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NOTKESTRASSE 85 · 2 HAMBURG 52

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Abstract:
We consider classical field theories of arbitrary space-time dimension d+1 ($d \geq 1$), where the fields take their values on an arbitrary manifold of dimension m ($m \geq 1$). Then those conserved topological currents $j^\mu(x)$ which depend only on the fields and their first space-time derivatives at x, are determined by those closed differential forms on the manifold, whose degree does not exceed the minimum min(m,d) of m and d.

1. Introduction

We consider a classical field theory of the following kind: Space-time is of arbitrary dimension d+1 ($d \geq 1$). The fields are points on an arbitrary manifold M of dimension m ($m \geq 1$). A law of motion is given for the fields, and energy conservation holds. All functions occuring in this paper are assumed to be sufficiently often continuous differentiable.

Following Coleman[1] one can divide the space of finite-energy solutions at a fixed time t into homotopy classes. Two solutions belong to the same class, if for either the initial-value data at time t can be continuously changed into that of the other, so that the energy remains bounded during the deformation. Since time evolution is continuous and leaves the energy constant, a solution stays in its class during its time evolution.

Now let us explain what a "topological conservation law" is: A topological conservation law is a map from the space of solutions into the real line, which remains constant on every homotopy class. Many conservation laws can be derived from a conserved current. Then the conserved quantity, usually called charge, is the integral of the time-component of the current over the whole space. Thus we are led to the following definition: A "conserved topological current" is a map from the space of solutions into the space of conserved currents. This yields a map from the space of solutions onto the set of the corresponding charges, which is required to be a topological conservation law.[2,3]

In the following we restrict our interest to those conserved topological currents $j^\mu(x)$, which depend only on the fields and their first space-time derivatives at x. The integral of a topological charge density $\varsigma = j^0$ over the whole space at time t does not change, if the initial-value data at time t are deformed continuously in a way which leaves the energy bounded. In this paper however we shall find the set of those conserved currents, whose charges - contained in an arbitrary open and bounded region of space at any time t - do not change during continuous finite-energy deformations of the initial-value data at time t, which take place in an arbitrary compact subset of this region. The conserved topological currents then are certainly contained in this wider class of conserved currents.

The result is easy to understand. The integral of a charge density from this wider class of currents over an arbitrary open and bounded region of space can always be transformed into an integral over the surface of this region. Hence the charge which is contained in this region depends only on the values of the fields and their first derivatives on the surface of this region. Therefore a deformation of the initial-value data taking place inside this region cannot change this charge.

Our description of all conserved topological currents which depend only on the fields and their first derivatives is not complete. Two important problems are left. One question is whether a current <u>exists</u>, whose charge is <u>finite</u> and remains constant not only during those continuous finite-energy deformations taking place in a bounded region of space but during <u>all</u> continuous finite-energy deformations of the initial-value data at any time t. The other question is whether the homotopy classes are determined uniquely by the values of the charges. Let us ask this question in the following way: Does an ordered set of

conserved topological currents exist, so that the homotopy classes are determined uniquely by the values of the corresponding ordered set of charges?

As far as we know there are two attempts towards a systematic treatment of conserved topological currents. The work of Hertel [4] deals with currents which are not local functionals of the fields. This case is excluded from our investigations. However the work of Patani, Schlindwein and Shafi [5] is relevant for us. Indeed all examples given in [5] are contained in our result.

Section 2 of this paper is on the charge densities and section 3 on the corresponding currents. A summary of this paper is given in form of a theorem at the end of section 3.

Notations:

x^0 is the time coordinate and $x^1,\dots x^d$ are the space coordinates. $\varphi^1,\dots\varphi^m$ are the field coordinates belonging to a certain chart on $\mathbb{M}$. The abbreviations $\varphi^a_\mu = \frac{\partial\varphi^a}{\partial x^\mu}$ and $\varphi^a_{\mu\nu} = \frac{\partial^2\varphi^a}{\partial x^\mu \partial x^\nu}$ are used. Einstein's summation convention is applied. Indices a,b,... from the beginning of the alphabet range over 1... m, indices i,k... from the middle of the alphabet over 1... d and greek indices $\mu,\nu\dots$ over 0 ... d.

2. The charge density

The first step will be to derive the equations (2.5a,b,c) for the charge density ϱ. This can be done by using standard methods from the calculus of variations.

