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1/N Expansions in $P(\phi^2)_{4-\varepsilon}$ Theory

I. Massless Theory, $0 < \varepsilon < 2$

by

K. Symanzik

Abstract

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1/N expansions in O(N) invariant one-scale massless $P(\vec{\phi}^2)_{4-\varepsilon}$ theory are derived for finite and infinite bare coupling constant. Since superficially relevant $(\vec{\phi}^2)^m$, $m \geq 3$, interactions can hereby be transformed into interactions irrelevant for large-distance behaviour, the scale-invariant limit theory is universal.

Introduction

In this series of papers, we analyze the 1/N expansion in one-scale $P(\vec{\phi}^2)_{4-\varepsilon}$ theory [+] in more detail than seems to have been done so far. This expansion constitutes a frame in which a variety of structural results can be proven. These results are then naturally abstracted from that frame. They concern, in particular, critical phenomena. [++] In this first paper, we consider massless O(N) invariant $P(\vec{\phi}^2)_{4-\varepsilon}$ theory, for $0<\varepsilon<2$. This corresponds /4-6/ to critical-point ($T = T_c$) behaviour of e.g. $4-\varepsilon$ dimensional lattices of N-component spins in O(N) symmetric short-range coupling, with structural instability /1/ and crossover phenomena /7/ disregarded for brevity. Since the treatment is not restricted to small ε, the inclusion of continuous ε merely serves to make the behaviour at $\varepsilon = 1$ more transparent from the formal side. $\varepsilon = 2$ seems, within the massless theory, best to be treated as a limit from $\varepsilon<2$, which we have not analyzed yet. - The next paper will deal with the massive $P(\vec{\phi}^2)_{4-\varepsilon}$ theory for $0<\varepsilon\leq 2$ in the one-phase region ($T > T_c$) and with the problem of the limit of vanishing mass. We plan to treat in later papers broken O(N) symmetry, as well as the extensions of the model to quantum statistical mechanics.

[+] For references, see ref. /1/. More recent papers are, e.g., refs. /2,3/.

[++] For justification of use of the field theoretical model as mimikry of real substances under critical conditions, see refs. /4-6/.

In sect. 1, we discuss the region $0 < \varepsilon < 1$ and develop the general formalism of 1/N expansion. Our treatment is based on the Schwinger-Dyson equations [+].

[+] The author's starting point for this approach were papers by R.P. Van Royen /8/, which also used Schwinger-Dyson equations but in a different way.

The root of the scaling behaviour at large distances is the removability, upon multiplicative renormalization, of the cutoff stemming from the finiteness of the bare coupling constant. While the proof of this renormalizability can be founded directly on standard renormalization theory, we give in Appendix A a simple recursive proof for renormalizability for $0 < \varepsilon \leq 2/3$ by showing that the derivatives of amplitudes with respect to the cutoff can be reintegrated to infinity. Although the method used is too primitive to allow by itself to prove renormalizability for $\varepsilon > 2/3$, we present it since partial results are valid for all $\varepsilon < 2$.

In sect. 2 we discuss the new features arising for $\varepsilon = 1$, in a broad way, and in sect. 3 the region $1 \leq \varepsilon < 2$ in a more concise way. The emphasis hereby is on the universality of the scale-invariant limit theory itself. Our treatment is based on a cancellation between diagrams, discovered by G. Parisi. Proving and refining this cancellation, we show that naively relevant interaction terms, $(\vec{\phi}^2)^m$ with $m \geq 3$, can be transformed into irrelevant ones, which modify the approach to scaling but not the scale-invariant limit theory itself. Thus, the mentioned interaction terms are redundant ones in the sense of Wegner /9/. - In sect. 4, we discuss the corrections to scaling arising within the massless-theory. Appendices B and C concern technical matters. - In the

discussion, sect. 5, we point out a limit on apriori validity of qualitative deductions from 1/N expansions.

1. $0<\varepsilon<1$ and general formalism

1.1 Lagrangian and unrenormalized vertex functions

We choose the Lagrangian

$$(1.1)\quad \mathcal{L} = -\tfrac{1}{2}:\partial_\mu\vec{\phi}\,\partial_\mu\vec{\phi}: - \tfrac{1}{2}m_{Bo}^2:\vec{\phi}^2: - \tfrac{1}{8}g_B:(\vec{\phi}^2)^2: + \vec{J}_B(x)\vec{\phi} + \tfrac{1}{2}\bar{K}_B(x):\vec{\phi}^2: - \tfrac{1}{2}\bar{K}_B(x)^2 V_B$$

treated Euclidean throughout. Normal products here and later on are formed with true (source free) vacuum contractions. m_{Bo}^2 is the bare-mass squared of the massless theory. With $Z\{J,K\} = {}^{J,K}\langle\,|\,\rangle^{J,K} = \exp G\{J,K\}$ the vacuum transition element in the presence of sources, Green's functions (GFs) are defined by

$$G_{i_1\ldots i_{2n}}(x_1\ldots x_{2n}, y_1\ldots y_\ell) = [\delta/\delta J_{i_1}(x_1)]\ldots[\delta/\delta K(y_1)]\ldots G\{J,K\}|_{J=K=0}$$

and vertex functions (VFs), their amputated one-particle irreducible parts, by Legendre transformation:

$$G_{,x}\{J,K\} = A(x;J,K\},\quad G_{,y}\{J,K\} = B(y;J,K\}$$

$$\Gamma\{A,B\} = G\{J(\cdot;A,B\},K(\cdot;A,B\}\} - \int dx\,A(x)J(x;A,B\} - \int dy\,B(y)K(y;A,B\},$$

$$\Gamma_{i_1\ldots i_{2n}}(x_1\ldots x_{2n}, y_1\ldots y_\ell) = [\delta/\delta A_{i_1}(x_1)]\ldots[\delta/\delta B(y_1)]\ldots\Gamma\{A,B\}|_{A=B=0}$$

with Fourier transforms $(2\pi)^{4-\varepsilon}\delta(\Sigma p+\Sigma q)\Gamma_{i_1\ldots i_{2n}}(p_1\ldots p_{2n}, q_1\ldots q_\ell)$. For brevity, we shall often refer to the respective GFs and VFs by $G_{2n\ell}$ and $\Gamma_{2n\ell}$. Propagators are the negative inverses of two-point VFs:

$$G_{20}(p(-p),\,) = -\Gamma_{20}(p(-p),\,)^{-1},$$

$$G_{02}(\,,q(-q)) = -\Gamma_{02}(\,,q(-q))^{-1}.$$

We shall be using GFs and VFs (after subtractions are effected) only at nonexceptional momenta (i.e. no nontrivial partial sum of momenta must vanish), where they are even analytic functions /10/, also termwise in the 1/N expansion to be developed. We are suppressing parameter dependence (presently, on g_B, ε, and N) in the notation, in particular the fixed one on ε and N, unless desirable for clarity.

The last term in (1.1) has the effect that

$$(1.2a)\quad G_B(\,,y_1y_2) = \tfrac{1}{4}\langle T(:\vec{\phi}^2(y_1)::\vec{\phi}^2(y_2):)\rangle - V_B\,\delta(y_1-y_2)$$

where T denotes Euclidean "time" ordering. The first term on the r.h.s. possesses a Källén-Lehmann representation, and (1.2a) reads in Fourier form

$$(1.2b)\quad -q^2\int_0^\infty d\kappa^2[\kappa^2(q^2+\kappa^2)]^{-1}\rho(\kappa^2) = \int_0^\infty d\kappa^2(q^2+\kappa^2)^{-1}\rho(\kappa^2) - \int_0^\infty d\kappa^2(\kappa^2)^{-1}\rho(\kappa^2).$$

The subtraction at zero momentum, or by contact term in coordinate space, is introduced to allow the l.h.s. in (1.2b) to scale at small momenta since, as we shall see later, $\rho(\kappa^2)$ prop. $(\kappa^2)^c$ for $\kappa^2 \to 0$ with

$c = \frac{1}{2}\varepsilon + O(N^{-1}) > 0$ in 1/N expansion. Also for V_B we shall find a convenient expression.

We pass from (1.1) to the equivalent Lagrangian

$$(1.3)\quad \mathcal{L}' = \mathcal{L} + \tfrac{1}{2} V_B \left(\bar{K}_B(x) - \tfrac{1}{2} V_B^{-1} : \vec{\phi}^2 : + N^{-\frac{1}{2}} \sigma\right)^2$$

which, using (C.4) of appendix C for normal ordering and introducing the source function $K_B(x) = V_B N^{-\frac{1}{2}} \bar{K}_B(x)$, becomes

$$\mathcal{L}' = -\tfrac{1}{2} : \partial_\mu \vec{\phi} \partial_\mu \vec{\phi} : - \tfrac{1}{2}(m_{B0}^2 - E_1 V_B^{-1}) : \vec{\phi}^2 : +$$

$$(1.4)\quad + \tfrac{1}{2} V_B N^{-1} : \sigma^2 : - \tfrac{1}{2} N^{-\frac{1}{2}} : \sigma \vec{\phi}^2 : -$$

$$- \tfrac{1}{8}(g_B - V_B^{-1}) : (\vec{\phi}^2)^2 : + \vec{J}_B(x) \vec{\phi} + K_B(x) \sigma .$$

The Schwinger-Dyson (SD) equations to (1.4) are

$$(1.5a)\quad \Gamma_{Bij}(p(-p),) =$$

$$= -(p^2 + m_{B0}^2 - E_1 V_B^{-1}) + F_{ij}(p(-p), ; \Gamma_B),$$

$$(1.5b)\quad \Gamma_B(, q(-q)) = N^{-1} V_B + F(, q(-q); \Gamma_B),$$

$$(1.5c)\quad \Gamma_{Bi_1i_2}(p_1 p_2, q) = - N^{-\frac{1}{2}} \delta_{i_1 i_2} + F_{i_1 i_2}(p_1 p_2, q; \Gamma_B),$$

$$(1.5d)\quad \Gamma_{Bi_1i_2i_3i_4}(p_1 \dots p_4,) =$$

$$= -(g_B - V_B^{-1})(\delta_{i_1 i_2}\delta_{i_3 i_4} + \dots + \dots) + F_{i_1 \dots i_4}(p_1 \dots p_4; \Gamma_B).$$

The expressions F... on the r.h.s. are integrals multilinear in the $\Gamma_{B2n\ell}$ $(n+\ell > 1)$ and the G_{B20}, G_{B02}, arrived at by well-known manipulations (see, e.g., /11/) and written simplest by introducing functions with certain irreducibility properties (Bethe-Salpeter-kernels and their generalizations). We shall use only the homogeneity property: If

$$(1.6a)\quad \Gamma'((2n),(\ell)) = a^n b^\ell \Gamma((2n),(\ell))$$

then

$$(1.6b)\quad F((2n),(\ell); \Gamma') = a^n b^\ell F((2n),(\ell); \Gamma).$$

For masslessness of the theory described by (1.5), we must set

$$(1.7a)\quad (m_{B0}^2 - E_1 V_B^{-1})\delta_{ij} = F_{ij}(00, ; \Gamma_B)$$

and

$$(1.7b)\quad (g_B - V_B^{-1})(\delta_{i_1 i_2}\delta_{i_3 i_4} + \dots + \dots) = F_{i_1 \dots i_4}(0000; \Gamma_B).$$

We furthermore extract from $F(, q(-q); \Gamma_B)$ in (1.5b) the part of order N^0 corresponding to the two-corners ϕ-line loop:

$$(1.8a)\quad \tfrac{1}{2} N (N^{-1/2})^2 \int dk\, (2\pi)^{-4+\varepsilon} (k^2)^{-1} [(k+q)^2]^{-1} =$$

$$= a(\varepsilon) |q|^{-\varepsilon}$$

with

(1.8b) $$a(\varepsilon) = \tfrac{1}{2}(4\pi)^{-2+\frac{1}{2}\varepsilon}\,\Gamma(2-\varepsilon)^{-1}\,\Gamma(\tfrac{1}{2}\varepsilon)\,\Gamma(1-\tfrac{1}{2}\varepsilon)^2 .$$

In (1.8a) the factor N stems from $\delta_{i'i'} = N$, and the factor $(N^{-1/2})^2$ from the two $\sigma\phi^2$ vertices. The VFs in 1/N expansion can now be generated from (1.5) with zeroth-order in N^{-1}

(1.9a) $$\Gamma^0_{ij}(p(-p),;g_B) = -p^2\delta_{ij} ,$$

(1.9b) $$\Gamma^0(,q(-q);g_B) = V_B N^{-1} + a(\varepsilon)|q|^{-\varepsilon} ,$$

and lowest order

(1.9c) $$\Gamma^0_{i_1 i_2}(p_1 p_2, q; g_B) = -N^{-1/2}\delta_{i_1 i_2}$$

whereby $V_B N^{-1} = a(\varepsilon)\Lambda^{-\varepsilon}$ is an UV cutoff term held fixed in the expansion. The interaction is a trilinear one based on the $\sigma\phi^2$ term in (1.4). The ϕ^4 term there effects the subtraction of F_{40} at zero momenta in (1.5d) as required for masslessness just as does the self mass term in (1.5a): Straightforward power counting (e.g., /1/) yields for the VFs the IR degree

(1.10) $$\vartheta_{IR\,2n\ell} = 4 - 2n - 2\ell + \varepsilon(n-1)$$

shown in table 1. For $\vartheta_{IR} > 0$ we have (due to $0 < \varepsilon < 1$) the appropriate subtraction at zero momenta by the positions (1.7), while $\vartheta_{IR} \leq 0$ indicates IR singularity at zero momenta.

