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Absence of Spontaneous Gauge Symmetry Breaking in Hamiltonian Lattice Gauge Theories

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Absence of spontaneous gauge symmetry breaking in Hamiltonian lattice gauge theories.

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Abstract: Abelian as well as non-abelian Hamiltonian lattice gauge theories are shown to have a gauge invariant ground state for any value of the dimensionless (bare) coupling constant g. In fact, there is a gap Δ between the non gauge invariant states and the vacuum. Δ is of the order of $\frac{g^2}{a}$, where a is the lattice spacing. I also speculate that the θ-vacua in non-abelian lattice gauge theories are the lowest energy states with a certain charge distribution at infinity.

I. Introduction

Nowadays there are mainly two alternative methods available to discuss low energy properties of non-abelian gauge theories. These are the lattice gauge theories as formulated by K.G. Wilson [1] and by J. Kogut and L. Susskind [2] on the one hand whereas on the other hand one can resort to semiclassical methods as first advocated by A. Polyakov [3]. The consideration of classical, euclidean field configurations such as instantons [3,4] and, more recently, merons [5] , led to a beautiful heuristic picture of a possible quark confinement mechanism. Also, there is some evidence for the existence of a family of physically distinguishable vacua, labelled by an angle θ , that describe alternative realizations of the non-abelian gauge theory. These results have been established by semiclassical methods. Naturally, one would like to contrast them with the lattice formulation of QCD.

As a preliminary step for such an attempt we discuss in this paper some properties of the ground state of Hamiltonian lattice gauge theories (without quarks).

We will prove that the vacuum is invariant with respect to local as well as global gauge transformations. Moreover, it turns out that the non gauge invariant eigenstates of the Hamiltonian have an energy of order $\frac{g^2}{a}$ above the ground state. Here, g denotes the dimensionless quark-gluon coupling constant (i.e. the charge unit) and a is the lattice spacing. A lower bound on this energy gap Δ is first obtained for finite volume lattice gauge theories. Since it comes out to be independent of the volume and the boundary conditions chosen, the gap persists in the infinite volume limit.

This result can be easily interpreted heuristically. Gluon wave functions that are not gauge invariant can be looked at as representing some gluon state in the presence of some infinitely heavy (i.e. static) point like charges [1]. These charges require the presence of a Coulomb field. The energy of this Coulomb field blows up near the charges and it is finite only because it is cut off by the lattice. Since the energy of the Coulomb field of an elementary charge with radius r equals $\frac{g^2}{8\pi r}$, this explains the gap $\Delta \propto \frac{g^2}{a}$.

On grounds of this simple physical reasoning I believe that the gap Δ is also there if we add massive quarks. In the presence of massless quarks one should however reconsider the problem.

The paper is organized as follows. In sec. II Hamiltonian lattice gauge theories are briefly described. The gap Δ is then first established for the abelian case by means of an elegant symmetry consideration (sec. III). The non-abelian case requires more sophisticated arguments. Namely, in sec. IV, we will make use of a refinement (due to O.Mc Bryan and T. Spencer [6]) of the Mermin-Wagner argument [7] well-known in statistical mechanics. This trick has been previously applied to abelian (euclidean) lattice gauge theories by J. Glimm and A. Jaffe [8]. The last section is almost entirely devoted to a speculation about how to construct the Θ-vacua in lattice gauge theories. Finally, the heat equation on SU(2) is discussed in an appendix.

II. Review of Hamiltonian lattice gauge theories [2]

In this paper we consider a non-abelian SU(2) -colour Yang-Mills gauge theory whose continuum Lagrangian density reads:

$$\mathcal{L} = -\frac{1}{2g^2} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu}$$

where the field tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \quad ; \quad A_\mu = \sum_{a=1}^{3} A_\mu^a \frac{\tau^a}{2}$$

A_μ is the vector potential and τ^a are the three Pauli matrices.

Choosing the gauge

$$A_0 = 0$$

we can describe the time evolution of the canonically conjugate fields A_k (k = 1,2,3) and $\frac{1}{g^2}\partial_0 A_k = -E_k$ (the electric field) by the Hamiltonian:

$$(1) \qquad H = \int d^3x \left\{ g^2 \operatorname{Tr} E^2 + \frac{1}{g^2} \operatorname{Tr} B^2 \right\}$$

Here we have set: $B_k = \frac{1}{2} \epsilon_{k\ell j} F^{\ell j}$ (the magnetic field).

The Cauchy data $\{E_k, A_\ell\}_{t=0}$ can however not be chosen at will. They are constrained through Gauss' law:

$$(2) \qquad D_k E^k = \partial_k E^k - i[A_k, E^k] = 0$$

If we add external sources (and sinks) with a density $g \cdot \rho = g \cdot \rho^a \frac{\tau^a}{2}$, Gauss' law will be changed to read

$$(3) \qquad D_k E^k = \rho$$

The Hamiltonian is the same as in the sourceless case.