Let $\varphi = (\varphi^1, \dots \varphi^m)$ be the field coordinates belonging to a chart $\mathbb{U} \to \mathbb{M}$ on $\mathbb{M}$. φ varies in the open domain $\mathbb{U} \subseteq \mathbb{R}^m$. For $\psi \in \mathbb{U}$ there exist initial-value data at time x^0 of finite energy and a bounded open set $\mathbb{O} \subset \mathbb{R}^p$ which contains the origin $x^i = 0$, so that the restriction of the initial-value data at time x^0 onto $\mathbb{O}$ can be represented by $\varphi^a(x^0, x^1, \dots x^d)$, $\varphi^a_0(x^0, x^1, \dots x^d)$ and that $\varphi^a(x^0, 0, \dots 0) = \psi^a$.

Now a deformation of the initial-value data is introduced, which takes place only inside of $\mathbb{O}$. $\mathbb{K} \subset \mathbb{O}$ denotes an arbitrary compact subset of $\mathbb{O}$. $\delta\varphi^a(x^1, \dots x^d)$, $\delta\varphi^a_0(x^1, \dots x^d)$ are functions on $\mathbb{O}$, vanishing on $\mathbb{O} - \mathbb{K}$. For ε from some open interval containing 0 assume that $(\dots, \varphi^a(x^0, x^1, \dots x^d) + \varepsilon \cdot \delta\varphi^a(x^0, x^1, \dots x^d), \dots)$ for $(x^1, \dots x^d) \in \mathbb{O}$ is contained in $\mathbb{U}$, and that $\varphi^a + \varepsilon \cdot \delta\varphi^a$, $\varphi^a_0 + \varepsilon \cdot \delta\varphi^a_0$ represent a restriction onto $\mathbb{O}$ of deformed initial-value data with bounded energy. ε is the parameter of deformation. The restriction onto $\mathbb{O}$ of the charge density $j^0_\varepsilon(x^0, x^1, \dots x^d)$ at time x^0 belonging to the deformed initial-value data at time x^0 then can be written as

$$j^0_\varepsilon(x^0, x^1, \dots x^d) = \varrho(\dots \varphi^a + \varepsilon \cdot \delta\varphi^a, \dots \varphi^b_0 + \varepsilon \cdot \delta\varphi^b_0, \dots \varphi^c_i + \varepsilon \frac{\partial}{\partial x^i} \delta\varphi^c, \dots).$$

The charge contained in the region $\mathbb{O}$ must be independent of the deformation parameter ε:

$$(2.1) \qquad 0 = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon = 0} \int_{\mathbb{O}} dx^1 \cdots dx^d \cdot \varrho$$

The integrand is a sum of a total divergence and the rest:

$$0 = \int_{\mathbb{O}} dx^1 \cdots dx^d \, \frac{\partial}{\partial x^i} \left(\frac{\partial \varrho}{\partial \varphi^a_i} \cdot \delta\varphi^a \right) +$$

(2.2)

$$+ \int_{\mathbb{O}} dx^1 \cdots dx^d \left[\frac{\partial \varrho}{\partial \varphi^a_0} \cdot \delta\varphi^a_0 + \left(\frac{\partial \varrho}{\partial \varphi^a} - \frac{\partial}{\partial x^i} \frac{\partial \varrho}{\partial \varphi^a_i} \right) \delta\varphi^a \right]$$

$\delta\varphi^a$ is zero in $\mathbb{O} - \mathbb{K}$, and therefore the first summand vanishes. Now assume that the class of variations $\delta\varphi^a$, $\delta\varphi^a_0$ with support in $\mathbb{K}$ is large enough to conclude from (2.2) that for $(x^1, \dots x^d)$ inside of $\mathbb{K}$:

$$(2.3a) \qquad \frac{\partial \varrho}{\partial \varphi^a_0} = 0$$

$$(2.3b) \qquad \frac{\partial \varrho}{\partial \varphi^a} - \frac{\partial}{\partial x^i} \frac{\partial \varrho}{\partial \varphi^a_i} = 0$$

This is certainly the case, if the energy is not a too pathological functional of the initial-value data. As $\mathbb{K}$ ($\subset \mathbb{O}$ and compact) is arbitrary, the equations (2.3a,b) are valid for all $(x^1, \dots x^d) \in \mathbb{O}$. They can be written in the form

$$(2.4a) \qquad \frac{\partial \varrho}{\partial \varphi^a_0} = 0$$

$$(x^1_1 \cdots x^d) \in \mathbb{O}$$

$$(2.4b) \qquad \frac{\partial \varrho}{\partial \varphi^a} - \frac{\partial^2 \varrho}{\partial \varphi^a_i \, \partial \varphi^b} \varphi^b_i - \frac{\partial^2 \varrho}{\partial \varphi^a_i \, \partial \varphi^b_k} \varphi^b_{ki} = 0$$

