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by

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Limitation of the Luminosity by Satellite Resonances

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Experimental and theoretical investigations have shown that the luminosity of the storage ring DORIS is limited by betatron-synchrotron resonances which are produced by the beam-beam interaction. The resonance frequencies are given by $Q_\beta = (p+rQ_s)/q$, where p, q, r are integers and Q_β and Q_s are the betatron and synchrotron wave numbers, respectively. It is shown that these resonances are caused by the crossing angle. Analytical investigations as well as computer simulations are in good agreement with measurements made at DORIS. The analytical investigation further shows that also a dispersion at the interaction point will produce these satellite resonances.

1. Introduction

We shall investigate a coupling between the betatron and synchrotron oscillation of an electron or positron in a storage ring. The coupling is produced by the space charge forces of two colliding beams, if the beams cross at an angle or if the dispersion at the interaction point is not zero. In these cases betatron-synchrotron resonances or satellite resonances are excited, which are given by the relation

$$Q_\beta = (p + r Q_s)/q \ , \tag{1.1}$$

where p, r and q are integers and Q_β and Q_s are the betatron and synchrotron wave numbers, respectively.

It is assumed that the chromaticity of the storage ring is compensated, so that the well known satellite resonances as described in [1)] cannot appear. The dispersion in the accelerating units is assumed to be zero, so that also the satellite resonances considered in [2)] cannot be excited.

In the case of a crossing angle, the closed orbit of a particle with a longitudinal displacement does not pass through the center of the opposing bunch. The closed orbit is therefore distorted, and the distortion varies with the longitudinal displacement, i.e. with the synchrotron oscillation. In the storage ring DORIS the orbit distortion is vertical due to the vertical crossing angle, and it is smaller than 15 % of the beam height. This is negligibly small and has no consequences except on a resonance. On a betatron-synchrotron resonance the betatron amplitude can increase so far that the particle is lost.

In the case of a dispersion at the interaction point, the closed orbit is distorted by the energy deviation of a particle. This gives, in linear approximation, a change of the momentum compaction factor. Since the distortion is a strongly nonlinear function of the energy deviation, there occurs also a nonlinear coupling between the betatron and synchrotron oscillation.

In both cases not only the betatron amplitude but also the synchrotron amplitude is changed. On a resonance there exists an invariant of motion which is determined by

$$\hat{z}^2 - \frac{q \alpha_M \bar{R} \beta}{r Q_s} \left(\frac{\widehat{\Delta E}}{E} \right)^2 = \mathrm{const.} \qquad (1.2)$$

with $\hat{z}$ = amplitude of the betatron oscillation
$\widehat{\Delta E}$ = amplitude of the energy oscillation
α_M = momentum compaction factor
$\bar{R}$ = mean radius of the ring
β = amplitude function

Since r can be positive or negative, the betatron and synchrotron amplitudes can increase or decrease at the same time or they can exchange their oscillation energy periodically. For most cases, however, the second term in Eq.(1.2) is much larger than the first term, and the change of the betatron amplitude is much larger than the change of the synchrotron amplitude.

The effect of the dispersion can roughly be compared with the effect of a crossing angle. The beam-beam interaction with a dispersion excites a nonlinear satellite resonance with the same strength as the beam-beam interaction with a crossing angle if

$$D_o \approx \phi \frac{\alpha_M \bar{R}}{Q_s} \qquad (1.3)$$

with D_o = dispersion at the interaction point, 2ϕ = crossing angle.

Although the width of these satellite resonances is very small ($\tilde{<}$ 0.001) as

compared to the distance between the resonances, they limit the luminosity of the storage ring DORIS. The reason are the decoupling transmitter and the rf-quadrupole, which are needed to suppress instabilities. The decoupling transmitter gives to different bunches different synchrotron frequencies, with a spread of $\Delta Q_s = \pm 0.006$. The rf-quadrupole produces a spread in betatron frequencies of $\Delta Q_\beta = \pm 0.005$. At the present operating point between 6.24 and 6.16, there are always some bunches on a resonance.

Theoretical and experimental investigations have shown that the satellite resonances are weaker for Q_β - values closer to an integer. We therefore expect that shifting the operating point into a region near 6.1 will permit higher currents and thus increase the luminosity.

2. Linear Approximation for the First Satellites of an Integer

2.1 One Interaction Point

The first satellites of an integer are defined by

$$Q_\beta = p \pm Q_s \qquad (2.1)$$

These satellites are the strongest resonances and they can occur in storage rings with a large Q_s like PETRA. They are excited if, for example, a separation of the two beams in the rf-straight sections leads to a crossing angle at the experimental points.

In linear approximation, i.e. for small oscillation amplitudes, a dispersion at the interaction point changes the momentum compaction factor but not the amplitudes. Only if one takes into account the nonlinearity of the space charge forces one gets a change of the amplitudes, also for the first satellites of an integer. Therfore, we assume in this section that the dispersion at the interaction point is zero.

We describe the betatron oscillation with the two variables

$$z \quad \text{and} \quad y = z'\beta - z\beta'/2$$

with $(\ )' = \frac{d}{d\ell}$ (ℓ = longitudinal coordinate).

The transformation of the betatron oscillation for one revolution, without the beam-beam interaction, is then given by

$$\begin{pmatrix} z \\ y \end{pmatrix}_{m+1} = \begin{pmatrix} \cos\mu_\beta & \sin\mu_\beta \\ -\sin\mu_\beta & \cos\mu_\beta \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix}_m \qquad (2.2)$$

with μ_β = betatron phase advance.

The synchrotron oscillation can be described with the two variables

$\zeta = \frac{\Delta E}{E}$ = relative energy deviation,

s = longitudinal position of a particle with respect to the center of the bunch.

The transformation of the synchrotron oscillation for one revolution can be written in the form

$$\begin{pmatrix} \zeta \\ s \end{pmatrix}_{m+1} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} \zeta \\ s \end{pmatrix}_m , \qquad (2.3)$$

where the coefficients S_{ij} satisfy the relations

$$S_{11} S_{22} - S_{12} S_{21} = 1 \qquad (2.4)$$

$$S_{11} + S_{22} = 2\cos\mu_s \qquad (2.5)$$

with $\mu_s = 2\pi Q_s$ = synchrotron phase advance.

Due to the space charge forces the betatron coordinate y is changed, as is well known, by

$$\Delta y = \beta_0 \Delta z' = -\beta_0 B (z + s\phi) \qquad (2.6)$$

with

$$\beta_0 B = 4\pi \xi \qquad (2.7)$$

$$\xi = \frac{r_e N_b \beta_0}{2\pi\gamma\sigma_{zef}(\sigma_x + \sigma_{zef})} \qquad (2.8)$$

and

$$\sigma_{zef} = \sqrt{\sigma_z^2 + \phi^2 \sigma_s^2}$$

r_e = electron radius
N_b = number of particles per bunch
γ = particle energy divided by its rest energy
$\sigma_{x,z,s}$ = standard deviations of the density distribution in the bunch.

At the same time the energy is changed by [3)]

$$\Delta\zeta = \phi\Delta z' = -\phi B (z + s\phi) \qquad (2.9)$$

Eqs. (2.6) and (2.9) show that the betatron and synchrotron oscillations are coupled by the crossing angle. The transformation matrix for the coupled oscillations is given by

$$M_1(\phi) = \begin{pmatrix} \cos\mu_\beta & \sin\mu_\beta & 0 & 0 \\ -\sin\mu_\beta & \cos\mu_\beta & 0 & 0 \\ 0 & 0 & S_{11} & S_{12} \\ 0 & 0 & S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta_0 B & 1 & 0 & -\phi\beta_0 B \\ -\phi B & 0 & 1 & -\phi^2 B \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.10)$$

In order to investigate the stability of the coupled motion we have to calculate the eigenvalues λ of the matrix $M_1(\phi)$. If we write λ in the form

$$\lambda = e^{i\mu} \qquad (2.11)$$

we find for μ the relation

$$\left(\cos\mu - \cos\mu_\beta + \tfrac{1}{2}\beta_0 B \sin\mu_\beta\right)\left(\cos\mu - \cos\mu_s + \tfrac{1}{2}\phi^2 B S_{21}\right) = \tfrac{1}{4}\phi^2 \beta_0 B^2 S_{21} \sin\mu_\beta \qquad (2.12)$$

The resonance is given by

$$\cos\mu_\beta - \tfrac{1}{2}\beta_o B \sin\mu_\beta = \cos\mu_s - \tfrac{1}{2}\phi^2 B S_{21} \qquad (2.13)$$

or

$$\mu_\beta \approx 2\pi p \pm \mu_s \qquad (2.14)$$

if $\beta_o B \sin\mu_\beta$ and $\phi^2 B S_{21}$ are small.