We now put the gauge theory on a three-dimensional space lattice (time remains continuous). The lattice sites can be labelled by three integer numbers $n = (n_1, n_2, n_3)$. The unit vector in direction k, $k = 1,2,3$, will be denoted by $\hat{k}$. With each link $(n, n + \hat{k})$ we associate a string bit variable $U(n,k)$. This is an SU(2)-matrix. It is a substitute for the vector fields A_k, i.e. we have:

$$U(n,k) = e^{i a A_k(n \cdot a)} \tag{4}$$

The k'th component of the electric field E emerging from site n is given by

$$E(n,k) = \frac{i}{a g^2} \, \partial_0 U(n,k) \cdot U(n,k)^{-1} \tag{5}$$

The Lagrangian of the (still classical) lattice gauge theory is taken to be:[3)]

$$L(U,\dot{U}) = \sum_{n,k} \frac{(-a)}{g^2} \, \mathrm{Tr}\left[\dot{U}(n,k) \cdot U(n,k)^{-1}\right]^2 + \sum_{n,k \neq j} \frac{1}{a g^2} \, \mathrm{Tr}\left[U(n,k) \cdot U(n+\hat{k},j) \cdot U^{-1}(n+\hat{j},k) \cdot U^{-1}(n,j)\right] \tag{6}$$

Upon canonical quantization one gets a quantum theory whose space of states consists of all square integrable functions $\Psi(U)$ of the string bit variables $U(n,k)$:

$$\int \prod_{n,k} dU(n,k) \, |\Psi(U)|^2 < \infty \tag{7}$$

dU denotes the invariant measure on SU(2). We choose the normalization

such that $\int dU = 1$. From (6) one derives the Hamilton operator:

$$\mathbb{H} = \sum_{n,k} \frac{g^2}{2a} \vec{J}(n,k)^2 - \sum_{n,k \neq j} \frac{1}{a g^2} \mathrm{Tr}\left[U(n,k)\, U(n+\hat{k},j)\, U^{-1}(n+\hat{j},k)\, U^{-1}(n,j)\right] = \mathbb{H}_0 + V \tag{8}$$

The first term in $\mathbb{H}$ (which we called $\mathbb{H}_0$) represents the electric field energy whereas the "potential" V stands for the magnetic field energy. $\mathbb{H}_0$ acts as a second order differential operator on wave functions $\Psi(U)$. More precisely, $J^b(n,k)$ $(b = 1,2,3)$ is a differential operator with respect to the variable $U(n,k)$. Thus, if f is a function of $U(n,k)$, we have:

$$\left(J^b(n,k) f\right)(U(n,k)) = \frac{1}{i} \frac{d}{ds} f\left(e^{-is\frac{\tau^b}{2}} U(n,k)\right)\Big|_{s=0} \tag{9}$$

Therefore, $J^b(n,k)$ satisfies angular momentum commutation relations

$$\left[J^b(n,k), J^c(m,j)\right] = \delta_{nm} \delta_{kj} \, i \, \epsilon_{bcd} \, J^d(n,k) \tag{10}$$

The possible eigenvalues of $\vec{J}(n,k)^2$ equal $J(J+1)$, where $J = 0, \frac{1}{2}, 1, \ldots$. The relation between $J^b(n,k)$ and the electric field $E^b(n,k)$ is:

$$E^b(n,k) = \frac{1}{a^2} J^b(n,k) \tag{11}$$

So far we did not take into account the constraint equation (2). To find out what it means, we first have to discuss time independent gauge transformations. A gauge transformation corresponds to a change of colour reference frames at each site n. It is thus characterized by a set of matrices $\Lambda_n \in SU(2)$. Gauge transformations are unitarily represented in the space of wave functions $\Psi(U)$:

$$(12) \qquad (G(\Lambda)\Psi)(\{U(n,k)\}) = \Psi(\{\Lambda_n^{-1}\, U(n,k)\, \Lambda_{n+\hat{k}}\})$$

Of course, the operators $G(\Lambda)$ satisfy the multiplication law $G(\Lambda)\cdot G(\Lambda') = G(\Lambda\cdot\Lambda')$. Especially, if we consider a gauge transformation at site n, i.e. $\Lambda_n = \Lambda$, $\Lambda_m = 1$ for $m \neq n$, we obtain a representation of SU(2). Its generators G_n^b , b = 1,2,3, will be called the (external) charge operators at site n.

The link between Gauss'law and gauge invariance is the following. As a simple consideration of Poisson brackets shows, $(D_k E^k)^b(x)$ are just the classical generators of gauge transformations at x. In quantum theory we therefore identify $(D_k E^k)^b(n\cdot a)$ with $a^{-3} G_n^b$. The constraint equation (2) then says that only those wave functions $\Psi(U)$ should be called physical which are annihilated by the charges G_n^b, i.e. $\Psi(U)$ should be invariant under local gauge transformations.

In case there are external charges with density $\rho^b(n\cdot a) = a^{-3} q_n^b$ Gauss' law becomes: $G_n^b = q_n^b$. The charge operators G_n^b have angular momentum commutation relations. Therefore we may only

prescribe $(\vec{q}_n)^2$ and, say, q_n^3 . The external charges thus get "quantized". This is, of course, due to the compactness of the gauge group. To sum up, we may state that a wave function $\Psi(U)$ with

$$(13) \qquad (\vec{G}_n)^2 \Psi = \ell_n(\ell_n + 1)\Psi$$

describes the gluon field in the presence of some external charges of magnitude $(\vec{q}_n)^2 = \ell_n(\ell_n+1)$. The possible values of ℓ_n are: $\ell_n = 0, 1/2, 1, \ldots$

We finally note that the Hamiltonian and all other observables are gauge invariant. The sectors in the space of wave function $\Psi(U)$ corresponding to the various distributions of external charges hence decouple completely, i.e. all physical matrix elements between different sectors vanish.