Consider (2.4a,b) at time x^0 and at the point $x^i = 0$ in space. Notice that at this point these equations are valid for <u>all</u> initial-value data at time x^0 belonging to finite energy, so that at this point the fields can be represented by $\varphi^a(x^0,0,\dots 0) = \psi^a$. If the energy functional behaves well enough, then the class of initial-value data at time x^0 belonging to finite energy is so large that for every $(\varphi^1(x^0,0,\dots 0),\dots\varphi^m(x^0,0,\dots 0)) = (\psi^1,\dots\psi^m) \in \mathbb{U}$ the $\varphi^a_\mu(x^0,0,\dots 0) \in \mathbb{R}^1$ and $\varphi^a_{ik}(x^0,0,\dots 0) + \varphi^a_{ki} \in \mathbb{R}^1$ can be treated as independent variables in (2.4a,b). Keeping in mind that the charge density depends only on the fields and their first space-time derivatives we conclude from (2.4a,b):

(2.5a) $$\frac{\partial\rho}{\partial\varphi^a_0} = 0$$

(2.5b) $$\frac{\partial^2\rho}{\partial\varphi^a_i\,\partial\varphi^b_k} + \frac{\partial^2\rho}{\partial\varphi^a_k\,\partial\varphi^b_i} = 0 \qquad \left[\begin{matrix}(\varphi^1,\dots\varphi^m)\in\mathbb{U}\\ \varphi^i_\mu\in\mathbb{R}^1\end{matrix}\right]$$

(2.5c) $$\frac{\partial\rho}{\partial\varphi^a} - \frac{\partial^2\rho}{\partial\varphi^a_i\,\partial\varphi^b}\varphi^b_i = 0$$

where $\mathbb{U}\to\mathbb{M}$, $\mathbb{U}\subseteq\mathbb{R}^m$ is an arbitrary chart on $\mathbb{M}$.

The second step in this section will be to work out conditions equivalent to (2.5a,b,c) and to explain their differential geometric meaning.

Because of (2.5a) the charge density ρ is independent of the time derivatives φ^a_0. From (2.5b) we conclude that ρ is a polynomial

(2.6) $$\rho = \sum_{r=0}^{\min(m,d)} \rho^{i_1,\dots i_r}_{a_1,\dots a_r}(\varphi)\cdot\varphi^{a_1}_{i_1}\cdots\varphi^{a_r}_{i_r}$$

in the φ^a_i, whose degree does not exceed the minimum min(m,d) of m and d, where the $\rho^{i_1,\dots i_r}_{a_1,\dots a_r}(\varphi)$ are totally antisymmetric in their upper and lower indices and depend on the field coordinates φ^a only. To see this consider at first (2.5b) for a = b and i = k: The second order derivative of ρ with respect to any φ^a_i vanishes. Therefore ρ has to be linear in every φ^a_i. This means that ρ is a polynomial of the form (2.6) in the φ^a_i of a degree, which is finite but possibly higher than min(m,d). The $\rho^{i_1,\dots i_r}_{a_1,\dots a_r}$ can be assumed to remain constant when an arbitrary permutation is applied simultaneously onto their upper and lower indices. Using this symmetry property and (2.5b) we get for r = 2,3,... :

$$\left[\rho^{i,k,i_3,\dots i_r}_{a,b,a_3,\dots a_r} + \rho^{k,i,i_3,\dots i_r}_{a,b,a_3,\dots a_r}\right]\cdot\varphi^{i_3}_{a_3}\dots\varphi^{i_r}_{a_r} = 0 .$$

Thus $\rho^{i_1,i_2,\dots i_r}_{a_1,a_2,\dots a_r}$ is antisymmetric in i_1 and i_2. From the above mentioned property of symmetry then follows the total antisymmetry of $\rho^{i_1,\dots i_r}_{a_1,\dots a_r}$ in its upper and lower indices.