The only UV divergence, of degree $2-\varepsilon$ in lowest order, occurs in F_{20} in (1.5a) and is removed by the subtraction term (1.7a). With the Feynman rules following from (1.9), (1.7a) becomes

(1.11a) $$m^2_{B0} - E_1 V_B^{-1} = (V_B N^{-1})^{-2/\varepsilon}\,\mu(\varepsilon, N).$$

The function $\mu(\varepsilon, N)$ has simple poles with computable residua /12/ at $\varepsilon = 2/n$, n = 2, 3, 4 ... and is highly model dependent; it will not concern us here although it has also a 1/N expansion. (1.7b) becomes

(1.12b) $$g_B - V_B^{-1} = (V_B N^{-1})^{-1} f(\varepsilon, N)$$

where $$f(\varepsilon, N) = \sum_{K=2}^{\infty} N^{-K} f_K(\varepsilon)$$

with $$f_2(\varepsilon) = 2\,B(1-\tfrac{1}{2}\varepsilon, 1-\tfrac{1}{2}\varepsilon)^{-1} B(1+\tfrac{1}{2}\varepsilon, 2-\tfrac{1}{2}\varepsilon)^{-1}.$$

For the subtraction term in (1.2) we thus obtain

(1.13) $$V_B = g_B^{-1}\left[1 + N f(\varepsilon, N)\right].$$

1.2 Renormalization

We set, in obvious notation for the momenta sets and indices,

(1.14) $$\Gamma_{..}((2n),(\ell);\mu,u) =$$

$$= Z_3(g_B\mu^{-\varepsilon})^n Z_2(g_B\mu^{-\varepsilon})^\ell\,\Gamma_{B..}((2n),(\ell);g_B).$$

To define the Z factors, we introduce "symmetry points" to normalization mass μ: For Γ_{2n0} we choose

(1.15a) $p_1 = p_2 = \dots = p_{2n-1} = -(2n-1)^{-1} p_{2n}\,,\ p_1^2 = \mu^2$

and for Γ_{2n1}

(1.15b) $p_1 = p_2 = \dots = p_{2n} = -(2n)^{-1} q\,,\ p_1^2 = \mu^2$

denoting the functions at these points (with the O(N) index part, which is the sum of $\delta_{i_1 i_2}\delta_{i_3 i_4}\cdots\delta_{i_{2n-1} i_{2n}}$ over all (2n-1)!! pairings, factored out) as $\Gamma_{2n0}(\mu)$ and $\Gamma_{2n1}(\mu)$, respectively, and similarly for the F in (1.5a, c, d). We set

(1.16a) $Z_3(g_0\mu^{-\varepsilon}) = -\mu^2\,\Gamma_{B20}(\mu)^{-1}\,,$

(1.16b) $Z_3(g_0\mu^{-\varepsilon})Z_2(g_B\mu^{-\varepsilon}) = -N^{-1/2}\,\Gamma_{B21}(\mu)^{-1}$

and

(1.16c) $Z_2(g_B\mu^{-\varepsilon})^2 V_B N^{-1} = \mu^{-\varepsilon} u\,,$

introducing hereby a dimensionless parameter u as convenient substitute for (approximately, see (1.32) below) $\mu^{\varepsilon} N^{-1} g_B^{-1}$.

The SD equations for the Γ following from (1.14), (1.16), (1.5-7) are

(1.17a) $\Gamma_{ij}(p(-p),;\mu,u) =$

$$= -p^2\delta_{ij} + [F_{ij}(p(-p),;\Gamma) - \delta_{ij}F_{20}(0;\Gamma)]$$
$$-\delta_{ij}\mu^{-2}p^2[F_{20}(\mu;\Gamma) - F_{20}(0;\Gamma)]\,,$$

(1.17b) $\Gamma(,q(-q);\mu,u) = \mu^{-\varepsilon}u + F(,q(-q);\Gamma)\,,$

(1.17c) $\Gamma_{i_1 i_2}(p_1 p_2, q;\mu,u) = -N^{-1/2}\delta_{i_1 i_2} +$

$$+F_{i_1 i_2}(p_1 p_2, q;\Gamma) - \delta_{i_1 i_2}F_{21}(\mu;\Gamma)\,,$$

(1.17d) $\Gamma_{i_1 \dots i_4}(p_1 \dots p_4,;\mu,u) =$

$$= F_{i_1\dots i_4}(p_1\dots p_4;\Gamma) - (\delta_{i_1 i_2}\delta_{i_3 i_4} + \dots + \dots)F_{40}(0;\Gamma).$$

For u > 0, the Γ are multiples of the Γ_B. The 1/N expansion of the Γ can, however, be generated directly from the SD equations (1.17), with negative inverse zeroth-order propagators

(1.18a) $\Gamma^0_{ij}(p(-p),;\mu,u) = -p^2\delta_{ij}$

(1.18b) $\Gamma^0(,q(-q);\mu,u) = \mu^{-\varepsilon}u + a(\varepsilon)|q|^{-\varepsilon}$

and lowest-order vertex

(1.18c) $\Gamma^{0}_{i_1 i_2}(p_1 p_2, q; \mu, u) = -N^{-1/2}\delta_{i_1 i_2}$,

while (1.17d) indicates the IR (and, for u = 0, see below, also UV) subtraction on F_{40}.

The terms in the 1/N expansion of the Γ have separately limits as $u \searrow 0$. The reason is that in the SD equs. (1.17) the correct number of subtractions is built in to accomodate the UV divergence then arising: for u = 0, the IR divergence degree in table 1 is also the UV divergence degree, since the zeroth-order propagators are homogeneous and the IR homogeneity degree is preserved throughout the calculation, by virtue of the subtractions in (1.17a) and (1.17d) with, in 1/N expansion, only logarithmic modifications (see below). Standard renormalization theory (e.g., /13/) is applicable to the theory specified by equs. (1.17), whereby the subtractions are used in the familiar way to disentangle overlapping UV divergences. F_{02} in (1.17b) is superficially convergent with UV degree $-\varepsilon$ but has overlapping subdivergences; in the formula (written for $u \geq 0$)

(1.17c') $\Gamma(, q(-q); \mu, u) = \mu^{-\varepsilon} u + a(\varepsilon)|q|^{-\varepsilon} +$

$$+\left[\int_{\infty}^{1} d\lambda\, \lambda^{-1} q_\mu [\partial/\partial q_\mu] F(, \lambda q(-\lambda q); \Gamma) - a(\varepsilon)|q|^{-\varepsilon}\right]$$

the momentum derivative allows to disentangle those divergences. The momentum

cutoff rendering for $u > 0$ all VFs (except the Φ self mass) finite before subtraction is of order $\mu u^{-1/\varepsilon}$ in view of (1.18b).

In appendix A we give a simple direct recursive proof that the u-derivative in

(1.19) $\Gamma_{\dots}((2n), (\ell); \mu, 0) =$

$$= \Gamma_{\dots}((2n), (\ell); \mu, u) + \int_{u}^{0} du' [\partial/\partial u'] \Gamma_{\dots}((2n), (\ell); \mu, u')$$

has, for $0 < \varepsilon \leq 2/3$, in 1/N expansion only logarithmic singularities at $u' \searrow 0$ such that the integral converges absolutely, and also show that these logarithms sum up to powers with exponents $O(N^{-1})$. For this reason we report that method there, although it is too primitive to allow to prove convergence of the integral in (1.19) for $\varepsilon > 2/3$. We hope to present a more complete, though necessarily more complicated, argument applicable for also larger ε elsewhere.

1.3 Renormalization group equations

From (1.14) and (1.16) follows

(1.20a) $\Gamma_{\dots}((2n), (\ell); \bar{\mu}, \bar{u}) =$

$$= z_3(\bar{\mu}\mu^{-1}; u)^n z_2(\bar{\mu}\mu^{-1}; u)^{\ell}\, \Gamma_{\dots}((2n), (\ell); \mu, u)$$

provided

(1.20b) $$\bar{\mu}^{-\varepsilon}\bar{u} = z_2(\bar{\mu}\mu^{-1}; u)^2 \mu^{-\varepsilon} u.$$

Differentiating (1.20a) w.r.t. $\bar{\mu}$ and setting then $\bar{\mu} = \mu$ yields

(1.21a) $$Op_{2n\ell}\, \Gamma_{i_1 \ldots i_{2n}}((2n), (\ell); \mu, u) = 0$$

with

(1.21b) $$Op_{2n\ell} = \mu[\partial/\partial\mu] + \beta(u)[\partial/\partial u] - 2n\gamma(u) - \ell\eta(u)$$

where, from (1.20b)

(1.22) $$\beta(u) = (\varepsilon + 2\eta(u))\, u.$$

Using (1.17) and (1.21) the parametric functions $\gamma(u)$ and $\eta(u)$ are given by

(1.23a) $$2\gamma(u) = 2 + \mu^{-2} p[\partial/\partial p]\, \Gamma_{20}(p; \mu, u)|_{p=\mu}$$

and

(1.23b) $$2\gamma(u) + \eta(u) = N^{1/2} p[\partial/\partial p]\, \Gamma_{21}(p; \mu, u)|_{p=\mu}$$

where symmetry point notation is used.

Both functions $\gamma(u)$ and $\eta(u)$ are $O(N^{-1})$ and possess limits for $u \searrow 0$ since the $\Gamma_{\ldots}$ do, in 1/N expansion. For $u \nearrow \infty$, both functions are $O(u^{-1})$, by graph inspection, as a consequence of (1.18b). Thus, in view of (1.22), $u = 0$ is an IR fixed point and $u = \infty$ an UV fixed point in usual renormalization group terminology /1, 14/. Since in the 1/N expansion sense,

$|\eta(u)| = O(N^{-1}) \ll \varepsilon$, these are the only fixed points.

Reintegrating (1.21) we find the expressions for the factors in (1.20a):

(1.24a) $$z_3(\bar{\mu}\mu^{-1}; u) = \exp[2\int_u^{\bar{u}} du'\, \beta(u')^{-1} \gamma(u')]$$

(1.24b) $$z_2(\bar{\mu}\mu^{-1}; u) = \exp[\int_u^{\bar{u}} du'\, \beta(u')^{-1} \eta(u')] = (\bar{u}u^{-1})^{1/2} (\bar{\mu}\mu^{-1})^{-\varepsilon/2}$$

where

(1.25a) $$\bar{u} = \bar{U}(u, \bar{\mu}\mu^{-1}) = \rho^{-1}(\rho(u) + \ln(\bar{\mu}\mu^{-1}))$$

with

(1.25b) $$\rho(u) = \int^{u} du'\, \beta(u')^{-1}.$$

The formulae showing the behaviour of $\bar{U}(u,\lambda)$ for $\lambda \searrow 0$ and $\lambda \nearrow \infty$ are:

As $\lambda \searrow 0$:

(1.26a) $\bar{U}(u,\lambda) = F_0^{-1}(\lambda^{\varepsilon+2\eta(0)} F_0(u))$

with

(1.26b) $F_0(u) = u \exp[(\varepsilon + 2\eta(0))\, \rho_0(u)]$

and

(1.26c) $\rho_0(u) = \int_0^u du'\, u'^{-1} [(\varepsilon + 2\eta(u'))^{-1} - (\varepsilon + 2\eta(0))^{-1}] =$

As $\lambda \nearrow \infty$: $\qquad = O(u (\ln u)^{\cdot})$.

(1.27a) $\bar{U}(u,\lambda) = F_\infty^{-1}(\lambda^{\varepsilon} F_\infty(u))$

with

(1.27b) $F_\infty(u) = u \exp[\varepsilon\, \rho_\infty(u)]$

and

(1.27c) $\rho_\infty(u) = \int_\infty^u du'\, u'^{-1} [(\varepsilon + 2\eta(u'))^{-1} - \varepsilon^{-1}] = O(u^{-1})$.

Hereby the estimate in (1.26c) is proven in appendix A for $0 < \varepsilon \leq 2/3$ whereby the logarithms sum to powers if the 1/N expansion is summed. The estimate of $\rho_0(u)$ for larger ε will be remarked upon in sect. 4.

With $\bar{\mu} = \lambda\mu$, (1.20a) becomes

(1.28) $\Gamma_{\cdot\cdot}^{\cdot}(\lambda(2n), \lambda(\ell); \mu, u) = \lambda^{4-2n-2\ell+\varepsilon(n-1)} \cdot$

$$\cdot Z_3(\lambda,u)^{-n} Z_2(\lambda,u)^{-\ell}\, \Gamma_{\cdot\cdot}^{\cdot}((2n),(\ell); \mu, \bar{U}(u,\lambda)).$$

Letting $\lambda \nearrow \infty$ and exploiting that (for $u > 0$) the theory is super-renormalizable such that in equs. (1.5) for large (nonexceptional) momenta the Born terms dominate, we find by comparison of (1.28) with (1.14), using also (1.24) and (1.27)

(1.29a) $Z_3(g_0 \mu^{-\varepsilon}) = \exp[2 \int_\infty^u du'\, \beta(u')^{-1} \gamma(u')]$,

(1.29b) $Z_2(g_0 \mu^{-\varepsilon}) = \exp[\int_\infty^u du'\, \beta(u')^{-1} \eta(u')] =$

$$= u^{1/2} F_\infty(u)^{-1/2}$$

and, from (1.16c)

(1.30) $V_B = \mu^{-\varepsilon} N F_\infty(u)$.