Before concluding this section let us recall briefly what changes occur for an abelian lattice gauge theory. First of all the string bit variables $U(n,k)$ simply become phases:

$$(14) \qquad U(n,k) = e^{i\theta(n,k)} \quad ; \quad -\pi \leq \theta(n,k) \leq \pi$$

Correspondingly, the Hamiltonian is:

$$(15) \qquad H = -\sum_{n,k} \frac{g^2}{2a} \frac{\partial^2}{\partial\theta(n,k)^2} - \sum_{n,k>j} \frac{1}{a g^2} \cos\left[\theta(n,k) + \theta(n+\hat{k},j) - \theta(n+\hat{j},k) - \theta(n,j)\right]$$

It acts on differentiable periodic functions $\Psi(\theta)$. At each site n we have a (external) charge operator G_n:

(16) $$G_n = \sum_{k=1,2,3} \frac{1}{i} \left\{ \frac{\partial}{\partial\theta(n-\hat{k},k)} - \frac{\partial}{\partial\theta(n,k)} \right\}$$

Its possible eigenvalues ℓ_n are: $\ell_n = 0, \pm 1, \pm 2, \ldots$. Thus, an external charge distribution is described by saying how many elementary charges sit at each site n .

I would like to add one last remark: Hamiltonian lattice gauge theories contain the same physics as their euclidean counterparts. Specifically, with the help of the transfer matrix formalism [11] we can easily move between the two formulations and thus translate all the results obtained in this paper into the euclidean language.

III. Gauge invariance of the ground state: the abelian case

We are now going to prove that the ground state of the Hamiltonian (15) is gauge invariant. Moreover, it will be shown that the energy (above the ground state energy) of an arbitrary state in the presence of some external charges is at least $\frac{g^2}{12a}$. This gap does not depend on the volume nor on the charge distribution nor on boundary conditions. Therefore, there cannot be any spontaneous breakdown of local or nonlocal gauge symmetry.

The idea of the proof is the following. Consider the Hamiltonian (15) and imagine for a moment that the θ -variables would range from $-\infty$ to $+\infty$. Interprete them as the coordinates of a particle with mass $\frac{a}{g^2}$ that moves in a strange potential in a many dimensional world. Within this picture gauge invariance means that the potential is translation invariant in some directions. Thus in these directions the particle will move with constant momentum. The crucial observation is now that

due to the standard Schrödinger form of $\mathbb{H}$ one can boost these momenta by applying a Galilei transformation. Specifically, one can make all momenta equal to zero. In this way we get a new state with smaller energy that is translation (i.e. gauge) invariant.

Let us now fill in the details. To properly define charge boost operators we have to introduce an auxiliary Hilbert space $\mathcal{H}_a$. Its elements are the square integrable functions $\Psi(\theta)$ with $-3\pi \leq \theta(n,k) \leq 3\pi$. Our original space of states $\mathcal{H}$ can be identified with the subspace of $\mathcal{H}_a$ consisting of all wave functions $\Psi(\theta)$ that are periodic in all variables $\theta(n,k)$ with period 2π . Obviously, we can consider $\mathbb{H}$ and G_n as operators acting in $\mathcal{H}_a$. We are now well prepared to define a charge boost operator B_n at site n :

(17) $$(B_n\Psi)(\theta) = \exp\left\{\frac{i}{6} \sum_{k=1,2,3} [\theta(n-\hat{k},k) - \theta(n,k)]\right\} \Psi(\theta)$$

B_n shifts the charge at site n by one unit:

(18) $$B_n G_m B_n^{-1} = G_m - \delta_{nm} + \frac{1}{6} \sum_{k=1,2,3} [\delta_{n+\hat{k},m} + \delta_{n-\hat{k},m}]$$

Also

(19) $$B_n \mathbb{H} B_n^{-1} = \mathbb{H} - \frac{g^2}{6a}(G_n - \tfrac{1}{2})$$

Thus, if $\Psi(\theta) \in \mathcal{H}$ is a state with energy E and charge $\ell_n > 0$ at site n, we can construct a state $\Psi' = B_n^{-1}\Psi \in \mathcal{H}_a$ whose energy is $E - \frac{g^2}{6a}(\ell_n - \frac{1}{2})$. In other words, Ψ has an energy of at least $\frac{g^2}{12a}$ above the lowest energy state in $\mathcal{H}_a$. Of course, the same conclusion holds if $\ell_n < 0$.

The last step in our argumentation is the proof that the ground state in $\mathcal{H}_a$ is nondegenerate and, furthermore, that its wavefunction can be chosen positive (throughout the whole discussion we keep the volume fixed). For then it follows that the ground state wave function must be periodic in each $\theta(n,k)$ with period 2π and is therefore contained in $\mathcal{H}$ [4]. The ground state energy in the auxiliary Hilbert space $\mathcal{H}_a$ is hence the same as the one in the physical space of states $\mathcal{H}$. As shown above, this implies the existence of the gap $\Delta \geqslant \frac{g^2}{12a}$ announced in the introduction to this section.

The proof that the groundstate of $\mathbb{H}$ in $\mathcal{H}_a$ is nondegenerate is based on a standard trick [5) [12]. It goes as follows: we first show that $e^{-\mathbb{H}}$ is a bounded, positivity improving operator, i.e. if $\psi \geqslant 0$, $\psi \not\equiv 0$, we have

(20) $$(e^{-\mathbb{H}}\psi)(\theta) > 0 \text{ a.e.}$$

This property immediately implies that any groundstate wave function is proportional to a positive function (if this were not so, we could easily prove that $(|\psi_0|, e^{-\mathbb{H}}|\psi_0|) > (\psi_0, e^{-\mathbb{H}}\psi_0)$). Two such functions can however never be orthogonal, hence ψ_0 is unique.

We are thus left to prove that $e^{-\mathbb{H}}$ is positivity improving. To this end we write $\mathbb{H} = \mathbb{H}_0 + V$ where

$$\mathbb{H}_0 = -\sum_{n,k} \frac{g^2}{2a} \frac{\partial^2}{\partial\theta(n,k)^2} = \sum_{n,k} \frac{g^2}{2a} J(n,k)^2$$

$e^{-s\mathbb{H}_0}$ is a product of commuting operators, one factor for each link.