The exact value of S_{21} depends on the distribution of the cavities in the ring. In appendix A we show that for small synchrotron frequencies ($\mu_s^2 \ll 1$) S_{21} is given by

$$S_{21} \approx -\alpha_M C \qquad (2.15)$$

with $C = 2\pi\bar{R}$ = length of the equilibrium orbit.

In the case of a resonance we get from Eq(2.12)

$$\cos\mu = \cos\mu_\beta - \tfrac{1}{2}\beta_o B \sin\mu_\beta \pm \tfrac{1}{2}\phi B\sqrt{-\alpha_M C \beta_o \sin\mu_\beta} \qquad (2.16)$$

For $\sin\mu_\beta > 0$ μ has an imaginary part. Then two of the four eigenvalues λ have an absolute value larger than 1, and the coupled motion is unstable. With $\mu_\beta = 2\pi\, Q_\beta$ such an instability occurs for Q_β above an integer. For Q_β below an integer there are, at the resonance, two different real phases μ, and betatron and synchrotron oscillation exchange their oscillation energy periodically.

Decomposing the frequency shift in a real and an imaginary part

$$\mu = \mu_\beta + \delta\mu_r + i\,\delta\mu_i \qquad (2.17)$$

we get from Eq.(2.16)

$$\cos(\mu_\beta + \delta\mu_r) = \cos\mu_\beta - \tfrac{1}{2}\beta_o B \sin\mu_\beta \qquad (2.18)$$

and

$$\delta\mu_i = \pm\tfrac{1}{2}\phi B\sqrt{\alpha_M C \beta_o/(\sin\mu_\beta(1-\tfrac{1}{4}\beta_o^2 B^2) + \beta_o B\cos\mu_\beta)} \qquad (2.19)$$

with the condition

$$(\delta\mu_i)^2 \ll 1\,. \qquad (2.20)$$

The rise time T_r follows from Eqs.(2.19) and (2.7) to be

$$T_r = \frac{1}{\delta\mu_i f_o} = \frac{1}{2\pi\xi|\phi| f_o}\sqrt{\frac{\beta_o}{\alpha_M C}(\sin\mu_\beta + 4\pi\xi\cos\mu_\beta)} \qquad (2.21)$$

with $\quad f_o$ = revolution frequency.

Here we have neglected $4\pi^2\xi^2$ as a small quantity as compared to 1.

The width of the resonance $\pm\,\delta f_\beta$ is determined by the condition that in Eq.(2.12) μ is real and has only one solution. This condition yields

$$\begin{aligned}\delta f_\beta &= f_o\,\delta\mu_\beta \\ &= \pm f_o|\phi| B\sqrt{\alpha_M C \beta_o \sin\mu_\beta}/(\sin\mu_\beta + \tfrac{1}{2}\beta_o B\cos\mu_\beta) \\ &\approx \pm\frac{2}{T_r}\end{aligned} \qquad (2.22)$$

As an example we consider the storage ring DORIS, where an operation with only one interaction point is possible. The DORIS-parameters are

ϕ = 12 mrad, f_o = 1.04 MHz, α_M = 0.018, C = 288 m, β_{oz} = 1 m, ξ= 0.01, Q_s = 0.034.

This gives a rise time of 0.3 msec.

2.2 2n Interaction Points

An interesting case is a machine with two interaction points which have crossing angles with opposite signs. A machine with more than 2 interaction points but a periodic structure and crossing angles with alternating signs can easily be reduced to this case.

The revolution matrix for a machine with the superperiodicity 2 and with two equidistant interaction points is given by

$$M_2(\phi_1,\phi_2) = M_1(\phi_1)\cdot M_1(\phi_2) \qquad (2.23)$$

$M_1(\phi)$ is defined by Eq.(2.10), where we have to divide μ_β, μ_s and C by 2.

For the eigenvalues of M_2 one obtains with Eq.(2.11) the expression

$$\left(\cos^2\frac{\mu}{2} - a_\beta^2 + b\right)\left(\cos^2\frac{\mu}{2} - a_{s1}\,a_{s2} + b\right) =$$

$$= -4b\left(a_\beta^2 + a_{s1}a_{s2} + a_\beta a_{s1}\phi_2/\phi_1 + a_\beta a_{s2}\phi_1/\phi_2\right) \qquad (2.24)$$

with

$$a_\beta = \cos\frac{\mu_\beta}{2} - \frac{1}{2}\beta_0 B \sin\frac{\mu_\beta}{2}$$

$$a_{s1,2} = \cos\frac{\mu_s}{2} - \frac{1}{4}\phi_{1,2}^2\,\alpha_M C B$$

$$b = \frac{1}{8}\phi_1\phi_2\,\alpha_M C \beta_0 B^2 \sin\frac{\mu_\beta}{2}$$

The resonance condition is

$$a_\beta^2 = a_{s1}\,a_{s2} \qquad (2.25)$$

Since the relation

$$\cos\frac{\mu_s}{2} \gg \frac{1}{4}\phi_{1,2}^2\,\alpha_M C B \qquad (2.26)$$

is always satisfied, the resonance condition can also be written in the form

$$\pm a_\beta = a_{s1} = a_{s2} > 0 \qquad (2.27)$$

Introducing Eq.(2.27) into Eq.(2.24) one obtains

$$\left(\cos^2\frac{\mu}{2} - a_\beta^2 + b\right)^2 = \pm 4 a_\beta^2 b\,(\phi_1 \mp \phi_2)^2/\phi_1/\phi_2 \qquad (2.28)$$

The real and the imaginary part of the tune shift are defined by

$$\cos\frac{\mu_\beta + i\mu_i}{2} = \cos\frac{\mu_\beta}{2} - \frac{1}{2}\beta_0 B \sin\frac{\mu_\beta}{2} \qquad (2.29)$$

and

$$\delta\mu_i = \pm\frac{1}{2}\,|\phi_1 \pm \phi_2|\,B\sqrt{\frac{\pm\alpha_M C\beta_0}{\sin\frac{\mu_\beta}{2}\left(2 - \frac{1}{2}\beta_0^2 B^2\right) + 2\beta_0 B\cos\frac{\mu_\beta}{2}}} \qquad (2.30)$$

where we have applied Eq.(2.20).

For $\phi_2 = \phi_1$ one obtains an instability if $\sin\mu_\beta/2 > 0$, and for $\phi_2 = -\phi_1$ one obtains an instability if $\sin\mu_\beta/2 < 0$, which follows from Eq.(2.28). (The case $\beta_0 B\cos\mu_\beta/2 > \sin\mu_\beta/2\,(1 - \beta_0^2 B/4) > 0$ is not possible, according to Eq.(2.25).) With $\mu_\beta = 2\pi Q_\beta$ in the more interesting case $\phi_2 = -\phi_1$ an instability occurs for Q_β-values below an even integer or above an odd integer. The rise time is then given by

$$T_r = \frac{1}{4\pi\xi|\phi_1| f_0}\sqrt{\frac{-\beta_0}{\alpha_M C}\left(2\sin\frac{\mu_\beta}{2} + 8\pi\xi\cos\frac{\mu_\beta}{2}\right)} \qquad (2.31)$$

which is approximately half the rise time for one interaction point.

In the case of 2n interaction points with crossing angles with alternating signs the rise time is obtained from Eq.(2.31) by dividing μ_β, μ_s and C by n and multiplying f_0 with n.

$$T_r = \frac{1}{4n\pi\xi|\phi_1| f_0}\sqrt{\frac{-2n\beta_0}{\alpha_M C}\left(\sin\frac{\mu_\beta}{2n} + 4\pi\xi\cos\frac{\mu_\beta}{2n}\right)} \qquad (2.32)$$

As an example we consider the storage ring PETRA with 4 interaction points. A crossing angle of 1 mrad is assumed, which is caused by the separation of the beams in the cavities. The PETRA-parameters are

f_0 = 130 kHz, α_M = 0.004, C = 2304 m, β_{z0} = 0.3 m, ξ = 0.06, Q_s = 0.1,

$Q_{\beta z}$ = 22.1 or $Q_{\beta z}$ = 18.1.

This yields a rise time of

T_r = 1.7 msec,

which is much shorter than the damping time.

3. Analytical Investigation of the Nonlinear Satellite Resonances

3.1 Satellites due to a Crossing Angle

The satellite resonances which limit the luminosity of the storage ring DORIS have a q larger than 3 and an r from -3 to +3. These satellites are excited by the strong nonlinearity of the space charge forces. The behaviour of the oscillations on the resonances can be described exactly with the help of a series expansion. But we will limit the investigation to the first approximation and discuss only some basic properties. Thus, we want to compare the increase of the betatron amplitude and the increase of the synchrotron amplitude. We will also compare even and odd resonances and consider the effect of several interaction points.