Setting

$$\Gamma_{40}(\mu; \mu, u) = -\mu^{\varepsilon} f(u, \varepsilon, N)$$

where $f(u,\varepsilon,N) = O(N^{-2}) = u^{-1} f(\varepsilon,N) + O(u^{-2})$

as $u \nearrow \infty$, we find from (1.5d) in a similar way

(1.31) $$g_B - V_B^{-1} = f(\varepsilon,N) F_\infty(u)^{-1} \mu^\varepsilon$$

with, as comparison with (1.30) and (1.12b) shows, $f(\varepsilon,N)$ being identical with the function introduced in (1.12b).

Finally, combining (1.30) and (1.31) yields

(1.32) $$g_B = \mu^\varepsilon F_\infty(u)^{-1} (N^{-1} + f(\varepsilon,N))$$

which, in combination with (1.27b, c), shows that we are using u as a measure of $\mu^\varepsilon (g_B N)^{-1}$. GFs, VFs and thus $\gamma(u)$ and $\eta(u)$ have asymptotic expansions /12/ for $u \nearrow \infty$ in terms of powers $u^{-\kappa - 2r\varepsilon^{-1}}$, κ and r nonnegative integers (for $\varepsilon = 2n^{-1}$, n = 2, 3 .., also logarithms appear), whereby the constants that remained uncomputable in the perturbation theoretical approach of ref. /12/ are here obtained in 1/N expansion. This way of computing them was pointed out by G. Parisi /15/.

From (1.29), (1.26), (1.27) and (1.32) we find

(1.33a) $$\lim_{g_B \to \infty} \left\{ (N g_B \mu^{-\varepsilon})^{2\gamma(0)\varepsilon^{-1}} Z_3(g_B \mu^{-\varepsilon}) \right\} = \\ = const = 1 + O(N^{-1})$$

and

(1.33b) $$\lim_{g_B \to \infty} \left\{ (N g_B \mu^{-\varepsilon})^{\eta(0)\varepsilon^{-1}} Z_2(g_B \mu^{-\varepsilon}) \right\} = \\ = const' = 1 + O(N^{-1}).$$

1.4 Scaling and conformal invariance

For $\lambda \downarrow 0$, we rewrite (1.28) as

(1.34) $$\Gamma_{i_1 \ldots i_{2n}}(\lambda(2n), \lambda(\ell); \mu, u) = \\ = \lambda^{4-\varepsilon - 2n(1-\frac{1}{2}\varepsilon+\gamma(0)) - \ell(2+\eta(0))} \cdot \\ \cdot \exp\left\{ \int_{\bar{u}(\lambda,u)}^{u} du' \beta(u')^{-1} [2n(\gamma(u')-\gamma(0)) + \ell(\eta(u')-\eta(0))] \right\} . \\ \cdot \Gamma_{i_1 \ldots i_{2n}}((2n), (\ell); \mu, \bar{u}(\lambda,u)).$$

This formula together with (1.26) shows, within the 1/N expansion, scaling behaviour of the Γ at small momenta with IR dimensions $1 - \frac{1}{2}\varepsilon + \gamma(0)$ for the ϕ and $2 + \eta(0)$ for the σ field. Thus, in statistical-mechanics (SM) notation:

(1.35a) $\gamma(0) = \frac{1}{2}\eta_{SM}$,

(1.35b) $\eta(0) = 2 - \varepsilon - \nu_{SM}^{-1}$.

The exponent c introduced after (1.2b) is, from (1.34), $\frac{1}{2}\varepsilon + \eta(0)$, and thus $c > 0$ is equivalent to $\alpha_{SM} = -(\varepsilon + 2\eta(0))/(2 - \varepsilon - \eta(0)) < 0$. We shall discuss (1.34) further in sect. 4, and the relevance of $c > 0$ for 1/N expandibility in sect. 5.

For u = 0, in (1.34) the exponential on the r.h.s. becomes unity and the VFs there are the same as on the l.h.s. This shows that, by virtue of the summation of logarithms in 1/N expansion to powers, the VFs for u = 0 are strictly scale invariant. They are, in 1/N expansion, also conformal invariant as seen by examining the Ward identities of (for $g_B < \infty$, broken) conformal invariance, analogous to the ones used in ϕ_4^4 theory /17/.

Conformal invariance can be exploited[+] to facilitate the computation of the 1/N expansions of $\gamma(0)$ and $\eta(0)$ from (1.17) and (1.23). The author found the formulae of ref. /18/ for conformal invariant integration useful. In order to cancel logarithmic divergences correctly, it is preferable not to use the regularization (1.18b) but to keep propagators homogeneous and change the dimensions of suitably selected ones slightly, compensating the dimensional imbalance in formulae (1), (5) or ref. /18/ by letting one coordinate point go to infinity. Calculations are also simplified by employing some latitude in the choice of subtraction momenta, e.g. allowing one such momentum in (1.17c) to be zero if no IR divergence arises thereby. With homogeneous propagators,

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[+] An approach to the computation of critical exponents, based on conformal invariance but not using the 1/N expansion, was proposed by Parisi and Peliti /16/.

subtractions for divergence of positive (not even integer) power degree are unnecessary.

2. $\underline{\varepsilon = 1}$

2.1 u = 0 equations

In view of $\vartheta_{60} = 0$ for $\varepsilon = 1$, from table 1, the SD equations for renormalized VFs (1.17) must be supplemented by

(2.1) $$\Gamma_{i_1 \dots i_6}(p_1 \dots p_6; \mu, 0) = F_{i_1 \dots i_6}(p_1 \dots p_6; T) - (\delta_{i_1 i_2} \cdots \delta_{i_5 i_6} + \dots)[F_{60}(\mu; T) - A(N)]$$

where again symmetry point notation is used. To determine A(N), we demand (1.21) to hold for Γ_{60} and also use $\partial_{p_{60}} F_{60}(p..; T) = 0$ whereby the latter holds because of (1.6), employed before the limit $u \searrow 0$ is taken. Thus

(2.2) $$A(N) = -[6\gamma(0, \varepsilon = 1, N)]^{-1} p[\partial/\partial p] F_{60}(p; T)|_{p=\mu}$$

which is finite, whereby the derivative also allows to disentangle overlapping divergences.

The lowest-order diagram for F_{60} is shown in fig. 1 and is of order N^{-3}. (Here straight lines represent lowest-order ϕ propagators (from (1.18a)), curly lines lowest-order σ propagators (from (1.18b)) and corners carry a factor $N^{-1/2}$ (from (1.18c)). Closed ϕ-line loops, to appear later, carry a factor N.) Since $\gamma(0, \varepsilon=1, N)$ is known to its lowest order N^{-1} and the derivative in (2.2) computable to order N^{-3} without using Γ_{60} itself, A(N) is computable to its lowest order N^{-2}.

The contribution of this apparent bare ϕ^6 vertex to the r.h.s. of (2.2) itself must be considered, however: The diagrams fig. 2a-c each give also a contribution proportional $N(N^{-2})^2 N = N^{-2}$ to the r.h.s. of (2.2). Actually, there takes place a complete cancellation between all diagrams of this order, however, e.g. 2b cancels 2a, and 2c is cancelled by another diagram of the five diagrams not drawn (excluding crossed ones, i.e. with fixed distribution of the external momenta), and similarly for diagram 2d and the related ones. In all cases (for general)

$$1 - N a(\varepsilon)|q|^{-\varepsilon}(N^{-1/2})^2 (a(\varepsilon)|q|^{-\varepsilon})^{-1} = 0.$$

This cancellation was discovered by G. Parisi /15/ and will be extensively used in the following, and will be developed and proven in full generality in sect. 3. Here, it shows that the computation of A(N) from the diagram fig. 1 alone is correct to order N^{-2}.

Thus, Γ_{60} can be computed, using (2.1-2), recursively and uniquely along with the other VFs in 1/N expansion. Equs. (1.23) then give also unique expansions for $\gamma(0)$ and $\eta(0)$.

2.2 $u \geq 0$ equations

We write the equation supplementing equs. (1.17) instead of (2.1)

$$(2.3)\quad \Gamma_{i_1\ldots i_6}(p_1 \ldots p_6; \mu, u) = F_{i_1\ldots i_6}(p_1 \ldots p_6; T) - (\delta_{i_1 i_2} \cdots \delta_{i_5 i_6} + \ldots)[F_{60}(\mu; T) - A(u,N)]$$

and find, analogous to before,

$$(2.4)\quad Op_{60} A(u,N) = p[\partial/\partial p] F_{60}(p; T)|_{p=\mu} = B(u,N).$$

To discuss the solution for A(u,N), we define: A function f(u,N) is called "log-free" if in its 1/N expansion, as $u \searrow 0$, no logarithms appear, that is, only constants (in u) appear and positive (possibly broken) powers of u multiplied by powers of ln u.

2.2.1 Standard solution

Assume B(u,N) in (2.4) to be log-free. Supposing that $\beta(u,N)$ and $\gamma(u,N)$ are log-free,

$$(2.5)\quad A(u,N) = -\tfrac{1}{6}\gamma(u)^{-1} B(0,N) + \int_0^u du' \beta(u')^{-1} \exp\left[6\int_{u'}^{u} du'' \beta(u'')^{-1}\gamma(u'')\right] \cdot \left\{B(u';N) - B(0,N) + \tfrac{1}{6} B(0,N)\beta(u')[\partial/\partial u']\gamma(u';N)^{-1}\right\}$$

is the unique solution of (2.4) that is log-free, since any other solution differs from (2.5) by a multiple of the homogeneous solution

$\exp[6\int^{u} du'\beta(u')^{-1}\gamma(u')]$ which is, because of (1.22) and $\gamma(0) \neq 0$, not log-free. If B(u,N) is not log-free, A(u,N) as given by (2.5) may still exist outside the 1/N expansion provided B(0,N) exists, but will not be log-free because of (2.4). - Since, with (1.17) and (2.3), all superficially divergent graphs become properly subtracted, in the next 1/N order B(u,N) will again be log-free, etc.. Since B(u,N) is log-free in its lowest N^{-3} order, there is a unique solution for all VFs that is log-free. We call it the standard solution.

The construction (2.3) has the consequence that Γ_{60} does not vanish at infinite momenta, because of $A(u,N) \neq F_{60}(\mu;T)$ since $A(u,N)=O(N^{-2})$ while $F_{60}(\mu;T)=O(N^{-3})$. Thus, F_{60} is not finite already to order N^{-3} even for u > 0, since the diagrams figs. 2a,d are not while the others to this order, e.g. fig. 2b, c, are since the broken line is regularized, see (1.18b). This means that there is, formally, a counterterm ϕ^6 in the Lagrangian with coefficient finite to order N^{-2} but infinite to higher order, such that it does not possess a 1/N expansion. We would have to introduce a regularization into the Lagrangian (1.1), e.g. in terms of higher derivatives in the kinetic part, to be able to discuss the possible finiteness of the ϕ^6 coefficient in the standard-solution Lagrangian when a summation over all N^{-1} orders is attempted.

2.2.2 Parisi solution

We consider the possibility of no subtraction for Γ_{60}, i.e.

$$(2.6)\quad {}^{P}\Gamma_{i_1\ldots i_6}(p_1\ldots p_6;\mu,u) = F_{i_1\ldots i_6}(p_1\ldots p_6;{}^{P}T).$$

This is for u > 0 fully acceptable and leads to no divergences since ${}^{P}\Gamma_{60}$ vanishes at infinite momenta. As u ↘ 0, $F_{60}(\ldots;{}^{P}T)$ has lnu in its lowest order N^{-3} from fig. 1, and higher powers of lnu in higher orders, such that a u ↘ 0 limit does not exist in 1/N expansion for ${}^{P}\Gamma_{60}$ and the problem of existence of that limit for all VF diagrams with Γ_{60} as subdiagram arises.

It has been suggested by G. Parisi /15/ that the u ↘ 0 limit exists even in 1/N expansion for almost all Green's functions due to the cancellation already mentioned: The factor of lnu in the Γ_{60} subdiagrams is a constant i.e. a contact term, and the typical combination in figs. 3a and 3b is proportional to (the ε = 1 case of)

$$(2.7)\quad 1 - a(\varepsilon)|q|^{-\varepsilon}[a(\varepsilon)|q|^{-\varepsilon} + \mu^{-\varepsilon}u]^{-1} = \mu^{-\varepsilon}u[a(\varepsilon)|q|^{-\varepsilon} + \mu^{-\varepsilon}u]^{-1}$$

suppressing the lnu by a factor u. This Parisi cancellation will be proven in full generality in sect. 3 for any ε in $1 \leq \varepsilon < 2$. We here give one more example, for G_{22}, in figs. 4a, b, and will discuss for pedagogical reasons its effect in detail in the simplest case.

Parisi cancellation takes place also for all VFs except when one-particle irreducibility conditions forbid it (they exclude the diagram fig. 4b for Γ_{22} leaving the one fig. 4a uncancelled), which allows to locate noncancellation.