Insertion of (2.5c) into (2.6) with $a_0 = a$ and Kronecker symbol δ^i_k leads to

$$0 = \left(\frac{\partial}{\partial\varphi^{a_0}}\rho^{i_1,\dots i_r}_{a_1,\dots a_r}\right)\varphi^{a_1}_{i_1}\dots\varphi^{a_r}_{i_r} -$$

$$-\left(\frac{\partial}{\partial\varphi^b}\rho^{i_1,\dots i_r}_{a_1,\dots a_r}\right)\varphi^b_i\left(\delta^i_{i_1}\delta^{a_1}_{a_0}\varphi^{a_2}_{i_2}\dots\varphi^{a_r}_{i_r}\dots + \delta^i_{i_r}\delta^{a_r}_{a_0}\varphi^{a_1}_{i_1}\dots\varphi^{a_{r-1}}_{i_{r-1}}\right)$$

$$= \left(\frac{\partial}{\partial\varphi^{a_0}}\rho^{i_1,\dots i_r}_{a_1,\dots a_r}\right)\varphi^{a_1}_{i_1}\dots\varphi^{a_r}_{i_r} -$$

$$-\Big[\left(\frac{\partial}{\partial\varphi^b}\rho^{i_1,\dots i_r}_{a_0,a_2,\dots a_r}\right)\varphi^b_{i_1}\varphi^{a_2}_{i_2}\dots\varphi^{a_r}_{i_r}\dots +$$

$$+\left(\frac{\partial}{\partial\varphi^b}\rho^{i_1\dots i_r}_{a_1,\dots a_{r-1},a_0}\right)\varphi^b_{i_r}\varphi^{a_1}_{i_1}\dots\varphi^{a_{r-1}}_{i_{r-1}}\Big] =$$

$$0 = \varphi^{a_1}_{i_1}\cdots\varphi^{a_r}_{i_r}\left[\frac{\partial}{\partial\varphi^{a_0}}\rho^{i_1,\dots i_r}_{a_1,\dots a_r} - \left(\frac{\partial}{\partial\varphi^{a_1}}\rho^{i_1,\dots\ i_r}_{a_0,a_2,\dots a_r}\cdots + \frac{\partial}{\partial\varphi^{a_r}}\rho^{i_1,\dots\ i_r}_{a_1,\dots a_{r-1},a_0}\right)\right] =$$

$$0 = \varphi^{a_1}_{i_1}\cdots\varphi^{a_r}_{i_r}\left(\frac{\partial}{\partial\varphi^{a_0}}\rho^{i_1,\dots i_r}_{a_1,\dots a_r} - \frac{\partial}{\partial\varphi^{a_1}}\rho^{i_1,\dots\ i_r}_{a_0,a_2,\dots a_r} + \frac{\partial}{\partial\varphi^{a_2}}\rho^{i_1,\dots\ i_r}_{a_0,a_1,a_3,\dots a_r}\cdots(-1)^r\frac{\partial}{\partial\varphi^{a_r}}\rho^{i_1,\dots i_r}_{a_0,\dots a_{r-1}}\right)$$

The expression in the bracket (...) is totally antisymmetric in $i_1,\dots i_r$ and in $a_0,\dots a_r$. To write our formulas in a more compact form the symbol $\varepsilon^{b_1,\dots b_s}_{a_1,\dots a_s}$ is introduced, which is equal to +1, if $b_1,\dots b_s$ is an even permutation of $a_1,\dots a_s$, -1 for odd permutations and vanishing in the other cases. The last equation then can be written as

$$(2.7)\qquad 0 = \varphi^{a_1}_{i_1}\cdots\varphi^{a_r}_{i_r}\cdot\varepsilon^{b_0,\cdots b_r}_{a_0,\cdots a_r}\frac{\partial}{\partial\varphi^{b_0}}\rho^{i_1,\cdots i_r}_{b_1,\cdots b_r}$$

which is equivalent to

$$(2.8)\qquad 0 = \varepsilon^{b_0,b_1,\cdots b_r}_{a_0,a_1,\cdots a_r}\frac{\partial}{\partial\varphi^{b_0}}\rho^{i_1,\cdots i_r}_{b_1,\cdots b_r}\qquad\left[r = 0,1,\cdots\min(m,d)\right]$$

Equations (2.6) and (2.8) are the desired conditions equivalent to (2.5a,b,c). This result is repeated as

Lemma 2.1:

The equations (2.5a,b,c) for the charge density ρ are equivalent to the following statements: ρ is a polynomial

$$\rho = \sum_{r=0}^{\min(m,d)}\rho^{i_1,\dots i_r}_{a_1,\dots a_r}(\varphi)\,\varphi^{a_1}_{i_1}\cdots\varphi^{a_r}_{i_r}$$

in the φ^a_i. The degree of this polynomial does not exceed the minimum min(m,d) of the manifold dimension m and the space dimension d. The coefficients $\rho^{i_1,\dots i_r}_{a_1,\dots a_r}(\varphi)$ of this polynomial depend only on the field coordinates φ^a, are totally antisymmetric in their upper and their lower indices and obey the constraining equations

$$0 = \varepsilon^{b_0,b_1,\dots b_r}_{a_0,a_1,\dots a_r}\frac{\partial}{\partial\varphi^{b_0}}\rho^{i_1,\dots i_r}_{b_1,\dots b_r}\qquad\left[r = 0,1,\dots\min(m,d)\right]$$

This result can be expressed in the language of differential geometry. For a detailed explanation of the occuring special termini one may consult textbooks on this subject. We give the two references [6,7].

