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Classical Anticommuting Massive Thirring Model
in One Space Dimension

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A Bäcklund transformation is derived for the solutions of the classical anticommuting massive Thirring model in one space dimension. We apply the transformation to construct exact solutions and to show the existence of an infinite number of conserved currents.

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A BÄCKLUND TRANSFORMATION FOR THE
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Abstract:

A Bäcklund transformation is derived for the solutions of the classical anticommuting massive Thirring model in one space dimension. We apply the transformation to construct exact solutions and to show the existence of an infinite number of conserved currents.

During the last years Bäcklund transformations were studied intensively for various types of nonlinear partial differential equations in two variables, e.g. the sine-Gordon equation.[1] The Bäcklund transformation for some differential equation is a system of first order differential equations, which contains two functions, the original and the transformed one, and a parameter.

Starting from one solution of the differential equation the Bäcklund transformation generates a family of functions which can be characterised by two unrelated numbers; the explicit parameter in the Bäcklund transformation and one implicit parameter which can be expressed by the value of the function at the origin. Every member of this family of functions should be a solution of a partial differential equation, not necessarily the original one. This equation does not depend on the Bäcklund parameter and on the original function.

Such Bäcklund transformations exist for the above mentioned partial differential equations and they allow the algebraic construction of more and more solutions. Furthermore they are helpful for the construction of an infinite number of conserved currents.

Because of the equivalence of the sine-Gordon equation and the massive Thirring model as quantum field theories[2] we looked for a Bäcklund transformation for the anticommuting massive Thirring model. It was helpful that the infinite number of conserved currents had already been established by Kulish and Nissimov.[3]

The Thirring model has the field equation

$$i\gamma^\mu \partial_\mu \varphi - m\varphi = \frac{\lambda}{2} (\bar{\varphi}\gamma^\mu \varphi)\gamma_\mu \varphi$$

where $\varphi = (\varphi_1, \varphi_2)$ is a two component Dirac spinor field with anticommuting components. Introducing light-cone coordinates

$$\xi = \tfrac{1}{2}(x^0 + x^1) \quad , \quad \eta = \tfrac{1}{2}(x^0 - x^1)$$

the equations are

$$\begin{aligned} i\partial_\eta \varphi_1 &= m\varphi_2 + \lambda \varphi_2^+ \varphi_2 \varphi_1 \\ i\partial_\xi \varphi_2 &= m\varphi_1 + \lambda \varphi_1^+ \varphi_1 \varphi_2 \quad . \end{aligned} \tag{1}$$

In the case of the free field ($\lambda = 0$) a Bäcklund transformation is defined by

$$\begin{aligned} \partial_\xi(\varphi_1^{(a)}+\varphi_1) &= m a^{-1}(\varphi_1^{(a)}-\varphi_1) \\ \partial_\eta(\varphi_2^{(a)}-\varphi_2) &= -m a(\varphi_2^{(a)}+\varphi_2) \\ \varphi_2^{(a)}-\varphi_2 &= -ia(\varphi_1^{(a)}+\varphi_1) \quad . \end{aligned} \tag{2}$$

To avoid algebraic relations between the functions, i.e. to work only with independent functions, we divide the Bäcklund transformation into two parts. With the help of the auxiliary function

$$\chi^{(a)} = \varphi_1^{(a)}+\varphi_1 = ia^{-1}(\varphi_2^{(a)}-\varphi_2)$$

we get

$$\begin{aligned} \partial_\xi \chi^{(a)} &= m a^{-1}\chi^{(a)} - 2ma^{-1}\varphi_1 \\ \partial_\eta \chi^{(a)} &= -m a\, \chi^{(a)} - 2mi\varphi_2 \end{aligned} \tag{3}$$

and

$$\begin{aligned} \varphi_1^{(a)} &= \chi^{(a)} - \varphi_1 \\ \varphi_2^{(a)} &= \varphi_2 - ia\chi^{(a)} \quad . \end{aligned} \tag{4}$$

It is easy to verify that

- the integrability of (3) implies that the φ_i solve (1) with $\lambda = 0$
- $\varphi_i^{(a)}$ solve the same equations (1) for any "a".