Inspection of graphs shows that there remain logarithms only in Γ_{03} (and so in G_{03}) and in all VFs that, to be constructed from GFs, involve G_{03} as factor in a term to be separated out. The simplest such VFs are Γ_{60}, Γ_{41}, and Γ_{22}, and the structure of their lnu-carrying part is given in figs. 5a-d along with the one for Γ_{03}. In formulae:

(2.8a) $$\Gamma_{60} = \sum \Gamma_{21}\Gamma_{21}\Gamma_{21} C + \ln\text{-free},$$

(2.8b) $$\Gamma_{41} = \sum \Gamma_{21}\Gamma_{21}\Gamma_{02} C + \ln\text{-free},$$

(2.8c) $$\Gamma_{22} = \Gamma_{21}\Gamma_{02}\Gamma_{02} C + \ln\text{-free},$$

(2.8d) $$\Gamma_{03} = \Gamma_{02}\Gamma_{02}\Gamma_{02} C + \ln\text{-free}.$$

Here C is the contact term (constant in momentum space) indicated by a black dot in figs. 5a-d. These factorizations will be proven in sect. 3. They can here also be related /15/ to Wilson short-distance-expansion /19/ considerations, which for brevity we leave to the reader. Note that, for the corresponding GFs, a factor Γ_{02} in equs. (2.8) is converted into minus one, such that the σ-argument in the GF becomes a contact term argument, in the lnu-carrying part.

From (2.8a), $Op_{21}\Gamma_{21} = 0$ and $Op_{60}\Gamma_{60} = 0$ we find $Op_{0(-3)}\, C$ = ln-free, and using the earlier discussion of (2.4) we obtain

(2.9) $$C(u,N) = C(N)\exp\left[-3\int_{\infty}^{u} du'\beta(u')^{-1}\eta(u')\right] + \ln\text{-free}$$

where C(N) = $O(N^{-2})$ and computable in 1/N expansion. In view of (1.22) and $\eta(0) \neq 0$, the purely logarithmic part of C(u,N) sums, in 1/N expansion, to

(2.10) $$C(u,N)_{\log} = C'(N)\left\{\exp\left[-3\eta(0)(\varepsilon+2\eta(0))^{-1}\ln u\right] - 1\right\}.$$

Provided $\eta(0) < 0$, which holds to lowest N^{-1} order, as $u \searrow 0$ $C(u,N)_{\log} \to -C'(N) = O(N^{-2})$, which reproduces the term of order N^{-2} obtained in the standard solution. In fact, the "summation" of 1/N expansion implied in this limit reproduces the unique u = 0 solution of sect. 2.1. For u > 0, however, the Parisi ${}^{P}\gamma(u)$ and ${}^{P}\eta(u)$ differ from the ones of the standard solution (see sect. 4.2).

More generally, the lnu-carrying terms can be removed by redefining G_{03} by a contact term which obviously does not change long-distance behaviour. This is done systematically in sect. 3. If, however, one would analyze the small-momentum behaviour of ${}^{P}G_{03}$ in the Parisi solution, i.e. no ϕ^6 term in the Lagrangian, without removing the contact term, one would as a consequence of

(2.11) $${}^{P}G_{03}(\ldots;\mu,u) = -{}^{P}C(u,N) + \ln\text{-free} = -{}^{P}C(u,N) + {}^{P}G_{03\,reg}(\ldots;\mu,u)$$

from (2.8d), (2.9), (1.34) and (1.26) find in 1/N expansion "anomalous logarithms" /2/, i.e. seemingly nonexponentiating ones (they exponentiate for themselves, however, see (2.10)). We here see that they are irrelevant.

2.2.3 General solution

Assume B(u,N) in (2.4) to be log-free up to order N^{-K}. If we then choose a solution for A(u,N) differing from the one given by (2.5) in that order, Γ_{60} to order N^{-K-1} will involve a lnu with p-dependent factor due to a Γ_{60} subgraph and lack of Parisi cancellation, cp. fig. 5a. Thus, B(u,N) to order N^{-K-1} will not be log-free, and so forth to all orders. The logarithms will be confined to the same VFs and structures discussed for the Parisi solution, merely the C(N) in (2.9) will differ from the one of that solution (apart from the difference in also the $\gamma(u)$ and $\eta(u)$ for $u > 0$).

In general, we may parametrize all solutions by the function C(N) in (2.8), (2.9). Its vanishing to all orders in 1/N expansion defines the standard solution. The Parisi solution is the only one with, in that expansion, finite (and actually vanishing) coefficient of ϕ^6 in the Lagrangian, since all other solutions will have a ϕ^6 term with formally (in 1/N expansion) infinite coefficient by the same argument as given for the standard solution.

2.3 Singularities for $u = 0, \varepsilon \nearrow 1$

For $\varepsilon < 1$, Γ_{60} is constructed without subtraction as described in sect. 1. In 1/N expansion, its terms will have singularities up to $(\varepsilon-1)^{-K}$ in order N^{-K-2} for $\varepsilon \nearrow 1$, due to UV almost-divergence. Terms with such factors can be grouped, however, in precisely the way shown in fig. 5a with lnu replaced by $(\varepsilon-1)^{-1}$. The reason is that an $(\varepsilon-1)^{-1}$ factor arises primitively from an almost-divergent final integration and thus can be associated with a momentum-

-independent i.e. contact term in the same combinatorial way as the $u \searrow 0$ logarithms for $\varepsilon = 1$ in the Parisi solution. For contact terms, however, Parisi cancellation as in figs. 2a, b is exact for u = 0. Thus, we have as in (2.8a)

$$\Gamma_{60} = \sum \Gamma_{21}\Gamma_{21}\Gamma_{21}\,\mu^{2(\varepsilon-1)}\,\bar{C}(\varepsilon,N) + \left\{\text{regular at } \varepsilon = 1\right\} \tag{2.12}$$

where, again,

$$\mathcal{R}_{0(-3)}\left[\mu^{2(\varepsilon-1)}\bar{C}(\varepsilon,N)\right] =$$

$$= \mu^{2(\varepsilon-1)}\left[2(\varepsilon-1) + 3\eta(0,\varepsilon,N)\right]\bar{C}(\varepsilon,N) = \left\{\text{regular at } \varepsilon = 1\right\}.$$

Thus

$$\bar{C}(\varepsilon,N) = \left[2(\varepsilon-1) + 3\eta(0,\varepsilon,N)\right]^{-1} r(\varepsilon,N) \tag{2.13}$$

with $r(\varepsilon,N) = O(N^{-3})$ regular at $\varepsilon = 1$. Expanding here the denominator because of $\eta(0,\varepsilon,N) = O(N^{-1})$, we would obtain from (2.13) all the $(\varepsilon-1)^{-K}$ terms in (2.12), whereby nonleading ones require knowledge also of $r(\varepsilon,N)$ near $\varepsilon = 1$. If we formally sum the 1/N expansion, the singular factors can all be lumped into the function $\bar{C}(\varepsilon,N)$ of (2.13) that has, using $\eta(\varepsilon,N) < 0$ as commented before, for $\varepsilon \nearrow 1$ the limit

$$\bar{C}(1,N) = \tfrac{1}{3}\,\eta(0,1,N)^{-1}\,r(1,N) = O(N^{-2})$$

and leads to the same N^{-2} term as found in sect. 2.1. - For the critical exponents $\gamma(0)$ and $\eta(0)$ themselves no such singularities occur, since they are derived from Γ_{20} and Γ_{21} in (1.23) for which Parisi cancellation is complete.

Phenomena of similar type as described here were observed by Abe and Hikami /3/, whose work inspired the present treatment.

3. $1<\varepsilon<2$

3.1 u = 0 equations

For $1<\varepsilon<2$ the SD equations require, because of (1.10) which is also the UV divergence degree, to impose renormalization conditions on Γ_{2m0}, $m \leq 1+[2/(2-\varepsilon)]$. VFs with positive IR = UV divergence degree are subtracted at zero momenta. If $\varepsilon = 2-2(n-1)^{-1}$, Γ_{2n0} has zero divergence degree and must be subtracted at the symmetry point, and as in (2.1-2) a term must be added to restore the validity of (1.21a). This problem has a unique solution, whereby again Parisi cancellation as in sect. 2.1 removes ambiguity. Thus, the SD equations for u = 0 are unique and so are their 1/N expansion solutions and these expansions for $\gamma(0)$ and $\eta(0)$.

3.2 Extended σ-transformation

We consider the Lagrangian, with interaction terms that are permissible by canonical power counting inserted,

$$(3.1) \quad \mathcal{L} = -\tfrac{1}{2} :\partial_\mu \vec{\phi}\, \partial_\mu \vec{\phi}: - \tfrac{1}{2} m_{B0}^2 :\vec{\phi}^2: - \tfrac{1}{8} g_B :(\vec{\phi}^2)^2: -$$
$$- \sum_{m=3}^{n} c_m(\varepsilon,N)\, 2^{-m} (m!)^{-1} g_B^{(4-2m-\varepsilon+m\varepsilon)/\varepsilon} :(\vec{\phi}^2)^m: +$$
$$+ \vec{J}_B(x)\vec{\phi} + \tfrac{1}{2}\bar{K}_B(x) :\vec{\phi}^2: - \tfrac{1}{2} V_B \bar{K}_B(x)^2 -$$
$$- \sum_{m=3}^{\infty} f_m(\varepsilon,N)(m!)^{-1} \bar{K}_B(x)^m g_B^{(4-2m-\varepsilon)/\varepsilon} .$$

This Lagrangian we use for $2-2(n-1)^{-1} < \varepsilon < 2-2n^{-1}$; the case $\varepsilon = 2-2(n-1)^{-1}$ will be discussed later. The c-number terms involving $\bar{K}_B(x)$ redefine the GFs G_{0m}, $m \geq 2$, by contact terms, with a corresponding redefinition of the VFs (see Appendix B). The coefficients V_B and f_m will be specified shortly in terms on the c_m. The latter are here arbitrary. It is convenient, but not essential as will be seen later, to imagine the c_m as $N^{(4-2m-\varepsilon)/\varepsilon}$ times power series in N^{-1}.

In analogy to the step from (1.1) to (1.4) effected by (1.3), we add to (3.1) a term $\Delta\mathcal{L}$ that makes no change in the GFs or VFs:

$$(3.2) \quad \mathcal{L}' = \mathcal{L} + \Delta\mathcal{L} = \mathcal{L} +$$
$$+ \tfrac{1}{2} V_B \left(\bar{K}_B(x) - \tfrac{1}{2} V_B^{-1} :\vec{\phi}^2: + N^{-1/2}\sigma\right)^2 +$$
$$+ \sum_{m=3}^{n} b_m(\varepsilon,N)(m!)^{-1} g_B^{(4-2m-\varepsilon)/\varepsilon} \left(-V_B^{-1}\tfrac{1}{2} :\vec{\phi}^2: + N^{-1/2}\sigma\right)^m +$$
$$+ \sum_{m=3}^{\infty} f_m(\varepsilon,N)(m!)^{-1} \bar{K}_B(x)^m g_B^{(4-2m-\varepsilon)/\varepsilon} .$$

ΔL will make no change if the function $F(z, \bar{K}_B)$, obtained from ΔL by replacing $-\frac{1}{2} V_B^{-1} :\vec{\phi}^2: + N^{-1/2}\sigma$ by z, vanishes at its stationarity point in z (of order $O(\bar{K}_B)$) identically in $\bar{K}_B$. This specifies uniquely the $g_B^{-1}V_B^{-1}f_m$, $m = 3 \dots \infty$, as polynomials in the $g_B^{-1}V_B^{-1}b_m$, $m = 3 \dots n$ (see Appendix B). The choice of V_B and the b_m for given c_m will be described presently.

Setting $\bar{K}_B(x) V_B N^{-1/2} = K_B(x)$, we have

$$
\begin{aligned}
(3.3)\quad L' = & -\tfrac{1}{2} :\partial_\mu \vec{\phi}\,\partial_\mu \vec{\phi}: - \tfrac{1}{2}(m_{B0}^2 - E_1 V_B^{-1}) :\vec{\phi}^2: + \\
& + \tfrac{1}{2} V_B N^{-1} :\sigma^2: - \tfrac{1}{2} N^{-1/2} :\sigma \vec{\phi}^2: - \tfrac{1}{8}(g_B - V_B^{-1}) :(\vec{\phi}^2)^2: + \\
& + \sum_{m=3}^{n} 2^{-m}(m!)^{-1}\left[c_m(\varepsilon, N) - (-V_B g_B^{-1})^m b_m(\varepsilon, N)\right] \cdot \\
& \qquad \cdot g_B^{(4-2m+m\varepsilon-\varepsilon)/\varepsilon} :(\vec{\phi}^2)^m: + \\
& + \sum_{m=3}^{n} b_m(\varepsilon, N)\, g_B^{(4-2m-\varepsilon)/\varepsilon} \, . \\
& \cdot \sum_{k=0}^{m-1} (k!)^{-1}(-V_B)^k \left[(m-k)!\right]^{-1} N^{-\frac{1}{2}(m-k)} 2^{-k} :(\vec{\phi}^2)^k \sigma^{m-k}: + \\
& + \sum_{m=3}^{n} b_m(\varepsilon, N)\, g_B^{(4-2m-\varepsilon)/\varepsilon} \, . \\
& \cdot \sum_{k=1}^{[m/2]} \left[k!(m-2k)!\right]^{-1} E_k V_B^{-2k} :(-\tfrac{1}{2}\vec{\phi}^2 V_B^{-1} + N^{-1/2}\sigma)^{m-2k} (\vec{\phi}^2)^k: + \\
& \qquad + \vec{J}_B(x)\vec{\phi} + K_B(x)\sigma .
\end{aligned}
$$

We shall refer to the three sums herein as C-terms, B-terms, and E-terms, respectively. In forming the normal products, we have used (C.4) of Appendix C. The σ field equation deriving from (3.2) is

$$\sigma = N^{1/2} V_B^{-1} \tfrac{1}{2} :\vec{\phi}^2: \quad + \text{source terms}$$

such that the hypothesis in the derivation of (C.4) is satisfied.