Let us pick out one of these. It acts on square integrable functions of one string bit variable θ , $-3\pi \leqslant \theta \leqslant 3\pi$ and is given by

$$e^{-s\frac{g^2}{2a}J^2} \qquad \text{with} \qquad J = \frac{1}{i}\frac{\partial}{\partial\theta}$$

Its action on wave functions $\varphi(\theta)$ can be described by an integral kernel:

$$(e^{-tJ^2}\varphi)(\theta) = \int_{-3\pi}^{3\pi} \frac{d\theta'}{6\pi} K_t(\theta,\theta')\,\varphi(\theta')$$

where

$$K_t(\theta,\theta') = \sum_{j=-\infty}^{\infty} e^{-\frac{t}{9}j^2} e^{i\frac{j}{3}(\theta-\theta')}$$
$$= 3\left(\frac{\pi}{t}\right)^{1/2} \sum_{m=-\infty}^{\infty} e^{-\frac{1}{4t}(\theta-\theta'-6\pi m)^2}$$

Obviously, $K_t(\theta,\theta') > 0$ for all θ, θ' so that e^{-tJ^2} and hence $e^{-s\mathbb{H}_0}$ are positivity improving. To show that this property is shared by $e^{-\mathbb{H}}$ we use the fact that V is bounded, i.e. $|V| \leqslant M$, and the Trotter product formula ([13] p. 297). Thus, if $\psi(\theta) \geqslant 0$, $\psi(\theta) \not\equiv 0$ we have:

$$e^{-\mathbb{H}}\psi = \lim_{\nu\to\infty} \left(e^{-\frac{1}{\nu}\mathbb{H}_0} e^{-\frac{1}{\nu}V}\right)^{\nu} \psi$$

Writing out all the factors $e^{-\frac{1}{\nu}\mathbb{H}_0}$ as integral operators and estimating $e^{-\frac{1}{\nu}V} \geqslant e^{-\frac{1}{\nu}M}$ we see that

$$(e^{-\mathbb{H}}\psi)(\theta) \geqslant e^{-M}(e^{-\mathbb{H}_0}\psi)(\theta) > 0 \qquad \text{a.e.}$$

This concludes the proof of uniqueness of the ground state in $\mathcal{H}_a$.

The result we have established in this section can be sharpened by systematically boosting all charges down to zero. Consider for example the lowest energy state of the photon field in the presence of two elementary charges, one sitting at the origin and the other one keeping at a distance R from the origin. The energy of this state above the groundstate is the static quark-antiquark potential $V(R)$ [1,2] . By boosting the charges completely away we get the result that $V(R)$ is bounded from below by the classical Coulomb potential on the lattice [6]. Certainly, this is not a dramatic result in 3 + 1 dimensions, however in 2 + 1 dimensions the Coulomb potential rises as $\log R$ for $R \to \infty$ so that quarks will be confined.

IV. Gauge invariance of the ground state: the non-abelian case

The gap between the "charged" states and the gauge invariant ground state in non-abelian lattice gauge theories will now be established. The proof of this result is somewhat more technical than in the abelian case and is therefore carried out for an SU(2)-colour group only. The argument we are going to use should however be generalizable to any compact gauge group. In the SU(2) case the lower bound on the gap Δ comes out to be $\Delta \geq \frac{g^2}{48a}$.

Let $\Psi(U)$ be a wave function describing the gluon field on the lattice in the presence of some charges, i.e. without loss of generality we may assume that

$$(21) \qquad (\vec{G}_m)^2 \Psi = \ell_m(\ell_m+1)\Psi \quad ; \quad G_m^3 \Psi = \ell_m \Psi \quad (\ell_m > 0)$$

for some site m. We then have to prove that

$$(22) \qquad |(\Psi, e^{-tH}\Psi)| \leq C\, e^{-t(E_0+\Delta)} \quad , \quad \Delta \geq \frac{g^2}{48a}$$

E_0 is the ground state energy and C is a t-independent constant.

The general strategy for the proof of eq. (22) is as follows. Upon splitting the Hamiltonian as in eq. (8) and using Trotters product formula we get:

$$(23) \qquad (\Psi, e^{-tH}\Psi) = \lim_{\nu\to\infty} (\Psi, (e^{-\frac{t}{\nu}H_0}\, e^{-\frac{t}{\nu}V})^\nu \Psi)$$

For finite ν we then use an integral kernel representation for $e^{-\frac{t}{\nu}H_0}$ to write the right hand side of eq. (23) as a multiple integral. Within these integrals we make a convenient complex gauge transformation at site m. This yields a damping factor from the wave function Ψ. On the other hand the contribution from the integral kernels can be estimated and the result will be eq. (22).

The action of e^{-sH_0} on a wave function $\varphi(U)$ can be written as

$$(24) \qquad (e^{-sH_0}\varphi)(U) = \int \mathcal{D}U'\; K_s(U,U')\,\varphi(U')$$

where $\mathcal{D}U$ is shorthand for $\prod_{n,k} dU(n,k)$. Since H_0 is a sum of commuting operators, we have:

$$(25) \qquad K_s(U,U') = \prod_{n,k} G_{\frac{sg^2}{2a}}(U(n,k), U'(n,k))$$

$G_\lambda(U,U')$ denotes the integral kernel for $e^{-\lambda\vec{J}^2}$, an operator acting in the one link Hilbert space $L^2(SU(2))$.