We generalize (3) for $\lambda \neq 0$, assuming that no extra derivatives appear:

$$\begin{aligned} \partial_\xi \chi^{(a)} &= m a^{-1}\chi^{(a)} - 2ma^{-1}\varphi_1 + \lambda f(\chi^{(a)},\chi^{(a)+},\varphi,\varphi^+) \\ \partial_\eta \chi^{(a)} &= -m a\,\chi^{(a)} - 2mi\varphi_2 + \lambda g(\chi^{(a)},\chi^{(a)+},\varphi,\varphi^+) \quad . \end{aligned}$$

The integrability condition for these equations

$$\partial_\eta(\partial_\xi \chi^{(a)}) = \partial_\xi(\partial_\eta \chi^{(a)})$$

should be Eq.(1). Therefore

$$\begin{aligned} f(\chi^{(a)},\chi^{(a)+},\varphi,\varphi^+) &= \tilde f(\chi^{(a)},\chi^{(a)+},\varphi_1,\varphi_1^+) = a_1\varphi_1^+\varphi_1\chi^{(a)} + b_1\chi^{(a)+}\chi^{(a)}\varphi_1 \\ g(\chi^{(a)},\chi^{(a)+},\varphi,\varphi^+) &= \tilde g(\chi^{(a)},\chi^{(a)+},\varphi_2,\varphi_2^+) = a_2\varphi_2^+\varphi_2\chi^{(a)} + b_2\chi^{(a)+}\chi^{(a)}\varphi_2 \quad . \end{aligned} \tag{6}$$

This very simple form follows from the anticommutativity, where the square of any of these fields vanishes. From (5) we calculate

$$\begin{aligned} \partial_\xi \chi^{(a)} &= m a^{-1}\chi^{(a)} - 2ma^{-1}\varphi_1 - i\lambda\varphi_1^+\varphi_1\chi^{(a)} - i\lambda\chi^{(a)+}\chi^{(a)}\varphi_1 \\ \partial_\eta \chi^{(a)} &= -m a\,\chi^{(a)} - 2im\varphi_2 - i\lambda\varphi_2^+\varphi_2\chi^{(a)} - \lambda a\chi^{(a)+}\chi^{(a)}\varphi_2 \end{aligned} \tag{7}$$

with real parameter "a".

These equations are equivalent to the recursive relations for the b_n defined by Kulish and Nissimov[3] when we expand $\chi^{(a)}$ as a formal power series in "a":

$$2\chi^{(a)} = \sum_{n=0}^{\infty} b_n a^n$$

It follows from Eq.(7) that

$$\partial_\xi\left[\tfrac{a}{4}(\varphi_2^+\chi^{(a)}+\chi^{(a)+}\varphi_2)\right] + \partial_\eta\left[\tfrac{i}{4}(\varphi_1^+\chi^{(a)}-\chi^{(a)+}\varphi_1)\right] = 0$$

and $\chi^{(a)}$ can be viewed as a generating function for the infinite number of conserved currents.

Secondly, we try to generalize (4) in such a way that $\varphi_i^{(a)}$ solves some differential equations which must reduce to (1) for $\lambda = 0$. With an ansatz similar to (6), it follows that the unique possibility is

$$\begin{aligned} \varphi_1^{(a)} &= \chi^{(a)} - \varphi_1 - \frac{i\lambda a}{2m}\chi^{(a)+}\chi^{(a)}\varphi_1 \\ \varphi_2^{(a)} &= \varphi_2 - ia\chi^{(a)} - \frac{i\lambda a}{2m}\chi^{(a)+}\chi^{(a)}\varphi_2 \quad . \end{aligned} \tag{8}$$

The so defined $\varphi_i^{(a)}$ is in fact a solution of (1) for any λ.

The Eqs.(7) and (8) together are the Bäcklund transformation because they have all the features specified in the introduction. The auxiliary function $\chi^{(a)}$

can be eliminated from the Bäcklund transformation by solving (8) for $\chi^{(a)}$:

$$-ia\chi^{(a)} = -ia\left\{\varphi_1^{(a)}+\varphi_1+\frac{i\lambda a}{2m}\varphi_1^{(a)+}\varphi_1^{(a)}\varphi_1-\frac{i\lambda a}{2m}\varphi_1^{+}\varphi_1\varphi_1^{(a)}\right\}$$
$$= \varphi_2^{(a)}-\varphi_2+\frac{i\lambda}{2ma}\varphi_2^{(a)+}\varphi_2^{(a)}\varphi_2+\frac{i\lambda}{2ma}\varphi_2^{+}\varphi_2\varphi_2^{(a)} \quad .$$

Thus we get the generalization of (2), which however is not as convenient for doing calculations.