We now choose m_{B0}^2, V_B, and the b_m, $m = 3 \dots n$ such that the Γ_{2m0}, $m = 1 \dots n$, which are for $V_B N^{-1} \searrow 0$ the VFs of positive divergence degree in the 1/N expansion, vanish at zero momenta to all orders in that expansion. Namely, this is sufficient for the constructibility of a massless theory, and it is convenient to arrange it so. The B terms in (3.3), which then in general are non-zero, are IR soft interactions, i.e. their IR degree (which is their operator dimension, ϕ counted with $1-\frac{1}{2}\varepsilon$, σ with 2) is larger than $4-\varepsilon$ by at least $2-\varepsilon$, the value for the $\sigma\phi^4$ term. Consequently, they do not inhibit for $g_B < \infty$ the massless theory in 1/N expansion to be constructed as we shall show later. - Given V_B and the b_m, the f_m in (3.1) are now fixed as explained before. Hereby the f_m terms with $m \geq n + 1$ play no actual role: they modify only GFs that are already singular at zero momenta, and they will vanish in the renormalization ($g_B \nearrow \infty$) limit.

If one applies to (3.1) the simple σ-transformation (1.3) one is left with the ϕ^{2m} interaction terms, and it seems surprising that the Γ_{2m0} vanish at zero momenta for arbitrary choice of the c_m. The reason for this is the Parisi cancellation already encountered in sect. 2.2.2. Consider again the diagrams figs.2a-c three of the eight contributions of related type to Γ_{60} from two ϕ^6 vertices. Each diagram is (for $g_B < \infty$ and $\varepsilon > 1$) singular at zero momenta. However, the sum of these eight diagrams is not singular there, since it has effectively one link

$$(3.4a)\quad a(\varepsilon)|q|^{-\varepsilon} - (a(\varepsilon)|q|^{-\varepsilon})^2[a(\varepsilon)|q|^{-\varepsilon} + V_B N^{-1}]^{-1} =$$

$$= [a(\varepsilon)^{-1}|q|^{\varepsilon} + V_B^{-1}N]^{-1}$$

and two form factors

$$(3.4b)\quad 1 - a(\varepsilon)|q|^{-\varepsilon}[a(\varepsilon)|q|^{-\varepsilon} + V_B N^{-1}]^{-1} =$$

$$= a(\varepsilon)^{-1}|q|^{\varepsilon}[a(\varepsilon)^{-1}|q|^{\varepsilon} + V_B^{-1}N]^{-1}$$

and thus altogether IR degree $4 - \varepsilon - 2 + 2\varepsilon = 2 + \varepsilon$. The subtraction at zero momentum can now be effected by one contribution of the type of the first term on the r.h.s. of fig. 4a, stemming from the contact term prop. f_3 in (3.1) (cp. Appendix B). That this Parisi shielding is a general phenomenon is proven by the extended σ-transformation. That this shielding is, in contrast to (2.7), not a complete cancellation, see (3.4), leads to the IR soft B-terms in (3.3).

3.3 Renormalization

The B-terms in (3.3) lead, treated separately, in 1/N expansion to expressions that are not UV finite even for $g_B < \infty$. A cancellation between the UV

divergences from the terms with equal m takes place, however, Consider the diagrams figs. 5a-f, which employ the m = 3 terms in (3.3) and in fig. 5f an E_1 term. The diagrams a,d,e and f are divergent even for $g_B < \infty$. However, the sum of these diagrams involves the propagator

$$-[a(\varepsilon)|q|^{-\varepsilon} + V_B N^{-1}]^{-1} + V_B^{-1}N =$$

$$= V_B^{-2}N^2[a(\varepsilon)^{-1}|q|^{\varepsilon} + V_B^{-1}N]^{-1}$$

twice and is finite. The E-terms in (3.3) thus merely restore contributions needed for cancellation of UV divergences when these had been removed inadvertently by the normal ordering prescription. The generality of this UV cancellation is shown by the evaluation of the same VF from (3.1) by simple σ-transformation (1.3): then the interaction vertices are ϕ^{2m} and do not give rise to UV divergences. The role of the f_m terms in (3.1) that then remain is described in Appendix B, whereby no momentum integration is involved.

The diagrams figs. 5a, d, e, f together, taking also the power of g_B in the coefficients in (3.3) into account, yield a contribution $O(g_B)$, and so do also the finite diagrams b, c, with the same sign. The subtraction at zero momenta, effected by the term prop. ϕ^4 in (3.3), is now possible and leads to a result with IR power $2 > \varepsilon$ rather than ε, whereby the IR-leading diagrams (of smallest IR power) are d and e. Thus, the diagrams fig. 5 altogether modify the approach to scaling (relative to, e.g., the standard solution, which is defined by $b_m = 0$, $m = 3 .. n$) but, already for dimensional reasons, they do not contribute to the scaling limit $g_B = \infty$.

More systematically, we write the SD equations for the Lagrangian (3.3) upon renormalization (1.14), using also (1.32) to convert from g_B to u. The equs. for Γ_{20}, Γ_{02}, Γ_{21}, and Γ_{40} are as in (1.17). In addition we have the ones for Γ_{2m0} , m = 3 ... n, subtracted once at zero momenta. The functions $\Gamma_{2m'(m-m')}$, m' = 0 ... m - 1, m = 3 .. n are superficially convergent. The Lagrangian (3.3) introduces Born terms for them, added to the $F_{sm'(m-m')}(\Gamma)$. Ordinarily, such Born terms destroy renormalizability, or even finiteness for u > 0. Here, we already showed that these terms give finite results for u > 0. The Born terms carry their (negative) dimension in terms of powers of $\mu^{-\varepsilon}u$ modified in 1/N expansion only by powers of lnu, stemming from Z factors, (1.32), (1.26) and (1.27). Since $\mu u^{-1/\varepsilon}$ is the effective cutoff itself, due to (1.18b), ultimate integration results using these Born terms can only diverge like powers of lnu.

Actually, all such logarithms cancel. Namely, by virtue of (1.21) the validity of which is secured throughout, and which implies (1.34), the question of the existence of the u ↘ 0 limit is identical with the one of proper scaling at small momenta of the u > 0 functions. Since the "dangerous" Born terms are IR softer than renormalizable, their insertion into diagrams gives improved IR behaviour provided (sub)diagrams that are superficially divergent for u ↘ 0 are always subtracted at zero momentum. This condition, however, was what determined the quantities m^2_{Bo}, V_B, and parameters b_m, m = 3 ... n. The diagrams for Γ_{21}, of divergence degree zero, need only be subtracted at finite momentum as they are.

As a consequence, for the question of the existence of the u ↘ 0 limit the

dangerous Born terms are irrelevant since their relative effect vanishes like $u^{(2-\varepsilon)/\varepsilon}$ times logarithms (in 1/N expansion) at least. Regarding renormalizability it thus suffices to consider the standard solution, with no dangerous Born terms. Here the problem is the same as the one discussed in sect. 1.2, however.

So far, we have excluded $\varepsilon = 2-2(n-1)^{-1}$. If this holds, Γ_{2n0} must be subtracted at symmetry momentum and a term added given by the formula analogous to (2.4), (2.5). Such formula is applicable since the analog of B(u,N) is log-free even for the general solution (3.1), due to removal of the lnu-carrying terms, analogous to the ones discussed in sects. 2.2.2-3, by the explicit contact terms.

3.4 A remark

The standard solution, defined by $b_3 = .. = b_n = 0$ in (3.3), implies certain coefficients c_m, m = 3 ... n in (3.1) and $f_m = 0$, $m \geq 3$. In 1/N expansion, these c_m are finite, except for c_n if $\varepsilon = 2-2(n-1)^{-1}$, the example n = 3 of which we discussed in sect. 2. c_m has the sign $(-1)^m$ in its lowest order N^{-m}, from the one-loop diagram with m σ-propagators (cp. fig. 1). If n is odd and $\varepsilon \neq 2-2(n-1)^{-1}$, one might worry about the sign of the highest interaction term $:\phi^{2n}:$ as it might rule out the existence of a ground state if N is too large (such that the leading 1/N term should dominate). The physical case, $\varepsilon = 1$, however, cannot be treated without introducing an extra regularization as discussed in sect. 2.2. No difficulty arises, for all ε, with the Parisi solution which has all c_m vanishing. For $\varepsilon = 2-2(n-1)^{-1}$, only the solutions with $c_n = 0$ have this coefficient well defined in 1/N expansion, as we showed in sect. 2.2 for $\varepsilon = 1$.

4. The approach to scaling

The approach to scaling is determined by the corrections to the scale invariant $u = 0$ theory arising from $u > 0$ ($g_B < \infty$) and from employing in (1.1) a bare mass squared $m_B^2 \neq m_{Bo}^2$ ($T \neq T_c$). The latter correction we shall treat in the next paper on the massive $P(\vec{\phi}^2)_{4-\varepsilon}$ theory. The corrections in the massless theory from $g_B < \infty$ correspond to those in SM caused by the finiteness of the cutoff (e.g., inverse lattice constant) and are model-dependent. In (1.34), they arise from $\bar{U}(u,\lambda) \neq 0$ in the exponential and in the VF on the r.h.s.. Due to (1.23), both effects are closely related.

4.1 Correction exponent ω

In the model considered here, ω is defined (see, e.g. /1/) by

(4.1) $$\omega = \bar{\beta}'(g^*)$$

Here g^* is the Wilson IR fixed point of a Gell-Mann-Low type β -function.

Thus, we define a renormalized coupling constant $g = g(u)$ by

(4.2a) $$-\mu^{\varepsilon}(\delta_{i_1 i_2}\delta_{i_3 i_4} + \cdot\cdot + \cdot\cdot)\, g(u) =$$

$$= \Big\{ \Gamma_{i_1 .. i_4}(p_1 .. p_4; \mu, u) +$$

$$+ \Gamma_{i_1 i_2}(p_1 p_2, (-p_1-p_2); \mu, u)\, G(,(-p_1-p_2)(-p_3-p_4); \mu, u)\,\cdot$$

$$\cdot \Gamma_{i_3 i_4}(p_3 p_4, (-p_3-p_4); \mu, u) + \cdot\cdot + \cdot\cdot \Big\}\Big|_{sy.pt.\, to\, \mu}$$

such that, from (1.18),

(4.2b) $$g(u) = N^{-1}\left[a(\varepsilon)\,2^{-\varepsilon} + u\right]^{-1} + O(N^{-2}).$$

Comparing (1.21) with the equs. of ref. /1/, we have

$$\bar{\beta}(g)^{-1} dg = \beta(u)^{-1} du$$

such that

$$\bar{\beta}(g) = \beta(u)\, g'(u)$$

Since $g^* = g(0)$, we find, using (1.22)

(4.3) $$\omega = (\varepsilon + 2\eta(0))\Big\{1 + \lim_{u\to 0}\left[g'(u)^{-1} u\, g''(u)\right]\Big\}.$$

Thus, if

(4.4a) $$g(u) = g(0) + \sum_i c_i\, u^{a_i} (\ln u)^{k_i}$$

then

(4.4b) $$\omega = (\varepsilon + 2\eta(0)) \min_i a_i .$$

While the function of $\bar{U}(u,\lambda)$ on the r.h.s. of (1.34) is not directly $g(\bar{U}(u,\lambda))$, the correction magnitude will be the same as on the r.h.s. of (4.4a), in view of the fact (cp., e.g., Appendix A) that the corrections to all GFs, and to the VFs appearing in (1.23), have the same exponents a_i as in (4.4a), and thus the same order of magnitude disregarding accidental vanishing of some c_i. Thus, using (1.26), the dominant relative correction in terms of λ to the r.h.s. of (1.34) is λ^{a}. In the two following sections, we give only a sketchy discussion of what determines the a_i in (4.4a) in the present model.

4.2 Parameter-dependent corrections ($\varepsilon \gtrsim 1$)

These corrections arise, for $\varepsilon \gtrsim 1$, in the Parisi or other nonstandard solution relative to the standard one, i.e. they are determined by the choice of the c_m in (3.1). In (3.3), the lowest-dimensional correction term is the $\sigma\phi^4$ interaction of canonical dimension $6-2\varepsilon$ and thus, as discussed in sect. 3.3, IR softer by $2-\varepsilon$ than renormalizable, implying a factor $u^{-1+2/\varepsilon}$ $(\ln u)^{\cdot\cdot}$ if logarithms are not summed.

However, also the $\sigma\phi^4$ term is subject to Parisi shielding in general. A way to obtain the net effect is to go back to (3.1) and use the simple σ-transformation (1.3) instead, with the same V_B as in (3.2). It leaves a ϕ^6 term in the Lagrangian, and also the corresponding contact term prop. $K(x)^3$ which is, however, irrelevant for large-distance behaviour. Estimates can then be based directly on the almost-cancellation between diagrams fig. 3a and 3b, described by (3.4b). The extended σ-transformation then merely assures the correctness of the combinatorics.

(3.4b) or (2.7), where they are applicable, give a factor $|g|^{\varepsilon}$ or, equivalently, u for each index-contracted pair of ϕ-lines. For a $(\bar{\phi}^2)^3$ vertex, this shielding takes place on all three pairs of ϕ-lines, leading to IR-softening by dimension 3ε, unless a condition on the formation of diagrams is violated. These conditions are, for a GF: (1) that a simple two-corner ϕ-line loop in a σ-line must not occur, cp. (1.17c'); (2) that no diagrams with tadpoles be used.