The function $G_\lambda(U,U')$ is studied in detail in the appendix. The outcome is that

a) $G_\lambda(U,U') > 0$ for all U, U'

b) $G_\lambda(U,U') = G_\lambda(U\cdot U'^{-1}, 1) = G_\lambda(U'^{-1}\cdot U, 1)$

c) $G_\lambda(U,U')$ can be analytically continued in U and U' to all of $S\ell(2,\mathbb{C}) \times S\ell(2,\mathbb{C})$. If X_1 and X_2 are hermitian, traceless 2×2 matrices, we have

$$(26)\quad |G_\lambda(e^{X_1} U e^{X_2}, U')| \leq \exp\frac{1}{2\lambda}\left[\sqrt{\mathrm{Tr}\, X_1^2} + \sqrt{\mathrm{Tr}\, X_2^2}\right]^2 \cdot G_\lambda(U,U')$$

Let us now insert the representation (24) into eq. (23):

$$(27)\quad (\Psi, e^{-t\mathbb{H}}\Psi) = \lim_{\nu\to\infty} \int \prod_{\mu=0}^{\nu} \mathcal{D}U_\mu \left\{\Psi^*(U_0)\, K_{\frac{t}{\nu}}(U_0,U_1)\, e^{-\frac{t}{\nu}V(U_1)} \ldots\right.$$
$$\left.\ldots e^{-\frac{t}{\nu}V(U_{\nu-1})} K_{\frac{t}{\nu}}(U_{\nu-1},U_\nu)\, e^{-\frac{t}{\nu}V(U_\nu)}\, \Psi(U_\nu)\right\}$$

Performing the gauge transformations

$$(28)\quad \Lambda_n^\mu = \begin{cases} \exp[i\,\frac{\mu}{\nu}\cdot\alpha\cdot\frac{\tau^3}{2}] & \text{for } n = m \\ 1 & \text{otherwise} \end{cases}$$

we get from (27):

$$(29)\quad (\Psi, e^{-t\mathbb{H}}\Psi) = \lim_{\nu\to\infty} \int \prod_{\mu=0}^{\nu} \mathcal{D}U_\mu \left\{\Psi^*(U_0)\, K_{\frac{t}{\nu}}(U_0,U_1^{\Lambda^1})\, e^{-\frac{t}{\nu}V(U_1)} \ldots\right.$$
$$\left.\ldots e^{-\frac{t}{\nu}V(U_{\nu-1})} K_{\frac{t}{\nu}}(U_{\nu-1}^{\Lambda^{\nu-1}}, U_\nu^{\Lambda^\nu})\, e^{-\frac{t}{\nu}V(U_\nu)}\, e^{i\alpha\ell_m}\,\Psi(U_\nu)\right\}$$

Here, we have used the abbreviation

$$U^\Lambda(n,k) = \Lambda_n^{-1}\, U(n,k)\, \Lambda_{n+\hat{k}}$$

and the fact that there is a charge at site m, i.e.

$$\Psi(U_\nu^{\Lambda^\nu}) = e^{i\alpha\ell_m}\, \Psi(U_\nu)$$

So far, α was taken real. However, due to the analyticity properties of the kernels involved (property c) above) the finite ν integrals in eq. (29) will not change their values if we choose α to be complex. Specifically, we can take $\alpha = i\beta$, $\beta > 0$. The optimal value for β will be determined later.

Using the estimate (26) we find

$$|K_{\frac{t}{\nu}}(U_{\mu-1}^{\Lambda^{\mu-1}}, U_\mu^{\Lambda^\mu})| \leq \exp\left(\frac{3a}{g^2\nu t}\beta^2\right) K_{\frac{t}{\nu}}(U_{\mu-1}, U_\mu)$$

and therefore:

$$|(\Psi, e^{-t\mathbb{H}}\Psi)| \leq (|\Psi|, e^{-t\mathbb{H}}|\Psi|)\cdot\exp\left\{\frac{3a}{g^2 t}\beta^2 - \ell_m\beta\right\}$$

Noting that

$$(|\Psi|, e^{-t\mathbb{H}}|\Psi|) \leq \|\Psi\|^2\, e^{-E_0 t}$$

and choosing

$$\beta = \frac{g^2 t}{3a}\cdot\frac{\ell_m}{2}$$

yields

$$|(\Psi, e^{-t\mathbb{H}}\Psi)| \leq \|\Psi\|^2\, e^{-E_0 t}\cdot\exp\left\{-\frac{g^2\ell_m^2}{12a}\cdot t\right\}$$

This concludes the proof of gauge invariance of the ground state in non-abelian lattice gauge theories.

V. Conclusions

In this paper it has been shown that the ground state of abelian and non-abelian Hamiltonian lattice gauge theories is invariant with respect to local as well as non-local gauge transformations. This result certainly excludes the possibility to identify the photon as the Goldstone boson of a spontaneously broken gauge symmetry. It also supports one of the basic assumptions when doing hadron mass calculations [14], namely the conjecture that the low energy behavior of non-abelian gauge theories is qualitatively correctly displayed by the strong coupling limit.

Let us now speculate how the θ-vacua [4] could be interpreted in the framework of lattice gauge theories. These states are conjectured to be the lowest energy states in a realization of the gluon field theory that is physically different from the common one. Let Ψ_θ denote the wave function of such a vacuum. Ψ_θ should be locally gauge invariant i.e.

$$G_n^b \Psi_\theta = 0$$

because this is required by the field equations. However, under certain non-local gauge transformations Ψ_θ is believed not to be invariant. Specifically, we should have:

$$G(\Lambda)\Psi_\theta = e^{-i\theta}\,\Psi_\theta \tag{30}$$

for

$$\Lambda(\vec{x}) = \frac{\vec{x}^2 - \lambda^2}{\vec{x}^2 + \lambda^2} - 2i\lambda\,\frac{\vec{x}\cdot\vec{\tau}}{\vec{x}^2 + \lambda^2} \tag{31}$$

There is an obvious way of how to construct states with such gauge transformation properties. Namely, we just take the ground state of a finite volume lattice gauge theory in the presence of some charges at the boundary and let the volume tend to infinity. Of course, the energy of this new vacuum relative to the vacuum without charges at infinity is may be infinite. However, this is of no concern to us, since the new vacuum is the ground state of a new sector orthogonal to the old physical Hilbert space. Hence, we may choose its energy to vanish.