With the usual transformation

$$\eta = \frac{z}{\sqrt{\beta}} \tag{3.1}$$

$$\Theta = \frac{1}{Q_\beta} \int \frac{dl}{\beta} \tag{3.2}$$

the differential equation for the betatron oscillation, including the space charge forces, can be written in the form

$$\frac{d^2\eta}{d\Theta^2} + Q_\beta^2 \eta = \lambda Q_\beta \sqrt{\beta} \sum_j f(z + s\phi_j)\, \delta(\Theta - \Theta_j) \tag{3.3}$$

with

$$f(z + s\phi) = \frac{\Delta p_z(z + s\phi)}{p_t}$$

$\Delta p_z(z + s\phi)$ = change of the transverse momentum,
p_t = absolute value of the particle momentum,
$\delta(\theta)$ = delta function,
θ_j = position of the j^{th} interaction point,
λ describes the order of approximation. The final value for λ is 1.

Near a satellite resonance the betatron frequency can be written as

$$Q_\beta^2 = \left(\frac{p + rQ_s}{q}\right)^2 + \lambda \Delta Q_\beta^2 \tag{3.4}$$

and Eq.(3.3) becomes

$$\frac{d^2\eta}{d\Theta^2} + \left(\frac{p+rQ_s}{q}\right)^2 \eta = \lambda \left[Q_\beta \sqrt{\beta} \sum_j f(z + s\phi_j)\, \delta(\Theta - \Theta_j) - \eta \Delta Q_\beta^2 \right] \tag{3.5}$$

The change of the energy due to the space charge forces is, with Eq.(2.9),

$$\Delta \frac{\Delta E}{E} = \phi_j f(z + s\phi_j) \tag{3.6}$$

The total change of the energy due to the accelerating voltage and due to the space charge forces is then given by

$$\frac{d}{d\Theta}\frac{\Delta E}{E} = \frac{Q_s^2}{\alpha_M \bar{R}} s + \lambda \sum_j \phi_j f(z + s\phi_j) \tag{3.7}$$

Here we have assumed that the change of the synchrotron coordinates is small for one revolution, i.e. $4\pi^2 Q_s^2 \ll 1$. Also we have used the variable θ, which is proportional to the betatron phase advance, instead of the azimuth. This is correct if θ_j coincides with the azimuth at the interaction points, and that is the case if, for example, the interaction points are equidistant. The actual synchrotron coordinates between the interaction points can be found by transforming θ into the azimuth or into the longitudinal coordinate (Eq.(3.2)).

With the second equation for the synchrotron oscillation

$$\frac{ds}{d\Theta} = -\alpha_M \bar{R} \frac{\Delta E}{E} \tag{3.8}$$

one obtains for s the differential equation

$$\frac{d^2 s}{d\Theta^2} + Q_s^2 s = -\lambda \alpha_M \bar{R} \sum_j \phi_j f(z + s\phi_j)\, \delta(\Theta - \Theta_j) \tag{3.9}$$

We write the solutions of the two coupled equations Eq.(3.5) and Eq.(3.9) in the form

$$\eta(\Theta) = A_\beta(\Theta) \cos\psi_\beta(\Theta) + \sum_{k=1}^{\infty} \lambda^k u_{\beta k}(A_\beta, A_s, \psi_\beta, \psi_s, \Theta) \tag{3.10}$$

$$s(\Theta) = A_s(\Theta) \cos\psi_s(\Theta) + \sum_{k=1}^{\infty} \lambda^k u_{sk}(A_\beta, A_s, \psi_\beta, \psi_s, \Theta) \tag{3.11}$$

with

$$\psi_\beta(\Theta) = \frac{p + rQ_s}{q}\Theta + \alpha_\beta(\Theta) \tag{3.12}$$

$$\psi_s(\Theta) = Q_s \Theta + \alpha_s(\Theta) \tag{3.13}$$

The most important quantities are the amplitudes A_β and A_s which are, as well as the phases α_β and α_s, slowly varying with θ. The functions $u_{\beta k}$ and u_{sk}

give only a small contribution to the total oscillation amplitudes and they are neglected in the following consideration. In appendix B is shown that the behaviour of A_β and A_s can be described in a first approximation, i.e. linear in λ, with the help of the Hamiltonian

$$H(A_\beta, A_s, \alpha_\beta, \alpha_s) = \frac{1}{2} \sum_{k=-\infty}^{\infty} H_k(A_\beta, A_s)\, e^{ik(q\alpha_\beta - r\alpha_s)} \qquad (3.14)$$

with

$$H_k(A_\beta, A_s) = \frac{1+(-1)^{k(q+r)}}{4\pi^3} \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \sum_j F(A_\beta\sqrt{\beta}\cos\psi_\beta + \phi_j A_s \cos\psi_s)\cdot$$
$$\cdot\, \delta(\Theta-\Theta_j)\, e^{ik(q\psi_\beta - r\psi_s - p\Theta)}\, d\psi_\beta\, d\psi_s\, d\Theta - 2\Delta Q_\beta A_\beta^2 \delta_{ok} \qquad (3.15)$$

$$F(x) = \int_0^x f(y)\, dy \qquad \delta_{ok} = \begin{matrix} 1 & k = 0 \\ 0 & k \neq 0 \end{matrix}$$

The change of the amplitudes and phases is than given by

$$\frac{dA_\beta^2}{d\Theta} = \frac{\partial H}{\partial \alpha_\beta} \qquad (3.16) \qquad\qquad \frac{d\alpha_\beta}{d\Theta} = -\frac{\partial H}{\partial A_\beta^2} \qquad (3.17)$$

$$\frac{dA_s^2}{d\Theta} = -\frac{\alpha_M \bar{R}}{Q_s}\frac{\partial H}{\partial \alpha_s} \qquad (3.18) \qquad\qquad \frac{d\alpha_s}{d\Theta} = \frac{\alpha_M \bar{R}}{Q_s}\frac{\partial H}{\partial A_s^2} \qquad (3.19)$$

If we introduce new coordinates for the synchrotron oscillation by

$$\tilde{A}_s = \sqrt{\frac{Q_s}{\alpha_M \bar{R}}}\, A_s \;, \qquad \tilde{\alpha}_s = -\alpha_s$$

we get

$$\frac{d\tilde{A}_s^2}{d\Theta} = \frac{\partial H}{\partial \tilde{\alpha}_s} \qquad (3.20) \qquad\qquad \frac{d\tilde{\alpha}_s}{d\Theta} = -\frac{\partial H}{\partial \tilde{A}_s^2} \qquad (3.21)$$

instead of Eqs.(3.18) and (3.19). The canonical equations have now the same form for the betatron and synchrotron motion.

As a first result we obtain from the Hamiltonian

$$\frac{dA_\beta^2}{d\Theta} = \frac{qQ_s}{r\alpha_M \bar{R}}\frac{dA_s^2}{d\Theta}$$

Integrating with respect to θ and replacing A_β and A_s by the amplitudes $\hat{z}/\sqrt{\beta}$ and $\hat{s}$ one obtains

$$\hat{z}^2 - \frac{qQ_s\beta}{r\alpha_M \bar{R}}\hat{s}^2 = const. \qquad (3.22)$$

With

$$\hat{s} = \frac{\alpha_M \bar{R}}{Q_s}\frac{\widehat{\Delta E}}{E}\,, \qquad (3.23)$$

which follows from Eq.(3.8), one finally gets Eq.(1.2).

The invariant defined by Eq.(1.2) or Eq.(3.22) agrees with the invariant derived in [2] for a different excitation of satellite resonances, namely the excitation by a dispersion in the cavities. The invariant shows that for positive r the betatron and the synchrotron amplitudes increase or decrease at the same time. For negative r, only one of the two amlitudes can increase whereas the other one decreases. In that case betatron and synchrotron oscillation exchange their energy periodically. Numerical evalution shows that the second term in Eq.(3.22) is much larger than the first term, and thus the increase of the betatron amplitude is much larger than the increase of the synchrotron amplitude.