(2) would exclude e.g. diagram fig. 7b which appears to be needed for shielding in diagram fig. 7a. However, in this diagram there are subtractions on Γ_{40} subdiagrams, e.g. the one enclosed by a broken line, and this subtraction reproduces the missing contribution of diagram fig. 7b. Since this holds for all tadpole diagrams, only the condition (1) above matters. For a ϕ^6 vertex, an easy discussion shows: The gain in canonical IR dimension, from shielding, above $6-3\varepsilon$ is: For $G_{2n\ell}$, $n>0$: 2ε if $\ell=1$, ε if $\ell\geq 2$; for G_{02}: ε; for G_{03}: 0; for all other GFs: 3ε. Two or more ϕ^6 vertices lead to higher IR gain and need not be considered.

Using also (1.23), the result is: if logarithms are not summed, the first parameter-dependent correction is soft by a factor $u^{1+2/\varepsilon}$ for G_{2n0} and thus for $\gamma(u)$, by a factor $u^{2/\varepsilon}$ for G_{2n1} and thus for $\eta(u)$, and by a factor $u^{-1+2/\varepsilon}$ for $G_{2n\ell}$, $\ell\geq 2$, excluding the contact term in G_{03}. One observes that these corrections correspond to the excess in canonical dimensions of σ^3, $\sigma^2\phi^2$, and $\sigma\phi^4$, respectively, over $4-\varepsilon$. Thus, in e.g. the last case, if logarithms are to be summed, to determine the true IR dimension of the correction it would suffice to solve the renormalization problem of the operator $\sigma\phi^4$. The method hereto would be analogous to the one of Appendix A, where that problem was solved in principle for the operator σ^2 for $0<\varepsilon\leq 2/3$, but which is easily extendible.

4.3 Parameter-independent corrections

These are the (model dependent) corrections present in the standard solution and arising already for $\varepsilon > 0$. In Appendix A we show: For $\varepsilon \leq 2/3$, with logarithms not summed, $\Gamma(u) = \Gamma(0) + O(u(\ln u)^{\cdot\cdot})$, and thus, due to (1.23), similarly for $\gamma(u)$ and $\eta(u)$. The logarithms are due to the logarithmic superficial divergence of Γ'_{02} (sub)diagrams i.e. Γ_{02} diagrams containing a σ^2 vertex at zero momentum, and they exponentiate, when summing 1/N expansion, to powers of u with exponents $O(N^{-1})$. These exponents are the eigenvalues of certain matrices computable in 1/N expansion. For $\varepsilon = \frac{2}{3}$, also Γ'_{60} (sub)diagrams contribute. - For $\varepsilon > \frac{2}{3}$, higher than logarithmic divergences arise in Γ'_{60} (sub)diagrams, and for $\varepsilon > 1$ also in others.

These higher divergences, however, do not reduce in full the smallness of the $O(u\ln u)$ correction because of Parisi shielding of the divergent (contact term) part of subdiagrams in general. E.g., the corrections to most GFs and to $\gamma(u)$ and $\eta(u)$ presumably remain of the $O(u\ln u)$ smallness, logarithms not being summed. The author has not carried this analysis sufficiently far to be able to make comprehensive statements. It appears, however, that for $\varepsilon \geq 1$, the parameter-independent corrections lead in (4.4a) to lower a_i-values than the parameter-dependent ones, such that universality of ω presumably holds, within the model considered here.

5. Discussion

We have shown how for the model considered, 1/N expansions for all quantities defined within the model can be obtained, and that for the scale-invariant limit theory these expansions are (up to two amplitude renormalization conventions) unique.

The value of the 1/N expansion is not to be seen in its numerical applicability, but in the formal derivations it allows of structural relations: existence of the massless superrenormalizable theory and of its scaling limit (by virtue of complete exponentiation of logarithms in the 1/N expansion), the universality of that limit, factorizability of the corrections to scaling (with, in 1/N expansion, again exponentiation of logarithms) and the like.

In view of this, the question arises as to how far deductions from the 1/N expansions are trustworthy. The obvious consistency condition is that the powers obtained by summing logarithms are not in violation of the principles upon which the summation of the logarithms was based. It is well known that the N^{-1} parts of the exponents obtained (see, e.g., ref. /1/) conform with the inequalities for exponents deduced from general principles (e.g. Källén-Lehmann positivity, Lebowitz inequality). To go beyond this, information must be adduced from outside the 1/N expansion.

Consider the quantity V_B in (1.2), which appeared often thereafter. It is proportional to the specific heat at $T = T_c$, and its 1/N expansion is (1.13).

V_B increases beyond all bounds + if the exponent $c = 2 - \frac{1}{2}\varepsilon - \nu_{SM}^{-1}$

+ This increase is made more plausible by rewriting (1.13) as 1/N expansion for V_B^{-1}. However, the signs of the $f_k(\varepsilon)$, $k \geq 3$, are not easily ascertained. Cp. hereto refs. /20/.

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defined after (1.2) goes to zero through positive values, which is equivalent to $\alpha_{SM} \nearrow 0$. At $\alpha_{SM} \geq 0$, the interpretation of V_B implied by (1.2b) is untenable. Instead, $N^{-1}V_B$ should act as a subtraction at zero momentum in (1.5b), and unless it is then negative+ it causes a zero in $G_{02}(q)$ at positive q^2, and thus a pole in $\Gamma_{02}(q)$ which would be difficult to interpret.

+ The change of sign of V_B when going from large to small N would make the interpretation of (1.3) in terms of functional integrals easier. We do not know whether functional integrals such as used in constructive quantum field theory /21/ may allow to put the ϕ, σ theory on a rigorous basis. To derive, formally, consequences of (1.4) for large N, e.g. estimates on the growth of coefficients in 1/N expansions by functional saddle point methods /22/ one would have to replace σ by $i\sigma$.

Within our approach, for $\alpha_{SM} > 0$, (1.17b) gives $v = -\mu^{\varepsilon} F(,00;\Gamma)$. As $g_B \nearrow \infty$, from (1.33b) and (1.13) we find $|v| \nearrow \infty$ such that in the scaling limit F_{02} in (1.17b) acquires an UV subtraction (at zero momentum) as it will need one by power counting of conformal invariant skeleton expansion diagrams /23/. However, (1.18b) as "zeroth-order approximation" will have no relation to the behaviour for $g_B \nearrow \infty$ of Γ_{02}, which suggests strongly that the 1/N

expansion looses sense for $N \leq N_{crit}(\varepsilon) = N(\alpha_{SM} = 0, \varepsilon)$. In particular, we do not expect the expansions for η_{SM} and ν_{SM} to be meaningful for $N \leq N_{crit}$. For ε small,

$$N_{crit}(\varepsilon) = 4 - 4\varepsilon - 3.04\,\varepsilon^2 + O(\varepsilon^3)$$

while /24/ for $\varepsilon = 1$, $N_{crit} = 1.8$, for $\varepsilon = 2$, $N_{crit} = 1$.

This limitation to $N > N_{crit}$ need not necessarily be one on structural deductions, however. E.g., ω in (4.9) would remain positive due to change of sign of both factors (with min replaced by max) on the r.h.s.. One may even attempt to use the 1/N expansion numerically for small N via Padé approximants +, e.g. diagonal

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+ To improve the approximation, one would also exploit rigorous results for N negative even /25/, and for $\varepsilon = 3$, the linear chain /26/ or anharmonic oscillator /27/.

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ones may allow to set N = 0, corresponding to self-avoiding walk. However, it is preferable to base the structural relations, derived for large N from 1/N expansions, for small N on different expansions (see discussions in Ref. /28/).

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Appendix A. u-derivative of vertex functions

Taking the derivatives w.r.t. u of a VF requires, according to (1.4), sect. 1.2, and the Schwinger action principle, to form the matrix element of the local operator $\frac{1}{2}\mu^{-\varepsilon}\!:\!\sigma^2\!:$ + counter terms, at zero momentum transfer. The counter terms are of $\partial_\mu\phi\partial_\mu\phi$, ϕ^2, ϕ^4, and $\phi^2\sigma$ type (and for $\varepsilon \geq 1$ also of ϕ^6 etc. type, depending on the choice of solution and σ-transformation, see sects. 2, 3). The canonical dimension in the u = 0 limit of σ^2 is $4 = 4 - \varepsilon + \varepsilon$, such that, according to table 1, the superficial divergence degrees of Γ'_{20}, Γ'_{40}, Γ'_{21}, Γ'_{02}, Γ'_{60} are $2+\varepsilon$, 2ε, ε, 0, and $3\varepsilon - 2$, respectively. For reasons given at the end, we shall restrict the explicit discussion here to $0 < \varepsilon \leq 2/3$. The other Γ' are then superficially convergent, and Γ'_{60} will only be needed for $\varepsilon = 2/3$. We call Γ'_α the set of superficially divergent functions Γ'.

For each diagram contributing to Γ'_α we determine the largest proper subdiagram of also Γ'_α type. That largest subdiagram is not unique in cases of overlap. In figs. 8a-c we give three examples, the last one only relevant for $\varepsilon = 2/3$. The broken lines indicate 2-ϕ lines irreducibility, the cross indicates σ^2 insertion.

We can write an integral equation (IE) for Γ'_α of Bethe-Salpeter type. The largest sub-Γ'_α is folded with a kernel, having irreducibility properties, to yield Γ'_α if also the bare matrix element of the local operator (σ^2 or counter term) is added. For Γ'_{20}, Γ'_{40}, and Γ'_{21} the latter part implements the renormalization conditions implied in (1.17), for Γ'_{02} it is $\mu^{-\varepsilon}$, for Γ'_{60} it is zero. A slight complication arises in the cases of overlap: one then must employ certain simple devices to avoid double counting. E.g., neglecting Γ'_{60}, the IE for Γ'_{40} is shown in fig. 9. The seventh term removes the double counting from the sixth term, corresponding to the overlap fig. 8a.

In partly symbolic notation, we can write

$$\Gamma'_{2n\ell} = K_{2n\ell}\,\Gamma'_{\alpha} + P_{20}\,a_{20}\,\Gamma'_{\alpha} + P_{40}\,a_{40}\,\Gamma'_{\alpha} + P_{21}\,a_{21}\,\Gamma'_{\alpha} + P_{02}\,c_{02}. \tag{A.1}$$

$K_{2n\ell}$ is a set of homogeneous integral kernels acting on the set Γ'_{α}. Homogeneity here means that $K_{2n\ell}$ does not involve the normalization momenta explicitly, such that the kernel acting on $\Gamma'_{2n'\ell'}$ obeys

$$Op_{2(n-n')(\ell-\ell')}\,K_{2n\ell,2n'\ell'} = 0 \tag{A.2}$$

with Op from (1.21). $P_{2n'\ell'}$ is short for $\delta_{nn'}\delta_{\ell\ell'}$ and is a projection. Furthermore,

$$a_{20} = -K_{20}(0) - \mu^{-2}p^2[K_{20}(\mu) - K_{20}(0)], \tag{A.3a}$$

$$a_{40} = -K_{40}(0), \qquad a_{21} = -K_{21}(\mu).$$

Hereby symmetry point notation as in (1.17) is used, and the O(N) indices and factors are suppressed. Finally, $c_{02} = \mu^{-\varepsilon}$. For $2n\ell \in \alpha$, (A.1) is the IE described. For $2n\ell \notin \alpha$, (A.1) shows how Γ' is obtained

from Γ'_{α} by quadrature.

We now define the sets of functions Γ'_j, with j ranging over the set 20,21,02 and for $\varepsilon = 2/3$ also 60 as implied in d, but with 40 omitted, by the IEs

$$\Gamma'_j = K\,\Gamma'_j + \Sigma P_{\alpha}\,a_{\alpha}\,\Gamma'_j + P_j\,c_j. \tag{A.4}$$

Here, complementing (A.3a)

$$a_{02} = -K_{02}(\mu), \qquad a_{60} = -K_{60}(\mu) \quad (\varepsilon = 2/3) \tag{A.3b}$$

and

$$c_{20} = -\mu^{-2}p^2, \qquad c_{21} = -N^{-1/2}, \tag{A.5}$$

$$c_{02} = \mu^{-\varepsilon}, \qquad c_{60} = \mu^{2(\varepsilon-1)} \quad (\varepsilon = 2/3).$$

From (A.4) and (A.2) we find

$$Op\,\Gamma'_j = K\,Op\,\Gamma'_j + \Sigma P_{\alpha}\,a_{\alpha}\,Op\,\Gamma'_j + \Sigma P_{\alpha}\,Op\,a_{\alpha}\cdot\Gamma'_j + P_j\,Op\,c_j. \tag{A.6}$$

Comparing this with (A.4), we find (uniquely, in the sense of 1/N expansion)

(A.7) $$\mathrm{Op}\,\Gamma_j' = f_j \Gamma_j' + \sum_k \Gamma_k' A_{kj} .$$

Here k goes over the same set as j, with $\mathrm{Op}\, a_{40} = 0$, in the sense of (A.2), being exploited. Also

(A.8a) $$f_j = c_j^{-1}\,\mathrm{Op}\, c_j$$

such that

(A.8b) $$f_{20} = -2\gamma(u), \qquad f_{21} = -2\gamma(u) - \eta(u)$$

$$f_{02} = -\varepsilon - 2\eta(u), \qquad f_{60} = 2\varepsilon - 2 - 6\gamma(u) \quad (\varepsilon = 2/3).$$

Furthermore,

(A.9a) $$A_{kj} = c_k^{-1}\,\mathrm{Op}\, a_k \cdot \Gamma_j'$$

Hereby, explicitly,

(A.9b) $$c_{20}^{-1}\,\mathrm{Op}\, a_{20} = \mu^2 p[\partial/\partial p]\{p^{-2}[K_{20}(p) - K_{20}(0)]\}|_{p=\mu},$$

$$c_{21}^{-1}\,\mathrm{Op}\, a_{21} = N^{1/2} p[\partial/\partial p]\, K_{21}(p)|_{p=\mu},$$

$$c_{02}^{-1}\,\mathrm{Op}\, a_{02} = -\mu^{\varepsilon} q[\partial/\partial q]\, K_{02}(q)|_{q=\mu},$$

$$c_{60}^{-1}\,\mathrm{Op}\, a_{60} = -\mu^{2-2\varepsilon} p[\partial/\partial p]\, K_{60}(p)|_{p=\mu} \quad (\varepsilon = 2/3).$$

The IEs (A.4) are solved by iteration, which yields the 1/N expansion of the Γ_j'. These iteration solutions satisfy (A.7), and they have termwise limits in 1/N expansion as $u \searrow 0$. The reason for this is that the subtractions in (A.4), which are chosen such as to accomodate the superficial UV divergence at $u = 0$, can be worked into the K in such a way that the resulting K_{sub} possess skeleton expansions in terms of renormalized propagators and VFs, or subtracted such functions. If K_{sub} is finite for $u \searrow 0$ (this is equivalent to being log-free in the sense explained before sect. 2.2.1) up to L loops, then Γ_j' is finite for $u \searrow 0$ up to L + 1 loops.