By successive application of Bäcklund transformations it is possible to derive algebraically new solutions of the Thirring model. Consider the sequences of Bäcklund transformations

$$\varphi \xrightarrow{a_1} \chi^{(1)} \Leftrightarrow \varphi^{(1)} = \mathcal{B}(a_1)\varphi \xrightarrow{a_2} \chi^{(1,2)} \Leftrightarrow \varphi^{(1,2)} = \mathcal{B}(a_2)\mathcal{B}(a_1)\varphi$$

$$\varphi \xrightarrow{a_2} \chi^{(2)} \Leftrightarrow \varphi^{(2)} = \mathcal{B}(a_2)\varphi \xrightarrow{a_1} \chi^{(2,1)} \Leftrightarrow \varphi^{(2,1)} = \mathcal{B}(a_1)\mathcal{B}(a_2)\varphi \quad .$$

To show the commutativity of two Bäcklund transformations we construct $\varphi^{(1,2)}$ and $\varphi^{(2,1)}$ directly from φ, $\varphi^{(1)}$ and $\varphi^{(2)}$. The result is considerably simplified if we express $\varphi^{(i)}$ by $\chi^{(i)}$.

From the assumption that $\varphi^{(1,2)} = \varphi^{(2,1)}$ together with Eq.(8) it follows that

$$\varphi_1^{(1,2)} = \chi^{(1,2)}-\varphi_1^{(1)}-\frac{i\lambda a_2}{2m}\chi^{(1,2)+}\chi^{(1,2)}\varphi_1^{(1)} = \chi^{(2,1)}-\varphi_1^{(2)}-\frac{i\lambda a_1}{2m}\chi^{(2,1)+}\chi^{(2,1)}\varphi_1^{(2)} = \varphi_1^{(2,1)}$$

$$\varphi_2^{(1,2)} = \varphi_2^{(1)}-ia_2\chi^{(1,2)}-\frac{i\lambda a_2}{2m}\chi^{(1,2)+}\chi^{(1,2)}\varphi_2^{(1)} = \varphi_2^{(2)}-ia_1\chi^{(2,1)}-\frac{i\lambda a_1}{2m}\chi^{(2,1)+}\chi^{(2,1)}\varphi_2^{(2)} = \varphi_2^{(2,1)} .$$

Eliminating $\varphi^{(1)}$ and $\varphi^{(2)}$ we have

$$\chi^{(1,2)}-\chi^{(1)}+\varphi_1+\frac{i\lambda a_1}{2m}\chi^{(1)+}\chi^{(1)}\varphi_1-\frac{i\lambda a_2}{2m}\chi^{(1,2)+}\chi^{(1,2)}\left(\chi^{(1)}-\varphi_1-\frac{i\lambda a_1}{2m}\chi^{(1)+}\chi^{(1)}\varphi_1\right)$$
$$= \chi^{(2,1)}-\chi^{(2)}+\varphi_1+\frac{i\lambda a_2}{2m}\chi^{(2)+}\chi^{(2)}\varphi_1-\frac{i\lambda a_1}{2m}\chi^{(2,1)+}\chi^{(2,1)}\left(\chi^{(2)}-\varphi_1-\frac{i\lambda a_2}{2m}\chi^{(2)+}\chi^{(2)}\varphi_1\right)$$

$$-ia_2\chi^{(1,2)}-ia_1\chi^{(1)}+\varphi_2-\frac{i\lambda a_1}{2m}\chi^{(1)+}\chi^{(1)}\varphi_2-\frac{i\lambda a_2}{2m}\chi^{(1,2)+}\chi^{(1,2)}\left(\varphi_2-ia_1\chi^{(1)}-\frac{i\lambda a_1}{2m}\chi^{(1)+}\chi^{(1)}\varphi_2\right)$$
$$= -ia_1\chi^{(2,1)}-ia_2\chi^{(2)}+\varphi_2-\frac{i\lambda a_2}{2m}\chi^{(2)+}\chi^{(2)}\varphi_2-\frac{i\lambda a_1}{2m}\chi^{(2,1)+}\chi^{(2,1)}\left(\varphi_2-ia_2\chi^{(2)}-\frac{i\lambda a_2}{2m}\chi^{(2)+}\chi^{(2)}\varphi_2\right).$$

Solving for $\chi^{(1,2)}$ and $\chi^{(2,1)}$ one gets

$$\chi^{(1,2)} = \frac{1}{a_1-a_2}\left\{2a_1\chi^{(1)}-(a_1+a_2)\chi^{(2)}\right\}-\frac{i\lambda(a_1+a_2)}{2m(a_1-a_2)^2}\left\{a_1(a_1+a_2)\chi^{(1)+}\chi^{(1)}\chi^{(2)}+2a_1a_2\chi^{(2)+}\chi^{(2)}\chi^{(1)}\right\} \tag{10}$$
$$\chi^{(2,1)} = \quad (1 \leftrightarrow 2)$$