First let us explain some notations: $P(\varphi)\in\mathbb{M}$ denotes the point on the manifold belonging to the coordinates φ^a. At any point P the m-dimensional tangent space exists, spanned by the $\frac{\partial P}{\partial\varphi^a}$, and over this space the tensor calculus can be built up. The dual space exists, spanned by the dual basis u^a so that $\langle u^b|\frac{\partial P}{\partial\varphi^a}\rangle = \delta^b_a$. The space of antisymmetric tensors of degree r (r = 0,1,...m) is spanned by the exterior (totally antisymmetric) products $\frac{\partial P}{\partial\varphi^{a_1}}\wedge\dots\frac{\partial P}{\partial\varphi^{a_r}}$. Its dual space is spanned by the products $u^{a_1}\wedge\dots u^{a_r}$, and $\langle u^{b_1}\wedge\dots u^{b_r}|\frac{\partial P}{\partial\varphi^{a_1}}\wedge\dots\frac{\partial P}{\partial\varphi^{a_r}}\rangle = \varepsilon^{b_1,\dots b_r}_{a_1,\dots a_r}$.

A differential form of degree r (r = 0,1,...m) on $\mathbb{M}$ can be described by a tensorfield $\omega = \omega_{a_1,\dots a_r}(\varphi)\cdot u^{a_1}(\varphi)\wedge\dots u^{a_r}(\varphi)$,

where $\omega_{a_1,\dots a_r}$ is totally antisymmetric in its indices. The exterior derivative $d\wedge\omega$ of ω is defined independently of the choice of coordinates on $\mathbb{M}$ by:

$$d \wedge \omega_{a_1,\dots,a_r} u^{a_1} \wedge \dots \wedge u^{a_r} = \frac{1}{(r+1)!} \varepsilon^{b_0,\dots,b_r}_{a_0,\dots,a_r} \frac{\partial}{\partial \varphi^{b_0}} \omega_{b_1,\dots,b_r} u^{a_0} \wedge \dots \wedge u^{a_r},$$

and this is a differential form of degree r+1.

Now equation (2.8) says that the differential form $\varrho^{i_1,\dots,i_r}_{a_1,\dots,a_r} u^{a_1} \wedge \dots \wedge u^{a_r}$ has a vanishing exterior derivative:

$$(2.9) \qquad 0 = d \wedge \varrho^{i_1,\dots,i_r}_{a_1,\dots,a_r} \cdot u^{a_1} \wedge \dots \wedge u^{a_r}$$

We still need a coordinate independent representation of the charge density ϱ. Consider the expression $\langle \varrho^{i_1,\dots,i_r}_{a_1,\dots,a_r} u^{a_1} \wedge \dots \wedge u^{a_r} \mid \frac{\partial P}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial P}{\partial x^{i_r}} \rangle$, where the summation convention is not applied onto the indices $i_1,\dots,i_r$. It is equal to

$$\langle \varrho^{i_1,\dots,i_r}_{a_1,\dots,a_r} u^{a_1} \wedge \dots \wedge u^{a_r} \mid \frac{\partial P}{\partial \varphi^{b_1}} \wedge \dots \wedge \frac{\partial P}{\partial \varphi^{b_r}} \rangle \cdot \varphi^{b_1}_{i_1} \dots \varphi^{b_r}_{i_r} =$$

$$\varepsilon^{a_1,\dots,a_r}_{b_1,\dots,b_r} \varrho^{i_1,\dots,i_r}_{a_1,\dots,a_r} \varphi^{b_1}_{i_1} \dots \varphi^{b_r}_{i_r} = r!\, \varrho^{i_1,\dots,i_r}_{b_1,\dots,b_r} \varphi^{b_1}_{i_1} \dots \varphi^{b_r}_{i_r} \quad :$$

$$(2.10) \qquad \varrho^{i_1,\dots,i_r}_{a_1,\dots,a_r} \cdot \varphi^{a_1}_{i_1} \cdots \varphi^{a_r}_{i_r} = \frac{1}{r!} \langle \varrho^{i_1,\dots,i_r}_{a_1,\dots,a_r} \cdot u^{a_1} \wedge \dots \wedge u^{a_r} \mid \frac{\partial P}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial P}{\partial x^{i_r}} \rangle \qquad \text{(no sum on } i_1,\dots,i_r)$$