Now, how should we choose the charge distribution at infinity to obtain the θ-vacuum? The answer to this question can be guessed from eq. (31). Consider a large ball with radius R centered at the origin. The gauge transformation $\Lambda(\vec{x})$ at a point $\vec{x} = R\cdot\vec{n}$, $\vec{n}^2 = 1$, on the balls boundary represents a rotation by an angle of approximately $4\lambda/R$ about an axis parallel to $\vec{n}$. Hence, if we put there a (colour) spin 1/2 charge pointing into the direction $\vec{n}$, we will get states that transform as

$$G(\Lambda)\Psi = e^{-i\frac{2\lambda}{R}}\,\Psi$$

Thus, distributing many charges on the sphere of radius R we get

$$G(\Lambda)\Psi = e^{-i\,16\pi R\lambda\cdot\sigma}\,\Psi$$

where σ is the charge density on the sphere. Choosing

$$\sigma = \frac{\theta}{16\pi R\lambda} \tag{32}$$

yields states transforming like Ψ_θ under the gauge transformation Λ.

To sum up, we may say that Ψ_θ is the lowest energy state in the presence of a uniform distribution of elementary charges on a large sphere of radius R $\rightarrow \infty$. More precisely, the charge density $\rho^a(\vec{x})$ (cp. eq.(3)) is given by

$$(33) \qquad \rho^a(\vec{x}) = \delta(R - |\vec{x}|)\, \sigma \frac{x^a}{R}$$

Of course, the arguments given here do not prove that θ-vacua exist. For instance, one must check whether the charges at infinity are not cancelled by gluon pair production. However, I think that there is not much room to give a different physical interpretation of θ-vacua when working in the framework of lattice gauge theories. Also, the heuristic picture developped here fits perfectly with the two dimensional Schwinger model where S. Coleman [15] showed that the angle θ can be identified with a constant background electric field.

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Appendix: The heat equation on SU(2).

The heat equation on SU(2) reads:

$$(A1) \qquad \frac{\partial}{\partial\lambda} T(\lambda,U) + \vec{J}^2 T(\lambda,U) = 0 \quad ; \quad \lambda \geq 0 \quad , \quad U \in SU(2)$$

$T(\lambda,U)$ is the temperature at time λ and location U . $\vec{J}^2$ is the Laplacian on SU(2), i.e. the square of the angular momentum operator J^b (cp.(9)). If $T(0,U) = \varphi(U)$ is square integrable, (A1) has a unique solution:

$$(A2) \qquad T(\lambda,U) = (e^{-\lambda\vec{J}^2}\varphi)(U) = \int dU'\, G_\lambda(U,U')\, \varphi(U')$$

The kernel $G_\lambda(U,U')$ will now be studied carefully. For this purpose we first have to collect some formulas about the Fourier transform on SU(2) (e.g. [16 , 17]).

Let π^j denote the irreducible representation of SU(2) with angular momentum j, j = 0, 1/2, 1, ... For any two states $|a\rangle, |b\rangle$ in the representation space of π^j we find that $\langle a|\pi^j(U)|b\rangle$ is square integrable. The set of all these functions spans a $(2j+1)^2$-dimensional subspace of $L^2(SU(2))$ that will be called $\mathcal{H}_j$. The Peter-Weyl theorem states that

$$(A3) \qquad L^2(SU(2)) = \bigoplus_{j=0,1/2,1,\dots} \mathcal{H}_j \qquad \text{(orthogonal sum)}$$

The Fourier decomposition of a function $\varphi(U)$ is correspondingly:

$$(A4) \qquad \varphi(U) = \sum_{j=0,1/2,1,\dots} \varphi_j(U) \quad ; \quad \varphi_j \in \mathcal{H}_j$$

The Fourier components φ_j can be calculated with the help of the characters χ_j :

(A5) $$\chi_j(U) = \mathrm{Tr}\{\pi^j(U)\} \in \mathcal{H}_j$$

(A6) $$\varphi_j(U) = (2j+1)\int dU'\, \chi_j^*(U\cdot U'^{-1})\, \varphi(U')$$

The square of the angular momentum operator becomes diagonal in the Fourier representation, i.e.

$$(\vec{J}^2\varphi)(U) = j(j+1)\,\varphi(U) \qquad \text{for} \quad \varphi \in \mathcal{H}_j$$

Therefore we have

(A7) $$G_\lambda(U,U') = \sum_{j=0,1/2,1,\dots} e^{-\lambda j(j+1)}\,(2j+1)\,\chi_j^*(U\cdot U'^{-1})$$

From this formula follows immediately property b) of G_λ as stated in sec. IV. Also, $G_\lambda(U,U')$ can obviously be analytically continued to all of $S\ell(2,\mathbb{C}) \times S\ell(2,\mathbb{C})$ since the damping factor $e^{-\lambda j(j+1)}$ guarantees absolute and uniform convergence (in bounded regions of $S\ell(2,\mathbb{C}) \times S\ell(2,\mathbb{C})$) of the series (A7) (the characters $\chi_j^*(U\cdot U'^{-1}) = \chi_j(U'\cdot U^{-1})$ have an analytic continuation, see [17], 4.7.5).