Note that $\chi^{(1,2)}$ does not depend explicitly on φ. To show that the so defined $\chi^{(1,2)}$ and $\chi^{(2,1)}$ are the Bäcklund transforms of $\varphi^{(1)}$ and $\varphi^{(2)}$ one verifies that they solve Eq.(7). Since the family of functions $\varphi^{(1,2)}$ contains four unrelated parameters (two in $\chi^{(1)}$ and two in $\chi^{(2)}$) it is evident that

$$\varphi^{(1,2)} = \mathcal{B}(a_2)\mathcal{B}(a_1)\varphi = \mathcal{B}(a_1)\mathcal{B}(a_2)\varphi \quad .$$

With the help of Eqs. (9) and (10) we can construct solutions of the Thirring equations as polynomials in $\chi^{(j)}$, if we start with the initial solution $\varphi = 0$. Then Eq.(7) states

$$\partial_\xi \chi^{(j)} = m a_j^{-1}\chi^{(j)}$$
$$\partial_\eta \chi^{(j)} = -m a_j \chi^{(j)} \quad .$$

Hence

$$\chi^{(j)} = A^{(j)} \exp\left(m a_j^{-1}\xi - m a_j \eta\right). \tag{11}$$

Here the $A^{(j)}$ is an arbitrary constant anticommuting generator of a Grassmann algebra. Because of (8)

$$\varphi_1^{(j)} = \chi^{(j)}, \quad \varphi_2^{(j)} = -ia_j\chi^{(j)} \quad .$$

Using Eqs.(10) and (9) we get

$$\varphi_1^{(1,2)} = \frac{a_1+a_2}{a_1-a_2}\left\{\chi^{(1)}-\chi^{(2)}\right\}-\frac{i\lambda(a_1+a_2)}{2m(a_1-a_2)^2}\left\{a_1(a_1+3a_2)\chi^{(1)+}\chi^{(1)}\chi^{(2)}+a_2(a_2+3a_1)\chi^{(2)+}\chi^{(2)}\chi^{(1)}\right\}$$

$$\varphi_2^{(1,2)} = -i\,\frac{a_1+a_2}{a_1-a_2}\left\{a_1\chi^{(1)}-a_2\chi^{(2)}\right\}-\frac{\lambda a_1a_2(a_1+a_2)}{2m(a_1-a_2)^2}\left\{(a_2+3a_1)\chi^{(1)+}\chi^{(1)}\chi^{(2)}+(a_1+3a_2)\chi^{(2)+}\chi^{(2)}\chi^{(1)}\right\}.$$

By successive application of this method one achieves higher and higher polynomials in $\chi^{(j)}$ and $\chi^{(j)+}$ which solve the Thirring equations.

The main difference of the solutions obtained here to the ones in the commuting Thirring model is that the solutions are not bounded at infinity, as "a" is a real parameter. Nevertheless the solutions for the commuting and the anticommuting cases are closely related.

The one-soliton solution of Orfanidis and Wang[4] for the commuting Thirring model can be written as

$$\begin{aligned} \varphi_1^{(a)} &= \chi^{(a)}\left(1-\frac{i\lambda a}{4m}\chi^{(a)+}\chi^{(a)}\right)^{-1} \\ \varphi_2^{(a)} &= -ia\,\chi^{(a)}\left(1+\frac{i\lambda a}{4m}\chi^{(a)+}\chi^{(a)}\right)^{-1} \end{aligned} \tag{13}$$

where $\chi^{(a)}$ is given by Eq.(11) with $A^{(a)}$ to be considered as an arbitrary c-number. In the case of anticommuting fields $(A^{(a)})^2=(A^{(a)+})^2=0$ and therefore $(\chi^{(a)})^2=0$. It follows that

$$\left(1+\alpha\,\chi^{(a)+}\chi^{(a)}\right)^{-1} = 1-\alpha\,\chi^{(a)+}\chi^{(a)} \quad .$$

With $\chi^{(a)}\chi^{(a)+}\chi^{(a)}=0$ one arrives at the solution (12). Thus in the anticommuting case the denominators of (13) are effectively equal to 1 with the consequence that the solution becomes unbounded.

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References and Footnotes

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