A coordinate independent representation of the charge density ϱ in (2.6) therefore is

$$(2.11) \qquad \varrho = \sum_{r=0}^{\min(m,d)} \frac{1}{r!} \langle \varrho^{i_1,\dots,i_r}_{a_1,\dots,a_r} u^{a_1} \wedge \dots \wedge u^{a_r} \mid \frac{\partial P}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial P}{\partial x^{i_r}} \rangle$$

In the next section we show that a charge density of this kind

is always the time-component of a conserved current. Therefore (2.9) and (2.11) lead to

Lemma 2.2:

Every charge density, which belongs to a current from the class described in section 1, is determined by a set of differential forms

$$\varrho^{i_1,\dots,i_r} = \frac{1}{r!} \varrho^{i_1,\dots,i_r}_{a_1,\dots,a_r} \cdot u^{a_1} \wedge \dots \wedge u^{a_r} \qquad (r = 0,1,\dots \min(m,d))$$

on the manifold $\mathbb{M}$, which are closed:

$$d \wedge \varrho^{i_1,\dots,i_r} = 0$$

and antisymmetric in $i_1,\dots,i_r$. The charge density ϱ is equal to

$$\varrho = \sum_{r=0}^{\min(m,d)} \langle \varrho^{i_1,\dots,i_r} \mid \frac{\partial P}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial P}{\partial x^{i_r}} \rangle \quad .$$

Finally we show how an integral of the charge density over a region of space can be transformed into an integral over the surface of this region. (For the rest of this section the summation convention is not applied onto the space indices $i_1,\dots,i_r$.) Look at the integral

$$\int \langle \varrho^{i_1,\dots,i_r} \mid \frac{\partial P}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial P}{\partial x^{i_r}} \rangle \cdot dx^{i_1} \dots dx^{i_r} ,$$

where $(x^{i_1},\dots x^{i_r})$ varies in a region $\tilde{\mathbb{O}}$ of $\mathbb{R}^r$, while the other space coordinates are held fixed. $\frac{\partial P}{\partial x^{i_1}} \wedge \dots \wedge \frac{\partial P}{\partial x^{i_r}} dx^{i_1} \dots dx^{i_r}$ is the oriented volume element of the sub-manifold $\tilde{\mathbb{M}} \subset \mathbb{M}$ of $\mathbb{M}$ covered by all $P[\varphi(x^0,\dots x^d)]$ during the variation of $(x^{i_1},\dots x^{i_r})$ in $\tilde{\mathbb{O}}$. The integral runs over all these volume elements on $\tilde{\mathbb{M}}$, multiplied with the differential form $\varrho^{i_1,\dots,i_r}$. Now as this form is closed, there exists – at least locally – another form $\tilde{\varrho}^{i_1,\dots,i_r}$ of degree r-1, with an exterior derivative $d \wedge \tilde{\varrho}^{i_1,\dots,i_r}$

equal to $\varrho^{i_1,\ldots i_r}$. Therefore the above integral can be transformed into an integral over all oriented volume elements on the surface of $\tilde{M}$, multiplied with the form $\tilde{\varrho}^{i_1,\ldots i_r}$. As the surface of $\tilde{M}$ corresponds to the surface of $\tilde{\mathcal{O}}$ via $F[\varphi(x^0,\ldots x^d)]$, we arrive at an integral over the surface of $\tilde{\mathcal{O}}$. Hence an integral

$$\int \langle \varrho^{i_1,\ldots i_r} \Big| \frac{\partial F}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial F}{\partial x^{i_r}} \rangle \cdot dx^1 \ldots dx^d$$

over a region $\mathcal{O}$ of space $\mathbb{R}^d$ can be transformed into an integral over the surface of $\mathcal{O}$. As the integral of the charge density is a finite sum over integrals of this type, we have finished our task.