To establish property a) and the estimate (26) we write $e^{-\lambda \vec{J}^2}$ as a limit of more convenient operators:

(A8) $$e^{-\lambda\vec{J}^2} = s\text{-}\lim_{\nu\to\infty} A_\nu$$

where $A_\nu = \left(B_{\frac{\lambda}{\nu}}\right)^\nu$ and

$$(B_s\varphi)(U) = \int dU'\, B_s(U,U')\,\varphi(U')$$

(A9) $$B_s(U,U') = \frac{4\sqrt{\pi}}{s^{3/2}}\; e^{\frac{3}{16}s}\; \exp\left\{\frac{1}{2s}\,\mathrm{Tr}\left[UU'^{-1} + U'U^{-1} - 2\right]\right\}$$

The proof of eq. (A8) is based on an explicit calculation of the eigenvalues of B_s:

<u>Lemma:</u> The Fourier representation of $B_s(U,U')$ is:

(A10) $$B_s(U,U') = \sum_{j=0,1/2,1,\dots} b_j(s)\;(2j+1)\,\chi_j^*(U\cdot U'^{-1})$$

$$b_j(s) = 2\left(\frac{\pi}{s}\right)^{1/2} e^{\frac{3}{16}s}\; e^{-\frac{2}{s}}\; I_{2j+1}\left(\frac{2}{s}\right)$$

where $I_\delta(z)$ is a Bessel function of imaginary argument ([18] , § 8.406).

We postpone the proof of the lemma to the end of the appendix and verify first eq. (A8) . The lemma says that the eigenvalues of B_s are $b_j(s)$. The corresponding subspace of eigenvectors is just $\mathcal{H}_j$. Hence, A_ν has the same eigenvectors and its eigenvalues are:

(A11) $$a_j(\nu) = \left(b_j\left(\frac{\lambda}{\nu}\right)\right)^\nu$$

As $z \to \infty$, $I_\delta(z)$ behaves as ([18] § 8.451(5)):

$$I_\delta(z) = \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 - \frac{1}{2z} \frac{\Gamma(\delta + \frac{3}{2})}{\Gamma(\delta - \frac{1}{2})} + O(\tfrac{1}{z^2}) \right\}$$

so that

$$\lim_{\nu \to \infty} a_j(\nu) = \lim_{\nu \to \infty} \left\{ e^{\frac{3}{16}\frac{\lambda}{\nu}} \left[1 - \frac{\lambda}{\nu}\left(j(j+1) + \tfrac{3}{16}\right) + O(\tfrac{1}{\nu^2}) \right] \right\}^{\nu}$$

$$= e^{-\lambda j(j+1)}$$

Thus, if $\varphi(U)$ is a function with terminating Fourier expansion (A4) we conclude that:

(A12) $$\lim_{\nu \to \infty} A_\nu \varphi = e^{-\lambda \vec{J}^2} \varphi$$

However, the operators A_ν are uniformly bounded (as follows easily from $0 \leqslant I_{2j+1}(z) \leqslant I_0(z)$ and eqs. (A10), (A11)), so that by an $\epsilon/3$ -argument (A12) extends to all square integrable functions φ. This proves (A8).

Since $B_s(U,U') > 0$ it is obvious that (A8) implies $G_\lambda(U,U') \geqslant 0$ We already know that G_λ is analytic. Therefore, it vanishes at most on a set of measure zero. Also

$$G_\lambda(U,U') = \int dV \; G_{\frac{\lambda}{2}}(UV^{-1}, 1) \; G_{\frac{\lambda}{2}}(VU'^{-1}, 1)$$

and hence $G_\lambda(U,U') > 0$. We have thus verified property a) of G_λ.

To prove the estimate (26) we also make use of (A8). For any pair $\varphi \geqslant 0$, $\psi \geqslant 0$ of wave functions and matrices $V_1, V_2 \in SU(2)$ define

(A13) $$(\varphi, A_\nu \psi)_{V_1,V_2} = \int dU\, dU' \; \varphi(U) \, A_\nu(V_1 U V_2, U') \, \psi(U')$$

We know that $\lim_{\nu \to \infty} (\varphi, A_\nu \psi)_{V_1,V_2} = (\varphi, e^{-\lambda \vec{J}^2} \psi)_{V_1,V_2}$ where

(A14) $$(\varphi, e^{-\lambda \vec{J}^2} \psi)_{V_1,V_2} = \int dU\, dU' \; \varphi(U) \, G_\lambda(V_1 U V_2, U') \, \psi(U')$$

This matrix element continues to an analytic function of $V_1, V_2 \in S\ell(2,\mathbb{C})$. The same is true for $(\varphi, A_\nu \psi)_{V_1,V_2}$. Let $V_{1,2} = e^{X_{1,2}} \cdot U_{1,2}$, $X_{1,2}^+ = X_{1,2}$ and $U_{1,2} \in SU(2)$. We are going to prove below that

(A15) $$|(\varphi, A_\nu \psi)_{V_1,V_2}| \leqslant \exp\left\{ \frac{1}{2\lambda}\left[\sqrt{T_r X_1^2} + \sqrt{T_r X_2^2}\right]^2 + \frac{1}{\nu} F(X_1,X_2) \right\} \times (\varphi, A_\nu \psi)_{U_1,U_2}$$

where $F(X_1,X_2)$ is continuous in X_1,X_2.

By Vitalis theorem this estimate implies the convergence of the sequence $(\varphi, A_\nu \psi)_{V_1,V_2}$ for all $V_1, V_2 \in S\ell(2,\mathbb{C})$:

$$\lim_{\nu \to \infty} (\varphi, A_\nu \psi)_{V_1,V_2} = (\varphi, e^{-\lambda \vec{J}^2} \psi)_{V_1,V_2}$$

and eq. (26) follows.