In the case of one interaction point and in the case of an even number of equidistant interaction points that have crossing angles with alternating signs and the same absolute value, the Hamiltonian can be written in the form

$$H = \frac{1}{2}H_0 + H_1\cos(q\alpha_\beta - r\alpha_s) + H_2\cos 2(q\alpha_\beta - r\alpha_s) + \dots \qquad (3.24)$$

with

$$H_k = \frac{S_{\phi k}}{4\pi^3}\left(1+(-1)^{k(q+r)}\right)\int_0^{2\pi}\int_0^{2\pi} F(A_\beta\sqrt{\beta}\cos\psi_\beta + \phi A_s\cos\psi_s)\cdot$$
$$\cdot \cos kq\psi_\beta \cos kr\psi_s\, d\psi_\beta\, d\psi_s - 2\Delta Q_\beta A_\beta^2 \delta_{ok} \qquad (3.25)$$

and

$$S_{\phi k} = \begin{cases} 1 & \text{for 1 interaction point} \\ ((-1)^{kr} + (-1)^{2kp/N_i})\,\frac{N_i}{2} & \text{for } 2\,kp/N_i = \text{integer} \\ 0 & \text{for } 2\,kp/N_i \neq \text{integer} \end{cases}$$

N_i = number of interaction points

The most important term in Eq.(3.24) is H_1, which gives the largest change of the amplitudes. H_1 vanishes if one of the following conditions is satisfied.

a) $r + q$ = odd

b) $r + 2p/N_i$ = odd

c) $p \neq \text{integer} * N_i/2$

3.2 Satellites due to a Dispersion at the Interaction Point

In the case of a dispersion at the interaction point the satellites are excited by the nonlinear part of the space charge forces. We choose as the reference orbit for the betatron oscillation an off-energy orbit which obtains the kick due to the space charge forces. Further we assume that the N_i interaction points are equidistant and that all interaction points have the same β_o and D_o. The off-energy orbit is then given by

$$X_o(\ell) = D(\ell)\frac{\Delta E}{E} + C_o\sqrt{\beta(\ell)}\,\cos\left(\int_0^{\ell}\frac{d\ell}{\beta} - \frac{\mu_\beta}{2N_i}\right) \qquad (3.26)$$

The constant C_o is determined by the condition

$$\Delta X_o'(0) = f(X_o(0))$$

or

$$2C_o\sin(\mu_\beta/2N_i)/\sqrt{\beta_o} = f\left(D_o\frac{\Delta E}{E} + C_o\sqrt{\beta_o}\cos(\mu_\beta/2N_i)\right) \qquad (3.27)$$

$f(x_o)$ is a nonlinear function and C_o is a root of Eq.(3.27).

With the transformation Eqs.(3.1) and (3.2) one obtains the differential equation of the betatron oscillation

$$\frac{d^2\eta}{d\theta^2} + \left(\frac{p+rQ_s}{q}\right)^2\eta = \lambda\left\{Q_\rho\sqrt{\beta}\sum_j\left[f(x+x_o) - f(x_o)\right]\delta(\theta-\theta_j) - \eta\Delta Q_\beta^2\right\} \qquad (3.28)$$

As we have seen in the preceeding section the change of the betatron oscillation due to the beam-beam interaction is much larger than the change of the synchrotron oscillation. Therefore we will neglect here the change of the synchrotron oscillation, which is produced by the change of the path length due to the betatron oscillation[2]. We will assume that the synchrotron amplitude and the synchrotron frequency remain constant. The solution of Eq.(3.28) can then be written in the form

$$\eta(\theta) = A_\beta(\theta)\cos\psi_\beta(\theta) + \sum_{k=1}^{\infty}\lambda^k v_k(A_\beta, A_E, \psi_\beta, \psi_E, \theta) \qquad (3.29)$$

with

$$\frac{\Delta E}{E} = A_E\cos\psi_E(\theta) \qquad (3.30)$$

$$\psi_\beta(\theta) = \frac{p+rQ_s}{q}\theta + \alpha_\beta(\theta) \qquad (3.31)$$

$$\psi_E(\theta) = Q_s\theta \qquad (3.32)$$

Similar as in Appendix B one can find a Hamiltonian which describes in a first approximation, i.e. linear in λ, the behaviour of the betatron amplitude and phase. The Hamiltonian is given by

$$H = \frac{1}{2}H_o + H_1\cos q\psi_\beta + H_2\cos 2q\psi_\beta + \dots \qquad (3.33)$$

with

$$H_k = \frac{S_{Dk}}{4\pi^2}\left(1 + (-1)^{k(q+r)}\right)\int_0^{2\pi}\left[\frac{1}{\pi}\int_0^{2\pi}F(A_\beta\sqrt{\beta}\cos\psi_\beta + X_o)\cos kq\psi_\beta\, d\psi_\beta - A_\beta\sqrt{\beta}\,f(x_o)\,\delta_{kq,1}\right]\cos kr\psi_E\, d\psi_E + 2A_\beta^2\Delta Q_\beta \qquad (3.34)$$

$$S_{Dk} = \begin{cases} N_i & \text{for } kp/N_i = \text{integer} \\ 0 & \text{for } kp/N_i \neq \text{integer}\end{cases}$$

For a small dispersion or for small synchrotron oscillations, i.e.

$$\left(D_o\frac{\Delta E}{E}\right)^2 \ll \sigma_x^2$$

the function f(u) is approximately - Bu. Eq.(3.27) can now be solved and yields

$$C_o = - \frac{\sqrt{\beta_o}\, D_o B}{2 \sin(\mu_\beta/2N_i) + \beta_o B \cos(\mu_\beta/2N_i)} \frac{\Delta E}{E}$$

x_o can be written in the form

$$X_o(0) = \bar{D}_o \frac{\Delta E}{E} \qquad (3.35)$$

with the new dispersion

$$\bar{D}_o = \frac{D_o}{1 + 2\pi\xi \cot(\mu_\beta/2N_i)} \qquad (3.36)$$

Introducing Eq.(3.35) into Eq.(3.34) one finds that the two Hamiltonians Eqs.(3.25) and (3.34) are very similar. Only the H_1 are different for q = 1, and the combined action of several interaction points can give different H_1, H_3, H_5 etc., which follows from $S_{\phi k}$. Except for these cases the two Hamiltonians are equal if the arguments of F are equal, i.e. if

$$\phi A_s = \bar{D}_o A_E \qquad (3.37)$$

With Eq.(3.23) one finally obtains Eq.(1.3)

4. Computer Simulation

A simulation of the betatron and synchrotron oscillation has been done on a digital computer. The coupling of the oscillations due to the crossing angle, the nonlinearity of the space charge forces and the nonlinearity of the synchrotron potential have been taken into account. The following equations are used in the computer program.

Between two interaction points the coordinates of the betatron and synchrotron oscillation are transformed by

$$z_{n+1} = z_n \cos\mu_\beta + y_n \sin\mu_\beta \qquad (4.1)$$

$$y_{n+1} = - z_n \sin\mu_\beta + y_n \cos\mu_\beta \qquad (4.2)$$

$$\left(\frac{\Delta E}{E}\right)_{n+1} = \left(\frac{\Delta E}{E}\right)_n + \frac{2\pi Q_s^2}{h \alpha_M \cos\psi_e}\left(\sin(\psi_e + \frac{h}{R} s_n) - \sin\psi_e\right) \qquad (4.3)$$

$$S_{n+1} = S_n - \alpha_M C \left(\frac{\Delta E}{E}\right)_{n+1} \qquad (4.4)$$

with h = harmonic number, ψ_e = equilibrium phase

At the interaction point the transformation is given by

$$z_{n+2} = z_{n+1} \qquad (4.5)$$

$$y_{n+2} = y_{n+1} + \beta_o f(z_{n+1} + S_{n+1}\phi) \qquad (4.6)$$

$$\left(\frac{\Delta E}{E}\right)_{n+2} = \left(\frac{\Delta E}{E}\right)_{n+1} + \phi f(z_{n+1} + S_{n+1}\phi) \qquad (4.7)$$

$$S_{n+2} = S_{n+1} \qquad (4.8)$$

The function f gives the change of the betatron angle due to the space charge forces. Since in DORIS the effective beam cross section is approximately circular, i.e.

$$\sigma = \sigma_x = \sqrt{\sigma_z^2 + \phi^2\sigma_s^2}$$

an exact analytical expression for f can be used. It is given by

$$f(v) = 8\pi\xi \frac{\sigma^2}{v\beta_c}\left(e^{-\frac{v^2}{2\sigma^2}} - 1\right) \qquad (4.9)$$

With these equations the oscillations were simulated for 2000 revolutions. The initial phase between the betatron and synchrotron oscillation was varied in steps of 2π/100, to look for the maximum amplitude which can occur during the 2000 revolutions.