The manipulations in forming the K_{sub} are familiar from the treatment of coupled integral equations of Schwinger-Dyson type /11/, and their explicit description is omitted for brevity.

That the K_{sub} are finite for $u \searrow 0$ up to L loop is at this stage the induction assumption, extended to L + 1 loops by showing that the VFs occurring in K_{sub} are finite for $u \searrow 0$ up to L + 1 loops. Increasing number of loops means increasing order in N^{-1}, whereby uncorrected two-corner ϕ-line loops in σ-lines, which must not occur, are not generated, due to the one-particle-irreducibility of the Γ_j' and the fact that the $\partial_\mu\phi\,\partial_\mu\phi$ counter term acts only in a corrected ϕ-line.

The finiteness for $u \searrow 0$ of the Γ_j' is supported by (A.7): the A_{kj} are also finite for $u \searrow 0$ since the subtractions and differentiations in (A.9b) are sufficient to render the convolutions in forming the A_{kj} in (A.9a) finite in that limit, similarly as for $K_{sub}\,\Gamma_j'$.

Comparing (A.1) with (A.4), we find (uniquely, in the sense of 1/N expansion)

(A.10) $$T_\alpha' = \sum_j x_j T_j'$$

provided

(A.11) $$\sum_j [P_{02} a_{02} T_j' + P_{60} a_{60} T_j' + c_j P_j] x_j = P_{02} c_{02}$$

is satisfied, with j = 60 and P_{60} only present for $\varepsilon = 2/3$. The remaining task is to show that the x_j, obtainable from (A.11), have for $u \searrow 0$ such a behaviour that (A.10) is integrable as required in (1.19).

We define the 3 x 3 (for $\varepsilon = 2/3$, 4 x 4) matrix U by

(A.12) $$U_{ij} = \delta_{ij} + c_i^{-1} a_i T_j'.$$

Introducing the projection matrix

$$P_{ij} = (\delta_{i20} + \delta_{i21})\delta_{ij} = \delta_{ij} - \bar{P}_{ij}$$

and the vector $e_j = \delta_{j02}$, we can write (A.11) as

(A.13) $$\bar{P}(U-1)x + x = e.$$

Since $Pe = 0$, $Px = 0$ i.e. $x_{20} = x_{21} = 0$, and (A.13) can be written

(A.14) $$\bar{P} U \bar{P} . x = \bar{P} e$$

An easy calculation, using (A.7), (A.8a), (A.9a), and (1.21b) yields

(A.15a) $$\beta(u)[\partial/\partial u] U_{ij}(u) =$$
$$= (-f_i(u) + f_j(u)) U_{ij}(u) + \sum_k U_{ik}(u) A_{kj}(u).$$

The boundary condition on the U_{ij} is

(A.15b) $$U_{ij}(u) = \delta_{ij} + O(u^{-1}) \quad \text{as } u \nearrow \infty$$

This follows from (1.32), which with (1.27) shows that $u \nearrow \infty$ yields the free theory with $g_B = O(u^{-1})$. For this reason, also $A_{kj}(u) = O(u^{-1})$ as $u \nearrow \infty$ holds. The integral equation, to be solved by iteration which suffices for 1/N expansion, equivalent to (A.15) is

(A.16a) $$U_{ij}(u) = \delta_{ij} + a_{ij} u^{(f_j^0 - f_i^0)/\varepsilon} + $$
$$+ \int_{L_{ij}}^{u} du' \beta(u')^{-1} (u/u')^{(f_j^0 - f_i^0)/\varepsilon} .$$
$$\cdot \Big[\sum_k U_{ik}(u') A_{kj}(u') + (f_j(u') - f_j^0 - f_i(u') + f_i^0) U_{ij}(u')\Big].$$

Hereby

(A.16b) $f_{20}^o = f_{21}^o = 0, \quad f_{02}^o = -\varepsilon, \quad f_{60}^o = 2\varepsilon - 2 \quad (\varepsilon = 2/3)$

(A.16c) $a_{ij} = 0 \quad (i = j),$

$a_{20,21} = a_{21,20} = a_{02,20} = a_{02,21} = 0,$

$a_{20,02}$ and $a_{21,02}$ undetermined,

$$\left.\begin{array}{l} a_{60,20} = a_{60,21} = a_{60,02} = a_{02,60} = 0, \\ a_{20,60} \text{ and } a_{21,60} \text{ undetermined.} \end{array}\right\} \varepsilon = 2/3$$

(A.16d) $L_{ij} = 1$ for the pairs

20,02; 21,02; 20,60; 21,60

$L_{ij} = \infty$ for all other pairs.

The nonvanishing a_{ij} could be computed in 1/N expansion from (A.4), (A.12).

We only need the behaviour for $u \searrow 0$ of the $U_{ij}(u)$ occurring in (A.14). It is simplest obtained directly from (A.15a) by solving the eigenvalue equations, for fixed i = 02, and 60 if $\varepsilon = 2/3$:

$$\text{(A.17)} \quad \sum_k [(-f_i(0) + f_j(0))\delta_{kj} + A_{kj}(0) - \lambda\delta_{kj}] v_{ik} = 0.$$

The eigenvalues are

$$\lambda_j = f_j^o - f_i^o + O(N^{-1}).$$

Thus, the $U_{ik}(u)$ behave for $u \searrow 0$ as

$$\text{(A.18)} \quad U_{ik}(u) = \sum_j u^{(f_j^o - f_i^o)/\varepsilon} P_{ikj}(\ln u) +$$

+ smaller terms

where the $P_{ikj}(\ln u)$ are, in 1/N expansion, polynomials in lnu, and the "smaller terms" are so by positive powers of u. For the i needed in (A.14), $f_j^o - f_i^o \geqslant 0$. Thus, since the matrix to be inverted in (A.14) is unity in lowest 1/N order, the nonvanishing x have only logarithmic or smaller terms as $u \searrow 0$. Thus, using (A.10) and (A.1), the Γ' in (1.19) is integrable.

We briefly describe what inhibits this simple inductive proof for $\varepsilon > 2/3$. Then in (A.18) also negative powers of u occur, since $a_{60} = -K_{60}(\mu)$ cannot be replaced by $a_{60} = -K_{60}(0)$, Γ'_{60} being, inspite of superficial divergence, singular at zero momenta. These negative powers, although with small exponent, can build up in the matrix inversion to powers with large negative exponents.

Actually, they do not, because of cancellations, shown in the following way: Introduce the inverse matrix $W = U^{-1}$. From (A.15a), it satisfies

$$\text{(A.19)} \quad \beta(u)[\partial/\partial u] W_{ij}(u) =$$

$$= (-f_j(u) + f_i(u)) W_{ij}(u) - \sum_k A_{ik}(u) W_{kj}(u)$$

with the initial condition analogous to (A.15b). The conversion of (A.20) into integral equations is analogous to (A.16), and the analysis of the $u \searrow 0$ behaviour of W gives in the equs. analogous to (A.17) the same set of eigenvalues λ, and formulae analogous to (A.18). From (A.13) one easily finds

$$\text{(A.20)} \quad x = \bar{P}W\bar{P}e - \bar{P}WP\sum_{n=0}^{\infty}[P(1-W)P]^n PW\bar{P}e$$

the sum breaking off in 1/N expansion. (A.20) shows that in the matrix inversion for (A.14), the negative powers in (A.18) do not build up to large ones since there are no negative powers in P(1-W)P. However, (A.20) admits in $\bar{P}W\bar{P}e$ and $PW\bar{P}e$, the power u^{-1}, and what one learns from (A.14) with (A.18) appears, for $\varepsilon > 2/3$, not to be sufficient to enforce the absence of this power in (A.20), which would spoil integrability in (1.19).

Irrespective of this failure of direct renormalizability proof, (A.10) with x from (A.14) or (A.20) is valid, such that the corrections to scaling, given by the integral in (1.19) as discussed in sect. 4, are linear combinations of u-independent functions times only u-dependent coefficients that vanish (the convergence of the integral in (1.19) supposed) for $u \searrow 0$ with certain power laws, plus smaller corrections. These power laws, which arise from the summation of logarithms in 1/N expansion, are simply related to the solutions of the eigenvalue conditions (A.17).

(A.17), in turn, is identical to the eigenvalue problem to be solved when determining the "anomalous dimensions" at u = 0 of the nonmixing renormalized

operators $\partial_\mu\phi\partial_\mu\phi$, $\phi^2\sigma$, σ^2 (and, for $\varepsilon \geqslant 2/3$, ϕ^6) from (A.7), at zero momentum transfer. (A.10) with (A.1) states that forming the u-derivative corresponds to inserting these renormalized operators, at zero momentum transfer, into the VF with certain VF-independent laws for the coefficients.

It is easy to extend this part of the analysis to all $\varepsilon < 2$.

Appendix B. Combinatorics of contact terms

Let $G\{J,K\} = G^o\{J,K\} + \int dy\, F(K(y))$, with $F(K(y))$ a power series in K(y). The GFs $G_{o\ell}$ are obtained from the $G^o_{o\ell}$ by addition of contact terms (products of δ-functions). We want to exhibit the relation between the corresponding VFs Γ and Γ^o.

From (cp. sect. 1.1)

$$G_{x,}\{J,K\} = G^o_{x,}\{J,K\} = A(x;J,K\} = A^o(x;J,K\}$$

and

$$G_{,y}\{J,K\} = G^o_{,y}\{J,K\} + F'(K(y)) =$$
$$= B(y;J,K\} = B^o(y;J,K\} + F'(K(y))$$

follows for the generating functionals of VFs the relation

$$\Gamma_{,y}\{A,B\} = \Gamma^o_{,y}\{A, B - F'(-\Gamma_{,\cdot}\{A,B\})\}$$

which allows iterative solution for the Γ in terms of the Γ^o. One sees that the Γ are multilinear in the Γ^o, the Γ^o being stacked together, with σ-arguments, at articulation points (cp. figs. 5) of the order of the corresponding terms in the polynomial F. For the inverse relation, Γ^o expressed in terms of Γ, one changes the sign of F.

To determine the f_m in (3.2), we write, simplified,

$$\Delta L = \tfrac{1}{2}(x+a)^2 + B(x) + F(a)$$

with $B(x) = O(x^3)$. Stationarity holds at

$$x = -a - B'(x)$$

to be solved by iteration. For ΔL to vanish at the stationarity point, we have, by Cauchy's formula

$$F(a) = -(2\pi i)^{-1} \oint dz\, [z + B'(z-a)]^{-1} \cdot$$
$$\cdot [1 + B''(z-a)][\tfrac{1}{2}z^2 + B(z-a)],$$

the integration contour encircling the pole close to the origin if a is small.

Appendix C. Normal products

Normal products with true vacuum contractions for two fields σ and ϕ are defined by the generating function

$$(C.1)\quad :\exp[\vec{U}\vec{\phi} + s\sigma]: =$$

$$= \langle \exp[\vec{U}\vec{\phi} + s\sigma]\rangle^{-1} \exp[\vec{U}\vec{\phi} + s\sigma].$$

In our application, we can use the zero-sources relation $\sigma = \frac{1}{2}:\vec{\phi}^2:$ in evaluating the vacuum expectation value:

$$(C.2)\quad \langle \exp[\vec{U}\vec{\phi} + s\sigma]\rangle =$$

$$= \exp[-\tfrac{1}{2}s\langle\vec{\phi}^2\rangle] \int \prod_i^N dv_i \exp[-\tfrac{1}{2}s^{-1}v_i^2]\,.$$

$$.\,(2\pi s)^{-N/2} \langle \exp[(\vec{U}+\vec{V})\vec{\phi}]\rangle.$$

Setting

$$(C.3)\quad \langle \exp[\vec{w}\vec{\phi}]\rangle = E(\vec{w}^2) = \sum_{n=0}^{\infty} (n!)^{-1} (\vec{w}^2)^n E_n$$

the E_n are defined by

$$(C.4)\quad \langle \phi_{i_1}\phi_{i_2}\ldots\phi_{i_{2n}}\rangle = (\delta_{i_1 i_2}\ldots\delta_{i_{2n-1} i_{2n}} + \ldots)E_n,$$ with $(2n-1)!!$ terms in the bracket. Thus, $E_1 = N^{-1}\langle\vec{\phi}^2\rangle$. For a free field, $E_n = E_1^{\,n}$.

