products $\vec{\phi}(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi})$
We obtain then

$$f_{\mu\nu}f_{\rho\sigma} = \begin{vmatrix} \partial_\mu\vec{\phi}\,\partial_\rho\vec{\phi} & \partial_\mu\vec{\phi}\,\partial_\sigma\vec{\phi} \\ \partial_\nu\vec{\phi}\,\partial_\rho\vec{\phi} & \partial_\nu\vec{\phi}\,\partial_\sigma\vec{\phi} \end{vmatrix} \qquad (17c)$$

Inserting (eq.17c) into the identity

$$\det f_{\mu\nu} = (f_{01}f_{23} + f_{02}f_{31} + f_{03}f_{12})^2$$

it follows that $\det f_{\mu\nu}=0$.

Since $f_{\mu\nu}(x,t)$ is defined on $S^3\times\mathbb{R}$ and represents a skew-symmetric tensor, such that the dual is conserved, there exists (cf.IIIa) a potential $A_\mu(x,t)$ on $S^3\times\mathbb{R}$ satisfying $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ on $S^3\times\mathbb{R}$.

The Hopf current is defined by

$$j^\mu := \tilde{f}^{\mu\nu}A_\nu \qquad (18)$$

The conservation law is a consequence of (eq.16a,b): $\partial_\mu j^\mu = 0$.

The Hopf charge Q is given by

$$Q = \int d^3x\, j^0(x,t) \qquad (19a)$$

with the charge density :

$$j^0 = \vec{A}\,\mathrm{curl}\,\vec{A} \qquad (19b)$$

To evaluate the integral in (eq.19a) we choose a 2-form $\sigma \in \mathcal{F}^{(2)}(S^2_\phi)$ such that σ is the normed surface form on S^2_ϕ (sec.II).

Let σ be given by: $\quad 4\pi\sigma = \frac{1}{2}\epsilon_{abc}\,\phi^a d\phi^b \wedge d\phi^c \qquad (20)$

Using polar coordinates (ϑ, φ) on S^2_ϕ , the 2-form σ reads:

$$\sigma = \frac{1}{4\pi}\sin\vartheta\, d\vartheta\, d\varphi$$

The induced 2-form $\phi^*\sigma$ is given by

$$4\pi\,\phi^*\sigma = \frac{1}{2} f_{ij}\, dx^i \wedge dx^j \qquad (21)$$

with $\quad f_{ij} = \epsilon_{abc}\,\phi^a\partial_i\phi^b\partial_j\phi^c \qquad (i,j)\in(1,2,3)$

We map $\vec{A}$ into the 1-form $\alpha = A_i dx^i$; then $\mathrm{curl}\vec{A}$ is mapped into the 2-form

$$d\alpha = \frac{1}{2} f_{ij}\, dx^i \wedge dx^j = 4\pi\,\phi^*\sigma$$

Hence the charge density 3-form $j^0 d^3x$ in (eq.19b) is given by

$$j^0 d^3x = 4\pi(\alpha\wedge\phi^*\sigma) \qquad (22)$$

and therefore $\quad Q = 4\pi\,\gamma(\phi) \qquad (23)$

Thus Q is a topological charge, determining the homotopy classes of the map $\phi(t): S^3 \mapsto S^2_\phi$. The charge conservation $\dot{Q}=0$ turns out to be a consequence of the invariance of homotopy classes under continuous transformations, and time developement is of that kind.

IV Additivity of Hopf Charges

If the Hopf charge Q in (eq.23) is to occur as a physical charge (as suggested in the introduction), we have to say how to count them.

We say that a finite energy field $\vec{\phi}(x,t)$ or equivalently the related map $\phi(t)$ describes a "particle".

A single "particle of small extension" is described by a map $\phi(t)$ which is the constant map $\hat{\phi}$ outside a convex subdomain $D \subset \mathbb{R}^3$ on which the "particle" is concentrated. The Hopf charge Q is defined via (eq.6,23).

We imagine a situation of a finite number N of "particles of small extension", each one carrying a Hopf charge Q_i $(i=1,\dots,N)$

To describe this N particle configuration we consider N disjoint convex subdomains $D_i \subset \mathbb{R}^3$ and a map $\phi(t)$ defined by

$$\phi(t) = \hat{\phi} \text{ outside all } D_i \ ; \quad \phi_{|D_i} = \phi_{i|D_i} \qquad (24)$$

where each ϕ_i describes a "particle of small extension" with Hopf charge Q_i .