The following parameters from DORIS were used

ξ = 0.01, β_o = 1.0 m, σ = 0.23 mm, Q_s = 0.034, h = 480, α_M = 0.018, C = 288 m, $\psi_e = 3^o$, $\hat{z}_{ini}$ = 0.8 mm, $(\widehat{\Delta E}/E)_{ini} = 1.8 \cdot 10^{-3}$

The simulation showed that the nonlinearity of the synchrotron potential had no important influence on the results. To reduce the computer time Eq.(4.3) was therefore replaced by

$$\left(\frac{\Delta E}{E}\right)_{n+1} = \left(\frac{\Delta E}{E}\right)_n + \frac{2\pi Q_s^2}{\alpha_M R} S_n \qquad (4.10)$$

Figs. 1 to 4 show the behaviour of the amplitudes on the two satellite resonances $Q_\beta = (31 \pm Q_s)/5$ during the 2000 revolutions. The simulation considered only one interaction point. Figs. 1 and 2 show that for a positive r the two amplitudes increase or decrease at the same time. The growth of the amplitudes is limited by the nonlinearity of the space charge forces. Figs. 3 and 4 show for a negative r a periodical exchange of the oscillation energy.

Since the change of the betatron amplitude is much larger than the change of the synchrotron amplitude , the particles will be lost when they exceed the betatron admittance. It is, therefore, sufficient to consider only the betatron amplitude. To reduce the computer time, the change of the synchrotron oscillation due to the beam-beam interaction was neglected and Eq. (4.7) was replaced by $(\Delta E/E)_{n+2} = (\Delta E/E)_{n+1}$. This gives an error which is, in the investigated range of amplitudes, less than 5 %.

The maximum betatron amplitude was now determined for more than 1000 Q_β-values between 6.02 and 6.48. The results for Q_β between 6.5 and 7.0 are mirror symmetric, since the synchrotron oscillation is constant.

Fig. 5 shows the ratio of the maximum betatron amplitude which occurs during the 2000 revolutions, over the minimum or initial amplitude. The simulation was done for 1 and 2 interaction points. In the case of two interaction points, the two crossing angles have opposite signs. The width of the resonances is very small. The total width of most of the resonances is smaller than 0.001. It could not be included in fig. 5. Furthermore only those resonances are shown which reached, either for 1 or for 2 interaction points, an increase of more than 50 % of their initial amplitudes.

The property that the satellites with an even q + r are stronger than satellites with odd q + r agrees with the analytical investigations (Eq.(3.15)). It is caused by the symmetry of the potential of the space charge forces. Comparing the results for 1 and 2 interaction points one finds that satellites with an even p + r and an even q + r are strengthened by the second interaction point whereas satellites with odd p + r are weakened. This follows also from Eq.(3.25). In the case of 2 interaction points the Hamiltonian H_1 which yields the largest contribution to the change of the amplitudes is proportional to $(1+(-1)^{p+r})(1+(-1)^{q+r})$.

Besides the satellite resonances of one dimensional betatron oscillations also satellites of coupling resonances between horizontal and vertical betatron oscillations can occur. These satellites can be simulated by introducing a horizontal displacement x into the expression for the change of the vertical momentum due to the space charge forces. Eq.(4.9) is then replaced by

$$f(z+s\phi, x) = 8\pi \xi \frac{\sigma^2}{\beta_c} \left(e^{-\frac{u^2}{2\sigma^2}} - 1\right) \frac{z+s\phi}{u^2} \qquad (4.11)$$

with

$$u^2 = (z+s\phi)^2 + x^2$$

The change of the horizontal betatron oscillation due to the space charge forces was neglected, since the horizontal amplitude function is a factor of about 4 smaller than the vertical amplitude function.

The horizontal betatron frequency was $Q_{\beta x} = \mu_{\beta x}/2\pi = 7.223$ and was kept constant. The horizotnal betatron amplitude was 0.92 mm, i.e. 4 standard deviations. The other parameters had the same values as before, and the vertical betatron frequency was varied again from 6.02 to 6.48. For one interaction point the following 11 resonances led to an increase of more than 80 %.

Table 1

	Resonance	$\hat{z}_{max}/\hat{z}_{min}$
1), 2)	$Q_z = (19 \pm Q_s)/3$	3.1
3)	$6 + Q_s$	2.8
4)	$Q_x - 1$	2.4
5)	$27/2 - Q_x$	2.3
6), 7)	$(5 + 2Q_x \pm Q_s)/3$	2.0
8), 9)	$(33 - 2Q_x \pm Q_s)/3$	1.9
10), 11)	$(4 + 2Q_x \pm Q_s)/3$	1.9

The satellite resonances for one dimensional betatron oscillations, for example $(19 \pm Q_s)/3$, are weaker in this case, since the vertical space charge forces are weaker because of the large horizontal amplitude. Only particles with a small horizontal amplitude will get a large increase of the vertical amplitude on a vertical resonance.

5. Measurement at DORIS

The measurements in the storage ring DORIS were done at 1.8 GeV. The behaviour of weak positron bunches colliding with strong electron bunches was investigated. Since the exact dependence of the positron life time on the betatron and synchrotron frequency should be measured, the decoupling transmitter and the rf-quadrupol, which give a frequency spread, could not be used in the positron ring. To provide a central collision of the bunches also the decoupling transmitter in the electron ring could not be used, because it produces additionally a longitudinal displacement of the bunches. Under these conditions a maximum bunch current of about 1.3 mA could be stored in 30 electron bunches. The resulting ξ is

$$\xi = 0.009 \qquad (5.1)$$

with $\sigma_x = 0.23$ mm, $\sigma_z = 0.2$ mm, $\sigma_s = 1$ cm

In most of the measurements the vertical betatron frequency of the positrons was varied nearly continuously by a computer, whereas the synchrotron frequency was kept constant. The speed of the variation was 1 kHz per 10 sec. Simultaneously the current of the 30 positron bunches was plotted. The sharp beam losses yield exactly the frequencies of the resonances.

An example of such a measurement is shown in Fig. 6 for one interaction. To increase the effect, a vertical scraper was used which reduces the aperture. Because of the very small width of the resonances and because of the speed of the variation only those resonances could be observed which lead to a life time less than 15 min. Table 2 shows a comparison of the measured and calculated resonance frequencies ($Q_s = 0.024$).

Table 2

N	$Q_{\beta meas.}$	$Q_{\beta cal.}$	resonance
1	6.0932	-	-
2	.0961	.0960	$6 + 4Q_s$
3	.1212	.1190	$(49-2Q_s)/8$
4	.1243	.1250	49/8
5	.1274	.1280	$(49+Q_s)/8$
6	.1397	.1394	$(43-Q_s)/7$
7	.1628	.1627	$(37-Q_s)/6$
8	.1666	.1667	37/6
9	.1801	.1808	$(31-4Q_s)/5$
10	.1858	.1856	$(31-3Q_s)/5$
11	.1921	.1904	$(31-2Q_s)/5$
12	.1952	.1952	$(31-Q_s)/5$
13	.1994	.2000	31/5
14	.2043	.2048	$(31+Q_s)/5$

In many measurements with one interaction point the following 25 vertical resonances are observed.

$6 + 3Q_s$, $6 + 4Q_s$,
49/8, $(49+Q_s)/8$, $(49-2Q_s)/8$,
43/7, $(43\pm Q_s)/7$, $(43-2Q_s)/7$
37/6, $(37-Q_s)/6$, $(37-2Q_s)/6$
31/5, $(31\pm Q_s)/5$, $(31-2Q_s)/5$, $(31\pm 3Q_s)/5$, $(31\pm 4Q_s)/5$
25/4, $(25-Q_s)/4$, $(25-2Q_s)/4$,
$28-3Q_x$, Q_x-1-Q_s,

The lifetime of the positrons on a resonance was between a few seconds and 15 minutes. The strength of the resonances was not well reproducible because the variation of Q_β was not quite continuous. The power supplies of the quadrupoles are digitally controlled, so Q_β is varied in very small steps which have the same order of magnitude as the width of the resonances.

The width of the satellite resonances was measured in several cases. It is about 0.0005. For satellites of coupling resonances it can be larger by a factor of 2 or 3.

The beam dimensions were measured using the synchrotron light, when the betatron frequency was on or near a resonance. It turned out that the dimensions, i.e. the core of the bunches, were not changed by a resonance. That means that only particles with a large amplitude are strongly affected by a resonance. Particles with a small amplitude will leave the resonance very fast due to the strong nonlinear frequency shift in the core of a bunch.

A horizontal satellite resonance, which was only produced by the beam-beam interaction, was not found. Also with scrapers no reduction of the lifetime could be observerd.

Acknowledgements

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Appendix A

One can verify directly that the expression

$$I_1 = \zeta_n^2 + \frac{S_{22} - S_{11}}{S_{21}} \zeta_n s_n - \frac{S_{12}}{S_{21}} s_n^2 \qquad \text{(A.1)}$$

is an invariant with respect to the transformation Eq.(2.3).