Let us now establish the estimate (A15). With no loss of generality we may assume that $U_1 = U_2 = 1$. Expressing A_ν in terms of B_s we obtain:

$$(\varphi, A_\nu \Psi)_{V_1,V_2} = \int \prod_{\mu=0}^{\nu} dU_\mu \; \varphi(U_0)\, B_{\frac{\lambda}{\nu}}(V_1 U_0 V_2 , W_1^1 U_1 W_2^1) \cdot$$

$$\cdot\, B_{\frac{\lambda}{\nu}}(W_1^1 U_1 W_2^1 , W_1^2 U_2 W_2^2) \cdot \ldots$$

$$\ldots \cdot B_{\frac{\lambda}{\nu}}(W_1^{\nu-1} U_{\nu-1} W_2^{\nu-1} , U_\nu)\, \Psi(U_\nu)$$

Here we have introduced matrices $W^{\mu}_{1,2} \in SU(2)$. They do not contribute, because they can be removed by a substitution of integration variables. Since the kernel $B_s(U,U')$ is analytic, we are allowed to take $W^{\mu}_{1,2} \in S\ell(2, \mathbb{C})$. A convenient choice is:

$$W^{\mu}_{1,2} = \exp \frac{\nu-\mu}{\nu} X_{1,2}$$

It is an elementary exercise to show that

$$| B_{\frac{\lambda}{\nu}}(W_1^{\mu} U_\mu W_2^{\mu} , W_1^{\mu+1} U_{\mu+1} W_2^{\mu+1})| = | B_{\frac{\lambda}{\nu}}(e^{\frac{1}{\nu}X_1} U_\mu e^{\frac{1}{\nu}X_2} , U_{\mu+1})| \leq$$

$$\leq \exp\left\{ \frac{1}{2\lambda\nu}\left[\sqrt{\mathrm{Tr}\, X_1^2} + \sqrt{\mathrm{Tr}\, X_2^2}\right]^2 + \frac{1}{\nu^2} F(X_1, X_2)\right\} B_{\frac{\lambda}{\nu}}(U_\mu , U_{\mu+1})$$

thus proving (A15) and therefore the estimate (26).

We still have to calculate the eigenvalues of B_s , i.e. to prove the lemma above.

Because $B_s(U,U') = B_s(UU'^{-1},1)$ and $B_s(U,1)$ is a class function we can expand it in a series as written down in eq. (A10). The coefficients $b_j(s)$ must be calculated from

$$\text{(A16)} \qquad b_j(s) = (2j+1)^{-1} \int dU \; B_s(U,1)\; \chi_j(U)$$

To evaluate the integral we make use of the Weyl integral formula ([17] 4.8.2).

This formula says that for any integrable function $f(U)$ we have:

$$\text{(A17)} \qquad \int dU \; f(U) = \int_{-2\pi}^{2\pi} \frac{dt}{4\pi} \int dU \;\; 2 \sin^2 \frac{t}{2} \; f(U e^{it\frac{\tau^3}{2}} U^{-1})$$

Thus, (A16) becomes

$$b_j(s) = (2j+1)^{-1} \int_{-2\pi}^{2\pi} \frac{dt}{4\pi} \;\; 2 \sin^2 \frac{t}{2} \;\; B_s(e^{it\frac{\tau^3}{2}}, 1)\, \chi_j(e^{it\frac{\tau^3}{2}})$$

Obviously

$$\chi_j(e^{it\frac{\tau^3}{2}}) = \sum_{m=-j}^{j} e^{itm} = \frac{\sin(j+\frac{1}{2})t}{\sin\frac{t}{2}}$$

Performing the substitution $t = 2\vartheta$ we get:

$$b_j(s) = \frac{4\sqrt{\pi}}{s^{3/2}} \;\; e^{\frac{3}{16}s}\, (2j+1)^{-1} \int_{-\pi}^{\pi} \frac{d\vartheta}{2\pi} \;\; \sin(2j+1)\vartheta \;\; \sin\vartheta \;\; e^{\frac{2}{s}(\cos\vartheta - 1)}$$

Finally, making an integration by parts and using an integral representation for I_{2j+1} ([18], § 8.431(5)) yields eq. (A10).

Footnotes

1) This will be explained in detail in sec. II.

2) This is not a complete introduction to lattice gauge theories. The reader seeking a more explicit presentation should consult the original papers [1,2] or the review articles [9,10].

3) A finite volume is assumed here. Our results will not depend on boundary conditions. For the sake of definiteness we may take a cubic lattice of volume $V = a^3(2N+1)^3$, i.e. $|n_k| \leq N$ and require that all those terms in (6) should be omitted that involve string bit variables not belonging to the finite volume lattice.

4) Let $\Psi_o(\theta)$ be the groundstate wave function. We may assume that $\Psi_o(\theta) > 0$ a.e.. Because the groundstate is unique and $\mathbb{H}$ is periodic in $\theta(n,k)$ we have

$$\Psi_o(\{\theta(n,k) + z(n,k)2\pi\}) = \lambda(\{z(n,k)\})\, \Psi_o(\{\theta(n,k)\})$$

for all sets $\{z(n,k)\}$ of integer numbers. Obviously, $|\lambda| = 1$ and $\lambda > 0$ because Ψ_o is positive. Hence $\lambda = 1$, i.e. Ψ_o is periodic.

5) I am indebted to K. Fredenhagen for explaining me this argument.

6) This result has first been established by J.Glimm and A. Jaffe [8] for euclidean lattice gauge theories.

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