(eq.24) means, that the N particles are concentrated on the N subdomains D_i .

Let Q the Hopf charge determined by the field $\phi(t)$ in (eq.24).

We now show the "additivity of Hopf charges" i.e. $Q = \sum_i Q_i$

For the proof we need some arguments which are used to verify the group structure of the third homotopy group π_3 of S^2, [4] .

Since the charges Q_i are invariants of the homotopy classes $[\phi_i]$ of ϕ_i they are unchanged whenever ϕ_i is replaced by a homotopic map $\phi_i' \in [\phi_i]$ and $\phi_i' = \phi_i = \hat{\phi}$ outside D_i .

We take N=2; the case $N > 2$ follows by induction.

It is convenient to represent S^3 as a 3-cell I^3 such that the boundary ∂I^3 of I^3 is identified with the NP of S^3 . The 3-cell I^3 consists of 3-tupels (x_1, x_2, x_3) such that $0 \le x_i \le 1$; $(i=1,2,3)$

The boundary ∂I^3 contains 3-tupels for which $\prod_{i=1}^{3} x_i(x_i-1)=0$
Let $\phi_1(t), \phi_2(t)$ be two maps related to finite energy fields $\vec{\phi}_1(x,t), \vec{\phi}_2(x,t)$ resp. by (eq.2). We then have $\phi_{1,2}: I^3 \longrightarrow S^2_\phi$ and $\phi_{1,2}(\partial I^3)=\hat{\phi}$
A multiplication $\square$ is defined by:

$$\phi(x_1,x_2,x_3) \equiv (\phi_1 \square \phi_2)(x_1,x_2,x_3) = \begin{cases} \phi_1(2x_1, x_2, x_3) & 0 \le x_1 \le \frac{1}{2} \\ \phi_2(2x_1-1, x_2, x_3) & \frac{1}{2} \le x_1 \le 1 \end{cases} \quad (25)$$

This operation induces a multiplication o in $\pi_3(S^2)$ the set of all homotopy classes $[\phi]$ of ϕ such that $\phi(\partial I^3)=\hat{\phi}$: $[\phi_1] \circ [\phi_2] = [\phi_1 \square \phi_2]$, $(\pi_3(S^2), \circ)$ is an abelian group. $\quad$ (26)

Now a special homotopy is defined on a sub-cell D_1 of I^3 :

D_1 contains those 3-tupels for which $\lambda \le x_1 \le \mu$

$$0 \le \lambda < \mu \le 1$$

Define:

$$\tilde{\phi}_1(x_1,x_2,x_3) = \begin{cases} \phi_1\left(\frac{x_1-\lambda}{\mu-\lambda}, x_2, x_3\right) & (x_1,x_2,x_3) \in D_1 \\ \hat{\phi} & \text{otherwise} \end{cases} \quad (27)$$

Then: $\tilde{\phi}_1 \in [\phi_1]$

Similarly D_2 and $\tilde{\phi}_2 \in [\phi_2]$ are defined.
From (eq.24) the 2-particle field ϕ is given by:

$$\phi(x_1,x_2,x_3) = \hat{\phi} \qquad (x_1,x_2,x_3) \in I^3 \setminus (D_1 \cup D_2)$$

$$\phi_{|D_1} = \phi_{1|D_1} \in [\phi_1] \ ; \quad \phi_{|D_2} = \phi_{2|D_1} \in [\phi_2]$$

We choose special sub-cells D_1 and D_2 :

$$D_1 := \{(x_1,x_2,x_3) : 0 \le x_1 \le \tfrac{1}{2}\}$$

$$D_2 := \{(x_1,x_2,x_3) : \tfrac{1}{2} \le x_1 \le 1\}$$

Any other choice of D_1 can be reduced to this special one, [4] .