If the synchrotron frequency is small, i.e. if $\mu_s^2 \ll 1$, or if the accelerating units are distributed very uniformly about the orbit, the change of the energy and the change of the longutidinal displacement are given by

$$\frac{d\zeta}{dt} = \frac{f_0 h e U \cos\psi_e}{ER} s \qquad \text{(A.2)}$$

$$\frac{ds}{dt} = - f_0 \alpha_M C \zeta \qquad \text{(A.3)}$$

with f_o = revolution frequency, h = harmonic number
e = electron charge, U = rf voltage, ψ_e = synchronous phase

From Eqs.(A.2) and (A.3) follows that the expression

$$I_2 = \zeta^2 + \left(\frac{Q_s}{\alpha_M \bar{R}}\right)^2 s^2 \qquad \text{(A.4)}$$

with

$$Q_s^2 = \frac{h \alpha_M e U \cos\psi_e}{2\pi E}$$

is an invariant of motion. Comparing Eqs. (A.1) and (A.4) one obtains

$$S_{22} = S_{11} \qquad \text{(A.5)}$$

$$-\frac{S_{12}}{S_{21}} = \left(\frac{Q_s}{\alpha_M \bar{R}}\right)^2 \qquad \text{(A.6)}$$

With Eqs. (2.4) and (2.5) one obtains

$$S_{12} S_{21} = - \sin^2 \mu_s \qquad \text{(A.7)}$$

or, since μ_s is small,

$$S_{12} S_{21} = - \mu_s^2 \qquad \text{(A.8)}$$

With Eqs. (A.6) and (A.8) one finally gets

$$S_{21} = - \alpha_M C \qquad \text{(A.9)}$$

where the sign is determined by the definition of s according to Eq.(2.6).

Appendix B

We extend a method described in [4] to investigate two coupled differential equations. The amplitudes A_β and A_s and the phases α_β and α_s are defined as the solutions of the system

$$\frac{dA_{\beta,s}}{d\Theta} = \sum_{k=1}^{\infty} \lambda^k G_{\beta,s}(A_\beta, A_s, \alpha_\beta, \alpha_s) \qquad \text{(B.1)}$$

$$\frac{d\alpha_{\beta,s}}{d\Theta} = \sum_{k=1}^{\infty} \lambda^k g_{\beta,s}(A_\beta, A_s, \alpha_\beta, \alpha_s) \qquad (B.2)$$

In first approximation the function f can be written in the form

$$f(z + s\phi_j) = f(\sqrt{\beta} A_\beta \cos\psi_\beta + \phi_j A_s \cos\psi_s) + \lambda\{\ldots\}$$

Now we may expand f in a Fourier series.

$$\sum_j f(z + s\phi_j)\,\delta(\Theta - \Theta_j) = \sum_{\ell,m,n} f_{\ell mn}(A_\beta, A_s)\, e^{i(\ell\psi_\beta + m\psi_s + n\Theta)} + \lambda\{\ldots\} \qquad (B.3)$$

The functions $u_{\beta k}$ and u_{sk} in Eqs.(3.10) and (3.11) are periodic with respect to ψ_β, ψ_s and θ and we may write

$$u_{\beta,sk}(A_\beta, A_s, \psi_\beta, \psi_s, \Theta) = \sum_{\ell,m,n} u_{\beta,sk\ell mn}(A_\beta, A_s)\, e^{i(\ell\psi_\beta + m\psi_s + n\Theta)} \qquad (B.4)$$

Introducing Eqs.(B.1) to (B.4) into Eq.(3.5) and considering only terms linear in λ one obtains

$$-2Q_\beta[G_{\beta 1}\sin\psi_\beta + (g_{\beta 1} - \Delta Q_\beta)A_\beta\cos\psi_\beta]$$

$$+ \sum_{\ell,m,n} u_{\beta 1\ell mn}\left[\left(\frac{p+rQ_s}{q}\right)^2 - \left(\ell\,\frac{p+rQ_s}{q} + m Q_s + n\right)^2\right] e^{i(\ell\psi_\beta + m\psi_s + n\Theta)}$$

$$= Q_\beta\sqrt{\beta}\sum_{\ell,m,n} f_{\ell mn}\, e^{i(\ell\psi_\beta + m\psi_s + n\Theta)} \qquad (B.5)$$

with

$$\Delta Q_\beta^2 = 2Q_\beta\Delta Q_\beta + \lambda\{\ldots\}.$$

For

$$(\ell \pm 1)\,\frac{p+rQ_s}{q} + mQ_s + n \neq 0$$

one obtains

$$u_{\beta 1\ell mn} = Q_\beta\sqrt{\beta}\, f_{\ell mn}\left[\left(\frac{p+rQ_s}{q}\right)^2 - \left(\ell\,\frac{p+rQ_s}{q} + mQ_s + n\right)^2\right]^{-1} \qquad (B.6)$$

For

$$(\ell \pm 1)\,\frac{p+rQ_s}{q} + mQ_s + n = 0$$

one obtains from Eq.(B.5)

$$-G_{\beta 1}\sin\psi_\beta - (g_{\beta 1} - \Delta Q_\beta)A_\beta\cos\psi_\beta = \frac{\sqrt{\beta}}{2}\sum_{\ell,m,n} f_{\ell mn}\, e^{i(\ell\psi_\beta + m\psi_s + n\Theta)} \qquad (B.7)$$

With $\ell \pm 1 = kq$, $m = -kr$, $n = -kp$

and with Eqs.(3.12) and (3.13) one obtains from Eq.(B.7)

$$G_{\beta 1} = -\frac{i}{2}\sqrt{\beta}\sum_{k=-\infty}^{\infty}\left(f_{kq+1,-kr,-kp} - f_{kq-1,-kr,-kp}\right) e^{ik(q\alpha_\beta - r\alpha_s)} \qquad (B.8)$$

and

$$(g_{\beta 1} - \Delta Q_\beta)A_\beta = -\frac{1}{2}\sqrt{\beta}\sum_{k=-\infty}^{\infty}\left(f_{kq+1,-kr,-kp} + f_{kq-1,-kr,-kp}\right) e^{ik(q\alpha_\beta - r\alpha_s)} \qquad (B.9)$$

Using

$$f_{kq\pm 1,-kr,-kp} = \frac{1}{8\pi^3}\int_0^{2\pi}\!\!\int_0^{2\pi}\!\!\int_0^{2\pi} \sum_j f(A_\beta\sqrt{\beta}\cos\psi_\beta + \phi_j A_s\cos\psi_s)\,\delta(\Theta - \Theta_j)\cdot$$

$$e^{ik(r\psi_s + p\Theta - q\psi_\beta) \mp i\psi_\beta}\, d\psi_\beta\, d\psi_s\, d\Theta \qquad (B.10)$$

and integrating by parts one finally gets

$$\frac{dA_\beta^2}{d\Theta} = 2A_\beta G_{\beta 1} = \frac{\partial H}{\partial\alpha_\beta} \qquad (B.11)$$

$$\frac{d\alpha_\beta}{d\Theta} = g_{\beta 1} = -\frac{\partial H}{\partial A_\beta^2} \qquad (B.12)$$

where H is defined by Eq.(3.14). In H the symmetry of the function F is taken into account.

Introducing Eqs.(B.1) to (B.4) into Eq.(3.9) one obtains for the terms linear in λ

$$
\begin{aligned}
& -2Q_s\left(G_{s1}\sin\psi_s + g_{s1}A_s\cos\psi_s\right) \\
& + \sum_{\ell,m,n} u_{s1\ell mn}\left[Q_s^2 - \left(\ell\,\frac{p+rQ_s}{q} + mQ_s + n\right)^2\right] e^{i(\ell\psi_\beta + m\psi_s + n\theta)} \\
& = -\alpha_M \bar{R} \sum_j \phi_j\, f\!\left(A_\beta\sqrt{\beta}\cos\psi_\beta + \phi_j A_s\cos\psi_s\right)\delta(\Theta - \Theta_j)
\end{aligned}
$$