3. The current

Let us look for a conserved current j^μ, which depends only on the φ^a and φ^a_ν:

$$(3.1) \qquad 0 = \frac{\partial j^\mu}{\partial x^\mu} = \frac{\partial j^\mu}{\partial \varphi^a}\varphi^a_\mu + \frac{\partial j^\mu}{\partial \varphi^a_\nu}\varphi^a_{\nu\mu}$$

Notice that (3.1) holds for all initial value data φ^a, φ^a_0 and that no second order time derivatives occur in this equation, for we already know $\varrho = j^0$ to be independent of φ^a_0. Therefore we can proceed similarly as in section 2. Then from (3.1) follows:

$$(3.2a) \qquad \frac{\partial j^\mu}{\partial \varphi^a}\varphi^a_\mu = 0$$

$$(3.2b) \qquad \frac{\partial j^\mu}{\partial \varphi^a_\nu} + \frac{\partial j^\nu}{\partial \varphi^a_\mu} = 0$$

where the φ^a, φ^a_μ can be treated as independent variables.

To derive equations which are decoupled with respect to the different components j^μ, apply $\frac{\partial}{\partial \varphi^b_\nu}$ and (3.2b) onto (3.2a):

$$0 = \frac{\partial j^\nu}{\partial \varphi^b} + \frac{\partial^2 j^\mu}{\partial \varphi^a \partial \varphi^b_\nu}\varphi^a_\mu =$$

$$(3.3a) \qquad 0 = \frac{\partial j^\nu}{\partial \varphi^b} - \frac{\partial^2 j^\nu}{\partial \varphi^a \partial \varphi^b_\mu}\varphi^a_\mu$$

and consider $\frac{\partial^2 j^\mu}{\partial \varphi^a_\nu \partial \varphi^b_\varrho}$. Because of (3.2b) it is antisymmetric in μ,ν and in μ,ϱ, hence it must be antisymmetric in ν,ϱ too:

$$(3.3b) \qquad \frac{\partial^2 j^\mu}{\partial \varphi^a_\nu \partial \varphi^b_\varrho} + \frac{\partial^2 j^\mu}{\partial \varphi^a_\varrho \partial \varphi^b_\nu} = 0$$

We take (3.3a,b) and (3.2b) as new starting point. These conditions obviously are necessary for (3.2a,b). Later on we shall show them to be sufficient too, but first compare (3.2b) for $\mu = \nu$ and (3.3a,b) with (2.5a,c,b). Because of the obvious analogy lemma 2.1 holds for every j^μ instead of $j^0 = \rho$, where φ^a_0 has to be replaced by φ^a_μ:

Lemma 3.1

The current j^μ is equal to

$$j^\mu = \sum_{r=0}^{\min(m,d)} j^{\mu;\mu_1,\dots\mu_r}_{a_1,\dots a_r}(\varphi)\cdot\varphi^{a_1}_{\mu_1}\cdots\varphi^{a_r}_{\mu_r} \quad .$$

The $j^{\mu;\mu_1,\dots\mu_r}_{a_1,\dots a_r}$ are totally antisymmetric in $\mu_1,\dots\mu_r$ and in $a_1,\dots a_r$, vanish if one of the $\mu_1,\dots\mu_r$ is equal to μ, and obey the constraining equations

$$\varepsilon^{b_0,b_1,\dots b_r}_{a_0,a_1,\dots a_r}\frac{\partial}{\partial\varphi^{b_0}}\, j^{\mu;\mu_1,\dots\mu_r}_{b_1,\dots b_r} = 0$$

There is left the task to work out the conditions coming from (3.2b) for μ,ν arbitrary, but first let us show that (3.2b) and (3.3a,b) are sufficient for (3.2a,b). From (3.3a) and (3.2b) follows

$$0 = \frac{\partial j^\nu}{\partial\varphi^b} + \frac{\partial^2 j^\mu}{\partial\varphi^a\partial\varphi^b_\nu}\varphi^a_\mu = \frac{\partial}{\partial\varphi^b_\nu}\left(\frac{\partial j^\mu}{\partial\varphi^a}\varphi^a_\mu\right) \quad .$$

Therefore the left hand side in (3.2a) does not depend on φ^b_ν. From lemma 3.1 we know the j^μ to be polynomials in the φ^b_ν, hence the left hand side in (3.2a) is a polynomial with its constant coefficient vanishing and thus vanishes identically.