In sects. 1 and 3, we need only the following expression computable from (C.1-3):

$$\exp[s(-\tfrac{1}{2}:\vec{\phi}^2: + \sigma)] = \int \prod_i dv_i \exp[-\tfrac{1}{2}s^{-1}v_i^2]\,.$$

$$.\,(2\pi s)^{-N/2} :\exp[i\vec{U}\vec{\phi} + s\sigma]: .$$

$$.\int \prod_i dv_i \exp[-\tfrac{1}{2}s^{-1}v_i^2].\,(2\pi s)^{-N/2} E((i\vec{U}+\vec{V})^2) =$$

$$= \int \prod_i dv_i \exp[-\tfrac{1}{2}s^{-1}v_i^2].\,(2\pi s)^{-N/2} :\exp[s(-\tfrac{1}{2}\vec{\phi}^2 + \sigma)].$$

$$.\int \prod_i dv_i \exp[-\tfrac{1}{2}s^{-1}v_i^2].\,(2\pi s)^{-N/2} E((i\vec{U}+\vec{V}-s\vec{\phi})^2): =$$

$$= :\exp[s(-\tfrac{1}{2}\vec{\phi}^2 + \sigma)]\,E(s^2\vec{\phi}^2):,$$

the evaluation done simplest in polar coordinates. Herefrom,

$$(C.4)\quad (-\tfrac{1}{2}:\vec{\phi}^2: + \sigma)^n = \sum_{k=0}^{[n/2]} [k!\,(n-2k)!]^{-1}\, n!\,.$$

$$.\,:(-\tfrac{1}{2}\vec{\phi}^2 + \sigma)^{n-2k}(\vec{\phi}^2)^k: E_k.$$

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Figure Captions:

Fig. 1 Lowest 1/N order contribution to Γ_{60}

Fig. 2 Some contributions to Γ_{60} (see text)

Fig. 3 (For $u > 0$, almost) cancelling diagrams

Fig. 4 (Almost) cancelling diagrams for Γ_{22}

Fig. 5 Splitting of some VFs into lnu-carrying and lnu-free parts

Fig. 6 Altogether IR-soft contributions to Γ_{40} (see text)

Fig. 7 (Almost) cancelling diagrams for Γ_{20}

Fig. 8 Some cases of overlap for Γ'_{40}, Γ'_{20}, Γ'_{60}

Fig. 9 Integral equation for Γ'_{40}, symmetry factors omitted

Table Caption:

Table 1: IR (and, for $g_B \to \infty$, UV divergence) degree of $\Gamma_{2n\ell}$

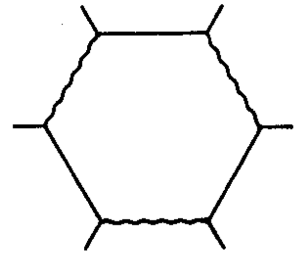

Fig.1

Fig. 2a

Fig. 2b

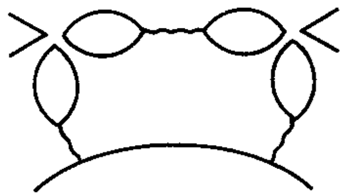

Fig. 2c

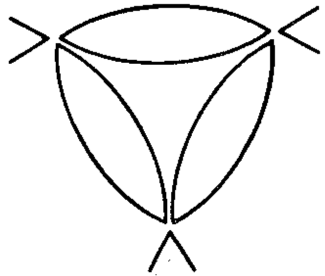

Fig. 2d

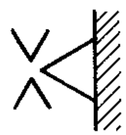

Fig. 3a

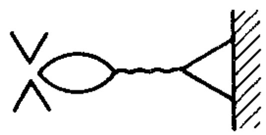

Fig. 3b

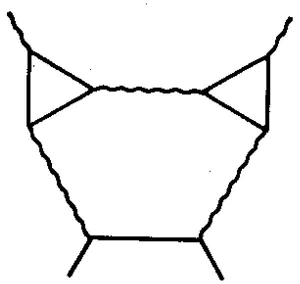

Fig. 4a

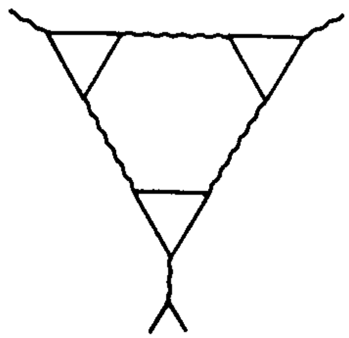

Fig. 4b

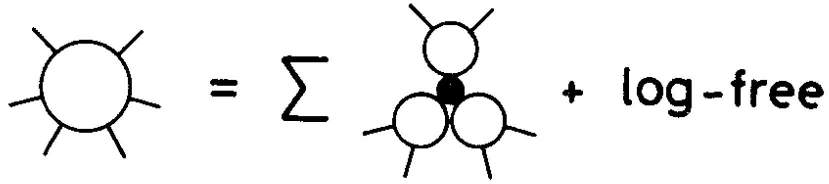

Fig. 5a

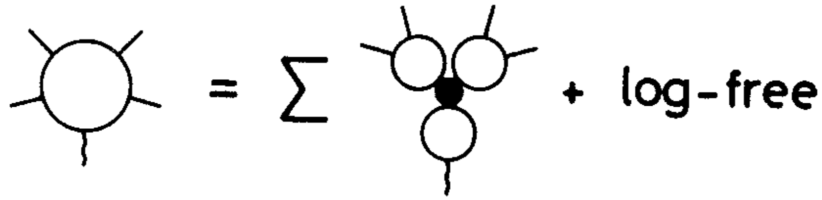

Fig. 5b

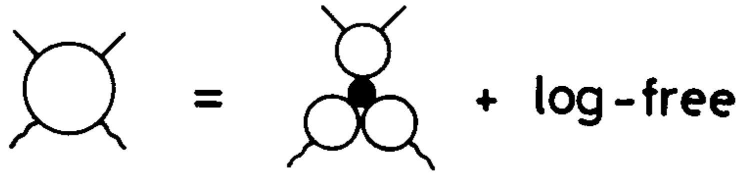

Fig. 5c

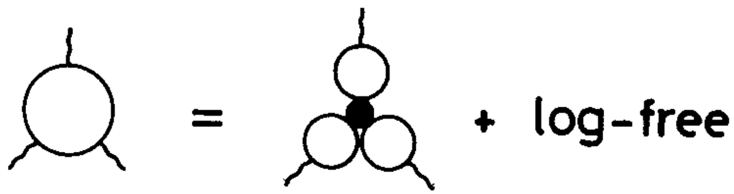

Fig. 5d

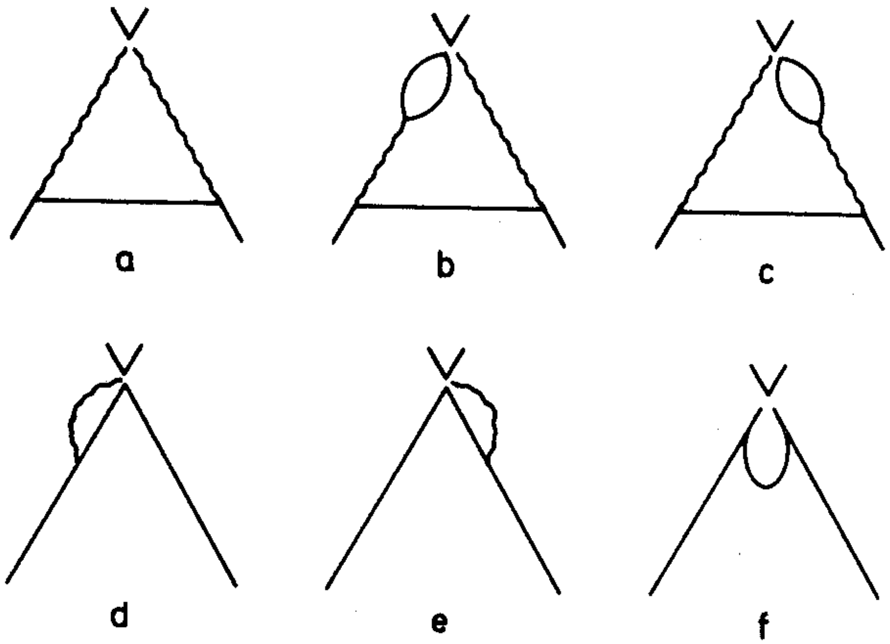

Fig. 6

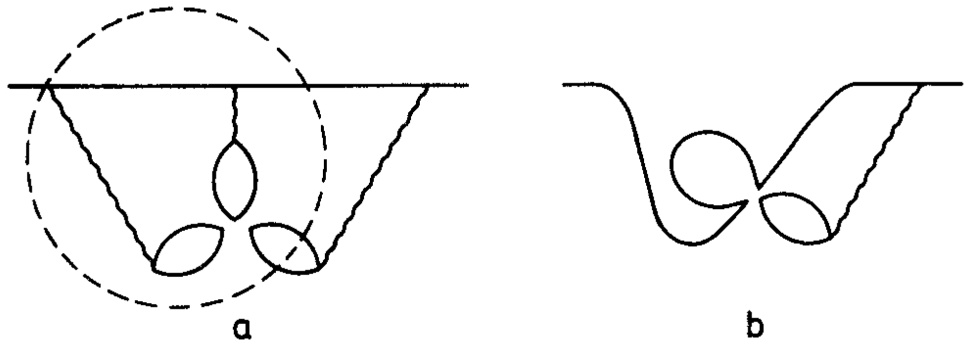

Fig. 7

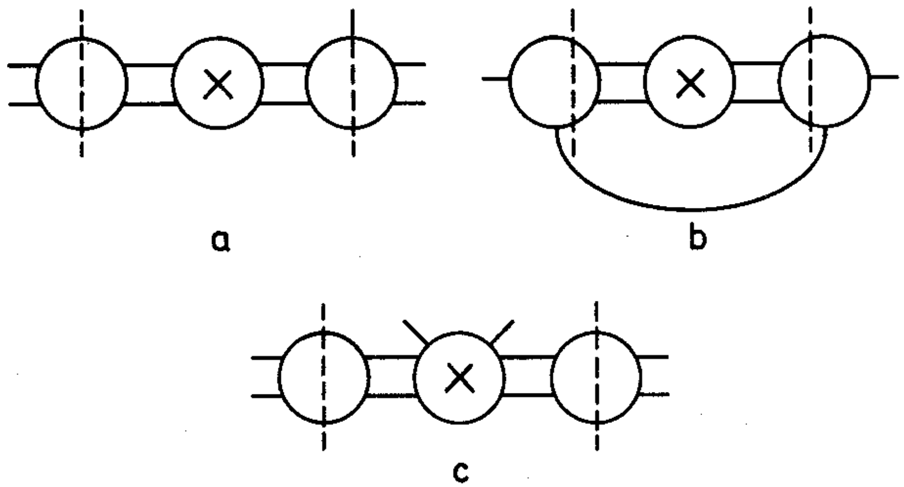

Fig. 8

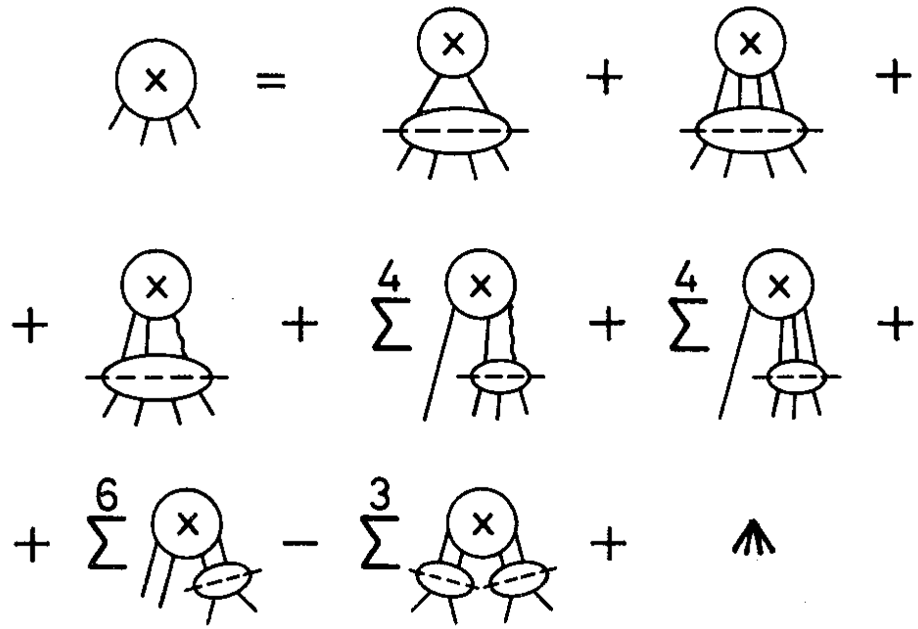

Fig. 9

l \ $2n$	0	2	4	6	8
0	$4-\varepsilon$	2	ε	$-2+2\varepsilon$	$-4+3\varepsilon$
1	$2-\varepsilon$	0	$-2+\varepsilon$	$-4+2\varepsilon$	$-6+3\varepsilon$
2	$-\varepsilon$	-2	$-4+\varepsilon$	$-6+2\varepsilon$	$-8+3\varepsilon$
3	$-2-\varepsilon$	-4	$-6+\varepsilon$	$-8+2\varepsilon$	$-10+3\varepsilon$

Table 1