The functions $u_{s1\ell mn}$ are determined by

$$
\begin{aligned}
u_{s1\ell mn} = -\frac{\alpha_M\bar{R}}{8\pi^3}\int_0^{2\pi}\!\!\int_0^{2\pi}\!\!\int_0^{2\pi} \sum_j \phi_j\, f\!\left(A_\beta\sqrt{\beta}\cos\psi_\beta + \phi_j A_s\cos\psi_s\right) e^{-i(\ell\psi_\beta + m\psi_s + n\theta)}\,\cdot \\
\cdot\, d\psi_\beta\, d\psi_s\, d\theta\left[Q_s^2 - \left(\ell\,\frac{p+rQ_s}{q} + mQ_s + n\right)^2\right]^{-1} \qquad \text{(B.13)}
\end{aligned}
$$

for

$$\ell\,\frac{p+rQ_s}{q} + mQ_s + n \neq Q_s\,. \qquad \text{(B.14)}$$

If Eq.(B.14) is not satisfied, we substitute

ℓ = kq, m= - kr ± 1, n = - kp

and obtain

$$
\begin{aligned}
G_{s1}\sin\psi_s + g_{s1}A_s\cos\psi_s = \frac{\alpha_M\bar{R}}{8\pi^2 Q_s}\sum_{k=-\infty}^{\infty}\int_0^{2\pi}\!\!\int_0^{2\pi}\!\!\int_0^{2\pi}\sum_j \phi_j\, f\!\left(A_\beta\sqrt{\beta}\cos\bar{\psi}_\beta + \phi_j A_s\cos\bar{\psi}_s\right)\cdot \\
\cdot\,\delta(\bar{\theta} - \theta_j)\, e^{ik\left(q(\psi_\beta - \bar{\psi}_\beta) + r(\bar{\psi}_s - \psi_s) + p(\bar{\theta} - \theta)\right)}\cos(\psi_s - \bar{\psi}_s)\, d\bar{\psi}_\beta\, d\bar{\psi}_s\, d\bar{\theta} \qquad \text{(B.15)}
\end{aligned}
$$

From Eq.(B.15) one gets

$$\frac{dA_s^2}{d\theta} = 2A_s G_{s1} = -\frac{\alpha_M\bar{R}}{Q_s}\frac{\partial H}{\partial \alpha_s} \qquad \text{(B.16)}$$

$$\frac{d\alpha_s}{d\theta} = g_{s1} = \frac{\alpha_M\bar{R}}{Q_s}\frac{\partial H}{\partial A_s^2} \qquad \text{(B.17)}$$

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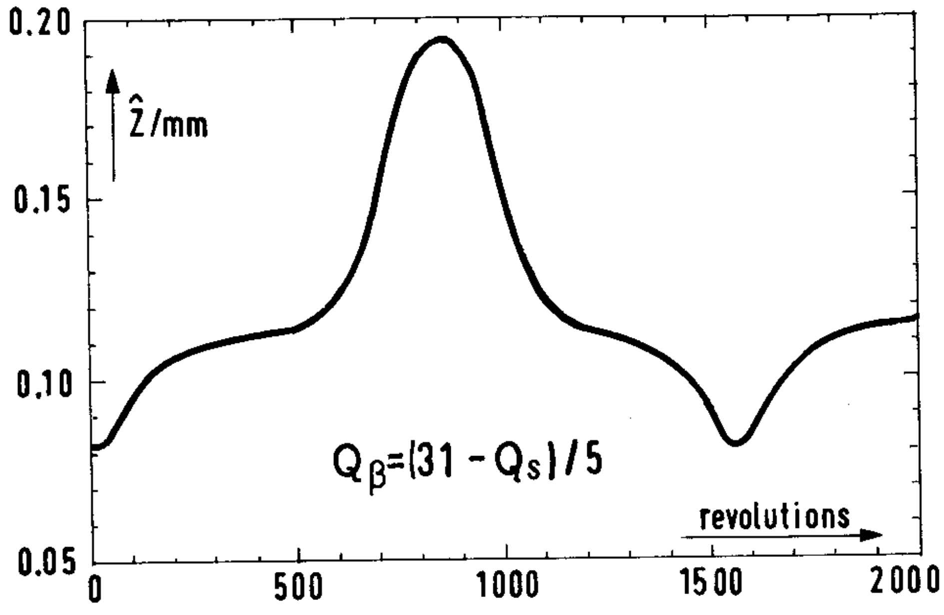