Finally (3.2b) must be worked out. Inserting it into the expression for j^μ, given in lemma 3.1, yields

$$0 = \frac{\partial}{\partial\varphi^a_\nu} j^{\mu;\mu_1,\dots\mu_r}_{a_1,\dots a_r}\cdot\varphi^{a_1}_{\mu_1}\cdots\varphi^{a_r}_{\mu_r} + \frac{\partial}{\partial\varphi^a_\mu} j^{\nu;\mu_1,\dots\mu_r}_{a_1,\dots a_r}\cdot\varphi^{a_1}_{\mu_1}\cdots\varphi^{a_r}_{\mu_r} =$$

$$0 = r\left(j^{\mu;\nu,\mu_2,\dots\mu_r}_{a,a_2,\dots a_r} + j^{\nu;\mu,\mu_2,\dots\mu_r}_{a,a_2,\dots a_r}\right)\cdot\varphi^{a_2}_{\mu_2}\cdots\varphi^{a_r}_{\mu_r} \quad ,$$

and $j^{\mu;\mu_1,\dots\mu_r}_{a_1,\dots a_r}$ is antisymmetric in μ,μ_1. The total antisymmetry in $\mu_1,\dots\mu_r$ then implies the total antisymmetry in $\mu,\mu_1,\dots\mu_r$.

The result of this paper is now repeated as

Theorem 3.1:

A current j^μ belongs to the class of currents described in section 1, if and only if it has the following properties:
j^μ is a polynomial

$$j^\mu = \sum_{r=0}^{\min(m,d)} j^{\mu,\mu_1,\dots\mu_r}_{a_1,\dots a_r}(\varphi)\ \varphi^{a_1}_{\mu_1}\cdots\varphi^{a_r}_{\mu_r}$$

in the field derivatives φ^a_μ, whose degree does not exceed the minimum min(m,d) of the manifold dimension m and the space dimension d. The coefficients $j^{\mu,\mu_1,\dots\mu_r}_{a_1,\dots a_r}(\varphi)$ depend only on the field coordinates φ^a, are totally antisymmetric in $\mu,\mu_1,\dots\mu_r$ and in $a_1,\dots a_r$ and obey the equations+)

$$\varepsilon^{b_0,b_1,\dots b_r}_{a_0,a_1,\dots a_r}\frac{\partial}{\partial\varphi^{b_0}}\, j^{\mu,\mu_1,\dots\mu_r}_{b_1,\dots b_r} = 0 \ .$$

Differential geometric formulation:
j^μ is determined by a set of differentialforms

$$j^{\mu,\mu_1,\dots\mu_r} = \frac{1}{r!}\, j^{\mu,\mu_1,\dots\mu_r}_{a_1,\dots a_r}\, u^{a_1}\wedge\dots\wedge u^{a_r} \quad (r = 0,1,\dots\min(m,d))$$

of degree r on the manifold $\mathbb{M}$, which are closed:

$$d\wedge j^{\mu,\mu_1,\dots\mu_r} = 0$$

+) $\varepsilon^{b_0,\dots b_r}_{a_0,\dots a_r}$ is equal to +1 if $b_0,\dots b_r$ is an even permutation of $a_0,\dots a_r$, -1 for odd permutations and vanishes in the other cases.

and totally antisymmetric in $\mu,\mu_1,\dots\mu_r$. The current j^μ is equal to

$$j^\mu = \sum_{r=0}^{\min(m,d)} \langle j^{\mu,\mu_1,\dots\mu_r} \mid \frac{\partial P}{\partial x^{\mu_1}} \wedge\dots\wedge \frac{\partial P}{\partial x^{\mu_r}} \rangle .$$

If a differential form ω of degree r is closed, then it can be represented - at least locally - as an exterior derivative of another form $\tilde{\omega}$ of degree r-1 (r = 1,...m), and conversely, if such a representation holds, then ω is closed. Therefore $j^{\mu,\mu_1,\dots\mu_r}_{a_1,\dots a_r}$ can be represented as

$$j^{\mu,\mu_1,\dots\mu_r}_{a_1,\dots a_r} = \varepsilon^{b_1,\dots b_r}_{a_1,\dots a_r} \frac{\partial}{\partial\varphi^{b_1}} \tilde{j}^{\mu,\mu_1,\mu_2,\dots\mu_r}_{b_2,\dots b_r} .$$

Then the general form of the current j^μ is

$$(3.4)\quad j^\mu = (\text{const.})^\mu + \sum_{r=1}^{\min(m,d)} \varepsilon^{b_1,\dots b_r}_{a_1,\dots a_r} \left(\frac{\partial}{\partial\varphi^{b_1}} \tilde{j}^{\mu,\mu_1,\mu_2,\dots\mu_r}_{b_2,\dots b_r} \right) \varphi^{a_1}_{\mu_1} \cdots \varphi^{a_r}_{\mu_r}$$

where the $\tilde{j}^{\mu,\mu_1,\mu_2,\dots\mu_r}_{a_2,\dots a_r}$ are arbitrary but totally antisymmetric in $\mu,\mu_1,\dots\mu_r$ and in $a_2,\dots a_r$.

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