Define a new map Υ_1 by $\Upsilon_1(x_1,x_2,x_3) \equiv \phi_{|D_1}(\frac{1}{2}x_1, x_2, x_3)$
Then $\phi_{|D_1} = \tilde{\Upsilon}_1$ where $\tilde{\Upsilon}_1$ and Υ_1 are related via (eq.27)
Similarly $\phi_{|D_2} = \tilde{\Upsilon}_2$ is found by definition of Υ_2 .
Thus we have

$$\phi(x_1,x_2,x_3) = \begin{cases} \Upsilon_1(2x_1, x_2, x_3) & 0 \le x_1 \le \frac{1}{2} \\ \Upsilon_2(2x_1-1, x_2, x_3) & \frac{1}{2} \le x_1 \le 1 \end{cases}$$

Hence $\phi \in [\Upsilon_1] \circ [\Upsilon_2]$ (cp.eq.25,26). Since $\Upsilon_1 \in [\tilde{\Upsilon}_1]$ and $[\tilde{\Upsilon}_1] = [\phi_{|D_1}] = [\phi_1]$ we find $\Upsilon_1 \in [\phi_1]$ and similarly $\Upsilon_2 \in [\phi_2]$ and therefore $\phi \in [\phi_1] \circ [\phi_2]$ which implies that $Q=Q_1+Q_2$,if Q, Q_1, Q_2 are the

Hopf charges determined by ϕ, ϕ_1, ϕ_2 resp.
Since $\pi_3(S^2)$ is cyclic infinite $\pi_3(S^2) \simeq \mathbb{Z}_\infty$ Hopf charges can take as values all integers.
The property that Hopf charges are always an integer supports the suggestion that they might represent a charge-like quantum number.
The following is a short summary of this chapter:

If the topological charge, determined by a given unit vector field, can take as values all integers, i.e. if $\pi_{D-1}(S^{m-1}) \simeq \mathbb{Z}_\infty$ and if a given N particle configuration is described by a field which results from "patching together" (cp.Coleman's lecture 6) the nontrivial parts of the single fields, then the total charge of the N particle configuration is the sum of all single charges.

V An Alternative Description of the Current

We give an alternative description of the current in (eq.18) for two reasons :the new formula confirms that the current is not a local functional in the fields, and is not gauge invariant; moreover it shows that, by using the gauge freedom ,the Hopf charge may be considered as concentrated on a closed 1-curve and not as continously distributed over the whole space
To any map $\phi(t): S^3 \mapsto S^2$ there exists arbitrarily close by a map $\tilde{\phi}: S^3 \mapsto S^2$ such that for a given but arbitrary point $\check{\phi} \in S^2$ the inverse image $\phi^{-1}(\check{\phi})$ is a 1-cycle z on S^3 .
(precisely: ϕ is homotopic to a fibre map $\tilde{\phi}$ over the fibre space S^3 with fibres S^1)
We approximate ϕ by $\tilde{\phi}$,i.e. we assume $\phi^{-1}(\check{\phi})$ to be a 1-cycle z .This 1-cycle z is determined by two equations:

$$P_a \equiv \phi^a(x) - \check{\phi}^a = 0 \qquad (a,b) \in (1,2,3) \qquad (28)$$
$$P_b \equiv \phi^b(x) - \check{\phi}^b = 0$$

We now use an integral transformation which is closely related to the original definition of Hopf's invariant as a certain mapping degree [5]

$$\int_{S^3} \alpha \wedge \phi^* \sigma = \int_z \alpha$$

Now define distributions $\delta_z^{\nu\rho}(x)$ which are concentrated on the 1-cycle z by:

$$\int_{Vol} d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \delta_z^{\nu\rho}(x)\, \varphi(x) = \int_z dx^\sigma \varphi(x) \qquad ; \varphi(x) \text{ testfunction}$$

Summing over the components of the potential we obtain

$$\int d\sigma^\mu j_\mu \equiv \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \delta_z^{\nu\rho} A^\sigma = \int_z dx^\sigma A_\sigma$$

with $j_\mu(x,t) \equiv \epsilon_{\mu\nu\rho\sigma} \delta_z^{\nu\rho}(x) A^\sigma(x,t)$

Since $dx^0 = 0$ on $z \subset S^3$ it follows that $\int_z dx^\sigma A_\sigma = \int_z \alpha$

and hence

$$4\pi \cdot \int d\sigma^\mu \epsilon_{\mu\nu\rho\sigma} \delta_z^{\nu\rho}(x) A^\sigma = Q \qquad (29)$$

With (eq.28) the distributions $\delta_z^{\nu\rho}(x)$ are given explicitly by

$$\delta_z^{jk}(x) = \begin{vmatrix} \partial_j \phi^a & \partial_k \phi^a \\ \partial_j \phi^b & \partial_k \phi^b \end{vmatrix} \delta(P_a)\, \delta(P_b)$$

where the δ -function $\delta(P_a)$ is concentrated on $P_a = 0$ and defined as in the standard text [7] .

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