Fig.1 Betatron amplitude

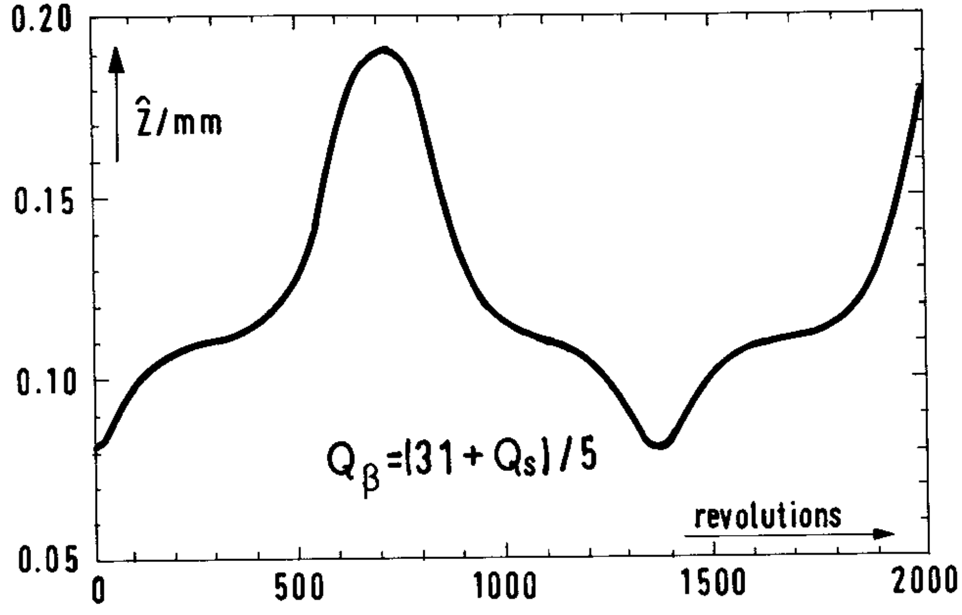

Fig.3 Betatron amplitude

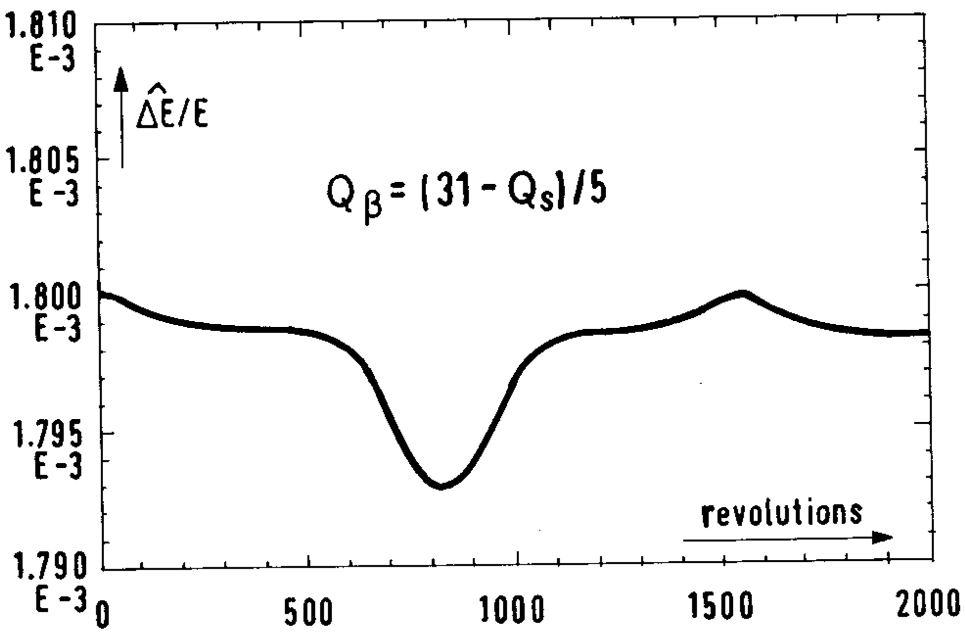

Fig.2 Synchrotron amplitude

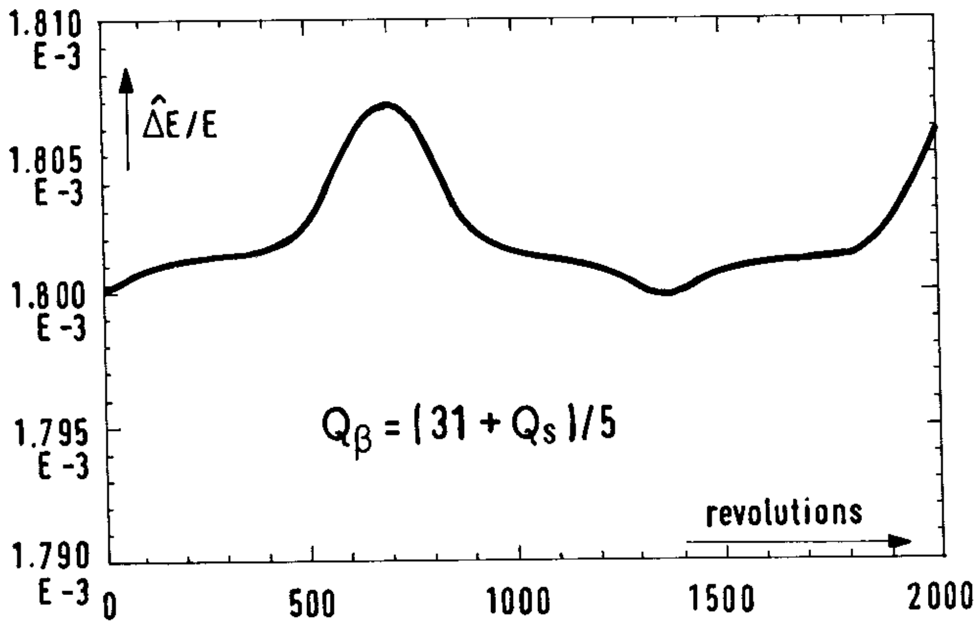

Fig.4 Synchrotron amplitude

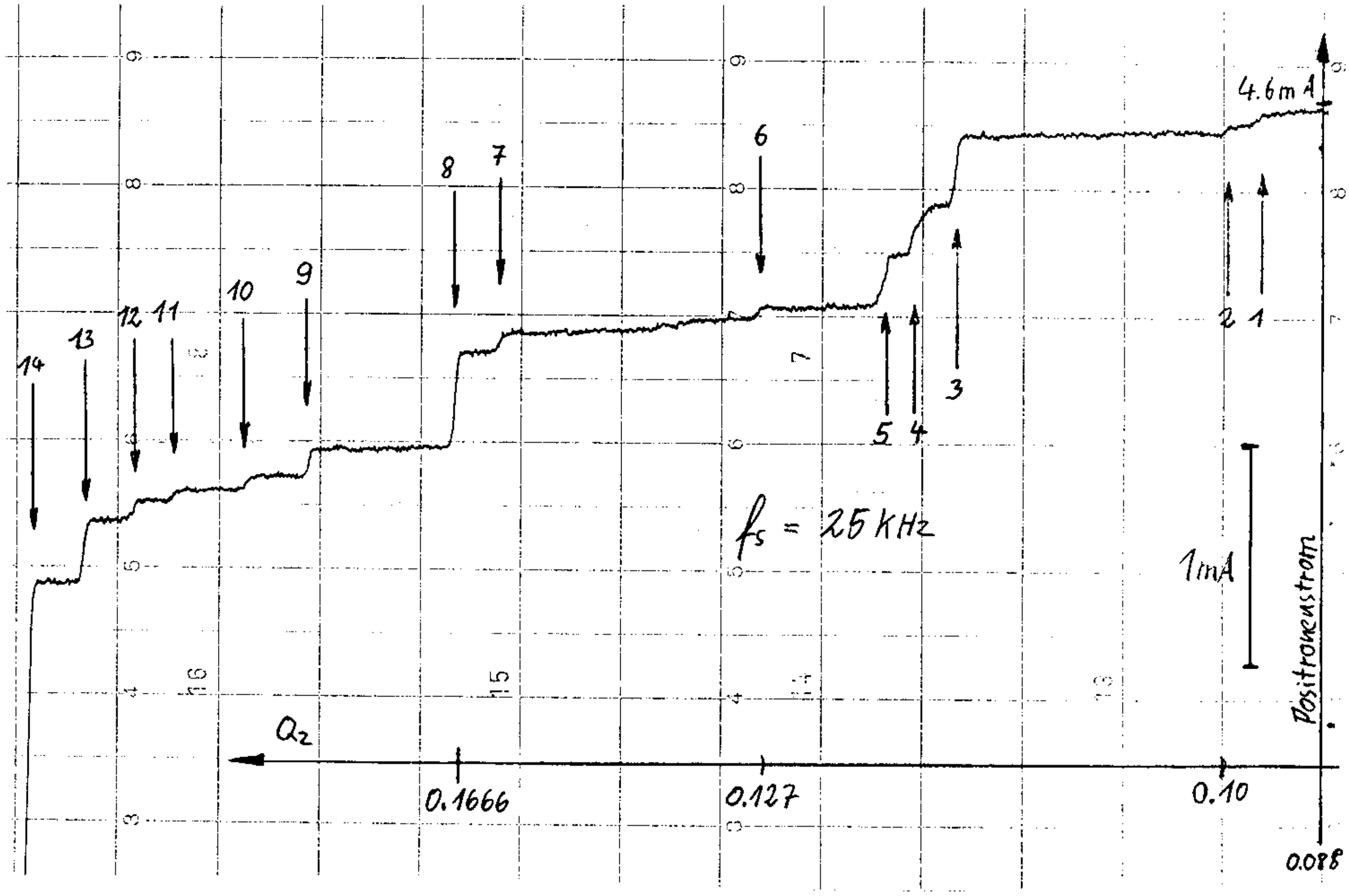

Fig.6 Variation of the beam current with the betatron frequency

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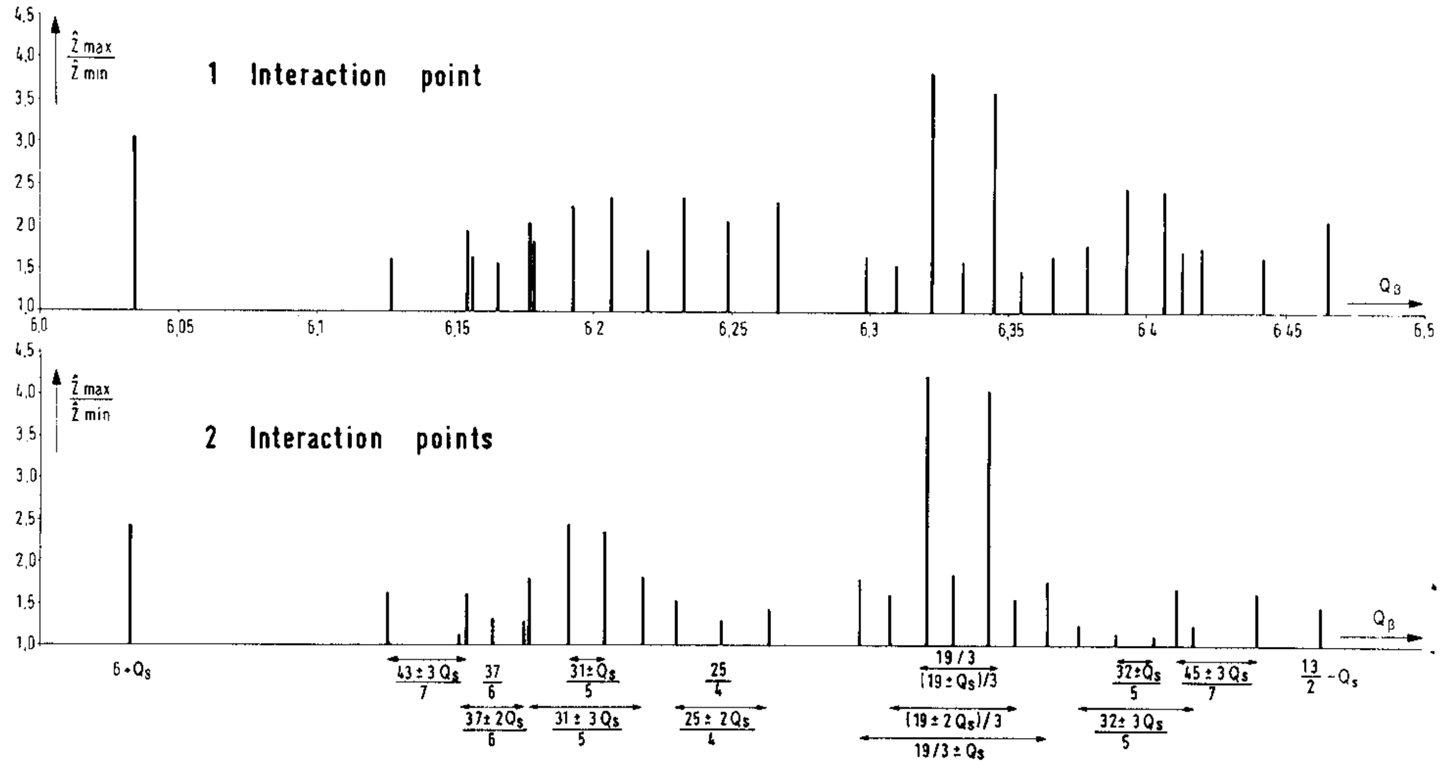

Fig.5 Maximum betatron amplitude as a function of the betatron frequency